Test Yourself

Answers to Test Yourself questions are located at the end of each section.

1. If R is a relation from A to B, $x \in A$, and $y \in B$, the notation $x\,R\,y$ means that _____.

2. If R is a relation from A to B, $x \in A$, and $y \in B$, the notation $x \not R\, y$ means that _____.

3. If R is a relation from A to B, $x \in A$, and $y \in B$, then $(y, x) \in R^{-1}$ if, and only if, _____.

4. A relation on a set A is a relation from _____ to _____.

5. If R is a relation on a set A, the directed graph of R has an arrow from x to y if, and only if, _____.

Exercise Set 8.1*

1. As in Example 8.1.2, the **congruence modulo 2** relation E is defined from **Z** to **Z** as follows: For all integers m and n,

$$m\,E\,n \quad \Leftrightarrow \quad m - n \text{ is even.}$$

   a. Is $0\,E\,0$? Is $5\,E\,2$? Is $(6, 6) \in E$? Is $(-1, 7) \in E$?
   b. Prove that for any even integer n, $n\,E\,0$.

H 2. Prove that for all integers m and n, $m - n$ is even if, and only if, both m and n are even or both m and n are odd.

3. The **congruence modulo 3** relation, T, is defined from **Z** to **Z** as follows: For all integers m and n,

$$m\,T\,n \quad \Leftrightarrow \quad 3 \mid (m - n).$$

   a. Is $10\,T\,1$? Is $1\,T\,10$? Is $(2, 2) \in T$? Is $(8, 1) \in T$?
   b. List five integers n such that $n\,T\,0$.
   c. List five integers n such that $n\,T\,1$.
   d. List five integers n such that $n\,T\,2$.
   H e. Make and prove a conjecture about which integers are related by T to 0, which integers are related by T to 1, and which integers are related by T to 2.

4. Define a relation P on **Z** as follows: For all $m, n \in \mathbf{Z}$,

$$m\,P\,n \quad \Leftrightarrow \quad m \text{ and } n \text{ have a common prime factor.}$$

   a. Is $15\,P\,25$? b. $22\,P\,27$?
   c. Is $0\,P\,5$? d. Is $8\,P\,8$?

5. Let $X = \{a, b, c\}$. Recall that $\mathscr{P}(X)$ is the power set of X. Define a relation **R** on $\mathscr{P}(X)$ as follows: For all $A, B \in \mathscr{P}(X)$,

$$A\,\mathbf{R}\,B \quad \Leftrightarrow \quad A \text{ has the same number of elements as } B.$$

   a. Is $\{a, b\}\,\mathbf{R}\,\{b, c\}$? b. Is $\{a\}\,\mathbf{R}\,\{a, b\}$?
   c. Is $\{c\}\,\mathbf{R}\,\{b\}$?

6. Let $X = \{a, b, c\}$. Define a relation **J** on $\mathscr{P}(X)$ as follows: For all $A, B \in \mathscr{P}(X)$,

$$A\,\mathbf{J}\,B \quad \Leftrightarrow \quad A \cap B \neq \emptyset.$$

   a. Is $\{a\}\,\mathbf{J}\,\{c\}$? b. Is $\{a, b\}\,\mathbf{J}\,\{b, c\}$?
   c. Is $\{a, b\}\,\mathbf{J}\,\{a, b, c\}$?

7. Define a relation R on **Z** as follows: For all integers m and n,

$$m\,R\,n \quad \Leftrightarrow \quad 5 \mid (m^2 - n^2).$$

   a. Is $1\,R\,(-9)$? b. Is $2\,R\,13$?
   c. Is $2\,R\,(-8)$? d. Is $(-8)\,R\,2$?

8. Let A be the set of all strings of a's and b's of length 4. Define a relation R on A as follows: For all $s, t \in A$,

$$s\,R\,t \quad \Leftrightarrow \quad s \text{ has the same first two characters as } t.$$

   a. Is $abaa\,R\,abba$? b. Is $aabb\,R\,bbaa$?
   c. Is $aaaa\,R\,aaab$? d. Is $baaa\,R\,abaa$?

9. Let A be the set of all strings of 0's, 1's, and 2's of length 4. Define a relation R on A as follows: For all $s, t \in A$,

$$s\,R\,t \quad \Leftrightarrow \quad \text{the sum of the characters in } s \text{ equals the sum of the characters in } t.$$

   a. Is $0121\,R\,2200$? b. Is $1011\,R\,2101$?
   c. Is $2212\,R\,2121$? d. Is $1220\,R\,2111$?

10. Let $A = \{3, 4, 5\}$ and $B = \{4, 5, 6\}$ and let R be the "less than" relation. That is, for all $(x, y) \in A \times B$,

$$x\,R\,y \quad \Leftrightarrow \quad x < y.$$

State explicitly which ordered pairs are in R and R^{-1}.

11. Let $A = \{3, 4, 5\}$ and $B = \{4, 5, 6\}$ and let S be the "divides" relation. That is, for all $(x, y) \in A \times B$,

$$x\,S\,y \quad \Leftrightarrow \quad x \mid y.$$

State explicitly which ordered pairs are in S and S^{-1}.

12. a. Suppose a function $F: X \to Y$ is one-to-one but not onto. Is F^{-1} (the inverse relation for F) a function? Explain your answer.
   b. Suppose a function $F: X \to Y$ is onto but not one-to-one. Is F^{-1} (the inverse relation for F) a function? Explain your answer.

*For exercises with blue numbers or letters, solutions are given in Appendix B. The symbol H indicates that only a hint or a partial solution is given. The symbol ✶ signals that an exercise is more challenging than usual.

Draw the directed graphs of the relations defined in 13–18.

13. Define a relation R on $A = \{0, 1, 2, 3\}$ by $R = \{(0, 0), (1, 2), (2, 2)\}$.

14. Define a relation S on $B = \{a, b, c, d\}$ by $S = \{(a, b), (a, c), (b, c), (d, d)\}$.

15. Let $A = \{2, 3, 4, 5, 6, 7, 8\}$ and define a relation R on A as follows: For all $x, y \in A$,

$$x\ R\ y \quad \Leftrightarrow \quad x \mid y.$$

H **16.** Let $A = \{5, 6, 7, 8, 9, 10\}$ and define a relation S on A as follows: For all $x, y \in A$,

$$x\ S\ y \quad \Leftrightarrow \quad 2 \mid (x - y).$$

17. Let $A = \{2, 3, 4, 5, 6, 7, 8\}$ and define a relation T on A as follows: For all $x, y \in A$,

$$x\ T\ y \quad \Leftrightarrow \quad 3 \mid (x - y).$$

18. Let $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ and define a relation V on A as follows: For all $x, y \in A$,

$$x\ V\ y \Leftrightarrow 5 \mid (x^2 - y^2).$$

Exercises 19–20 refer to unions and intersections of relations. Since relations are subsets of Cartesian products, their unions and intersections can be calculated as for any subsets. Given two relations R and S from A to B,

$$R \cup S = \{(x, y) \in A \times B \mid (x, y) \in R \text{ or } (x, y) \in S\}$$
$$R \cap S = \{(x, y) \in A \times B \mid (x, y) \in R \text{ and } (x, y) \in S\}.$$

19. Let $A = \{2, 4\}$ and $B = \{6, 8, 10\}$ and define relations R and S from A to B as follows: For all $(x, y) \in A \times B$,

$$x\ R\ y \quad \Leftrightarrow \quad x \mid y \quad \text{and}$$
$$x\ S\ y \quad \Leftrightarrow \quad y - 4 = x.$$

State explicitly which ordered pairs are in $A \times B$, R, S, $R \cup S$, and $R \cap S$.

20. Let $A = \{-1, 1, 2, 4\}$ and $B = \{1, 2\}$ and define relations R and S from A to B as follows: For all $(x, y) \in A \times B$,

$$x\ R\ y \quad \Leftrightarrow \quad |x| = |y| \quad \text{and}$$
$$x\ S\ y \quad \Leftrightarrow \quad x - y \text{ is even.}$$

State explicitly which ordered pairs are in $A \times B$, R, S, $R \cup S$, and $R \cap S$.

21. Define relations R and S on $\mathbf{R}$ as follows:

$$R = \{(x, y) \in \mathbf{R} \times \mathbf{R} \mid x < y\} \quad \text{and}$$
$$S = \{(x, y) \in \mathbf{R} \times \mathbf{R} \mid x = y\}.$$

That is, R is the "less than" relation and S is the "equals" relation on $\mathbf{R}$. Graph R, S, $R \cup S$, and $R \cap S$ in the Cartesian plane.

22. Define relations R and S on $\mathbf{R}$ as follows:

$$R = \{(x, y) \in \mathbf{R} \times \mathbf{R} \mid x^2 + y^2 = 4\} \quad \text{and}$$
$$S = \{(x, y) \in \mathbf{R} \times \mathbf{R} \mid x = y\}.$$

Graph R, S, $R \cup S$, and $R \cap S$ in the Cartesian plane.

23. Define relations R and S on $\mathbf{R}$ as follows:

$$R = \{(x, y) \in \mathbf{R} \times \mathbf{R} \mid y = |x|\} \quad \text{and}$$
$$S = \{(x, y) \in \mathbf{R} \times \mathbf{R} \mid y = 1\}.$$

Graph R, S, $R \cup S$, and $R \cap S$ in the Cartesian plane.

24. In Example 8.1.7 the result of the query SELECT Patient_ID#, Name FROM S WHERE Primary_Diagnosis $= X$ is the projection onto the first two coordinates of the intersection of the set $A_1 \times A_2 \times A_3 \times \{X\}$ with the database.
   a. Find the result of the query SELECT Patient_ID#, Name FROM S WHERE Primary_Diagnosis = pneumonia.
   b. Find the result of the query SELECT Patient_ID#, Name FROM S WHERE Primary_Diagnosis = appendicitis.

Answers for Test Yourself

1. x is related to y by R 2. x is not related to y by R 3. $(x, y) \in R$ 4. A; A 5. x is related to y by R

8.2 Reflexivity, Symmetry, and Transitivity

Mathematics is the tool specially suited for dealing with abstract concepts of any kind and there is no limit to its power in this field. — P. A. M. Dirac, 1902–1984

Let $A = \{2, 3, 4, 6, 7, 9\}$ and define a relation R on A as follows: For all $x, y \in A$,

$$x\ R\ y \quad \Leftrightarrow \quad 3 \mid (x - y).$$