Example 7.2.14 Finding an Inverse Function for a Function of Two Variables

Define the inverse function $F^{-1}: \mathbf{R} \times \mathbf{R} \to \mathbf{R} \times \mathbf{R}$ for the one-to-one correspondence given in Example 7.2.10.

Solution

The solution to Example 7.2.10 shows that $F\left(\frac{u+v}{2}, \frac{u-v}{2}\right) = (u, v)$. Because F is one-to-one, this means that

$\left(\frac{u+v}{2}, \frac{u-v}{2}\right)$ is the unique ordered pair in the domain of F that is sent to (u, v) by F.

Thus, F^{-1} is defined as follows: For all $(u, v) \in \mathbf{R} \times \mathbf{R}$,

$$F^{-1}(u, v) = \left(\frac{u+v}{2}, \frac{u-v}{2}\right).$$

■

Test Yourself

1. If F is a function from a set X to a set Y, then F is one-to-one if, and only if, _____.
2. If F is a function from a set X to a set Y, then F is not one-to-one if, and only if, _____.
3. If F is a function from a set X to a set Y, then F is onto if, and only if, _____.
4. If F is a function from a set X to a set Y, then F is not onto if, and only if, _____.
5. The following two statements are _____:

   $\forall\, u, v \in U,$ if $H(u) = H(v)$ then $u = v$.

   $\forall\, u, v \in U,$ if $u \neq v$ then $H(u) \neq H(v)$.
6. Given a function $F: X \to Y$ and an infinite set X, to prove that F is one-to-one, you suppose that _____ and then you show that _____.
7. Given a function $F: X \to Y$ and an infinite set X, to prove that F is onto, you suppose that _____ and then you show that _____.
8. Given a function $F: X \to Y$, to prove that F is not one-to-one, you _____.
9. Given a function $F: X \to Y$, to prove that F is not onto, you _____.
10. A one-to-one correspondence from a set X to a set Y is a _____ that is _____.
11. If F is a one-to-one correspondence from a set X to a set Y and y is in Y, then $F^{-1}(y)$ is _____.

Exercise Set 7.2

1. The definition of one-to-one is stated in two ways:

   $\forall x_1, x_2 \in X$, if $F(x_1) = F(x_2)$ then $x_1 = x_2$

   and $\forall x_1, x_2 \in X$, if $x_1 \neq x_2$ then $F(x_1) \neq F(x_2)$.

   Why are these two statements logically equivalent?

2. Fill in each blank with the word *most* or *least*.
   a. A function F is one-to-one if, and only if, each element in the co-domain of F is the image of at ______ one element in the domain of F.
   b. A function F is onto if, and only if, each element in the co-domain of F is the image of at ______ one element in the domain of F.

H 3. When asked to state the definition of one-to-one, a student replies, "A function f is one-to-one if, and only if, every element of X is sent by f to exactly one element of Y." Give a counterexample to show that the student's reply is incorrect.

H 4. Let $f: X \to Y$ be a function. True or false? A sufficient condition for f to be one-to-one is that for all elements y in Y, there is at most one x in X with $f(x) = y$.

H 5. All but two of the following statements are correct ways to express the fact that a function f is onto. Find the two that are incorrect.
   a. f is onto $\Leftrightarrow$ every element in its co-domain is the image of some element in its domain.
   b. f is onto $\Leftrightarrow$ every element in its domain has a corresponding image in its co-domain.
   c. f is onto $\Leftrightarrow \forall y \in Y, \exists x \in X$ such that $f(x) = y$.
   d. f is onto $\Leftrightarrow \forall x \in X, \exists y \in Y$ such that $f(x) = y$.
   e. f is onto $\Leftrightarrow$ the range of f is the same as the co-domain of f.

6. Let $X = \{1, 5, 9\}$ and $Y = \{3, 4, 7\}$.
   a. Define $f: X \to Y$ by specifying that

   $$f(1) = 4, \quad f(5) = 7, \quad f(9) = 4.$$

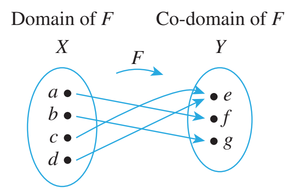
   Is f one-to-one? Is f onto? Explain your answers.

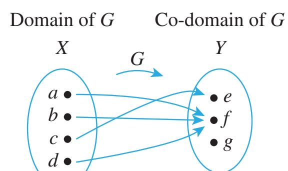
b. Define $g: X \to Y$ by specifying that

$$g(1) = 7, \quad g(5) = 3, \quad g(9) = 4.$$

Is g one-to-one? Is g onto? Explain your answers.

7. Let $X = \{a, b, c, d\}$ and $Y = \{e, f, g\}$. Define functions F and G by the arrow diagrams below.

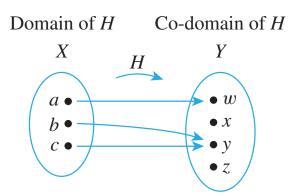


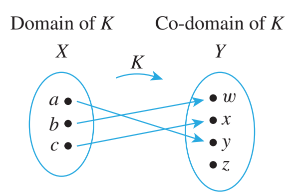


a. Is F one-to-one? Why or why not? Is it onto? Why or why not?

b. Is G one-to-one? Why or why not? Is it onto? Why or why not?

8. Let $X = \{a, b, c\}$ and $Y = \{w, x, y, z\}$. Define functions H and K by the arrow diagrams below.





a. Is H one-to-one? Why or why not? Is it onto? Why or why not?

b. Is K one-to-one? Why or why not? Is it onto? Why or why not?

9. Let $X = \{1, 2, 3\}$, $Y = \{1, 2, 3, 4\}$, and $Z = \{1, 2\}$.

a. Define a function $f: X \to Y$ that is one-to-one but not onto.

b. Define a function $g: X \to Z$ that is onto but not one-to-one.

c. Define a function $h: X \to X$ that is neither one-to-one nor onto.

d. Define a function $k: X \to X$ that is one-to-one and onto but is not the identity function on X.

10. a. Define $f: \mathbf{Z} \to \mathbf{Z}$ by the rule $f(n) = 2n$, for all integers n.

(i) Is f one-to-one? Prove or give a counterexample.

(ii) Is f onto? Prove or give a counterexample.

b. Let $2\mathbf{Z}$ denote the set of all even integers. That is, $2\mathbf{Z} = \{n \in \mathbf{Z} \mid n = 2k$, for some integer $k\}$. Define $h: \mathbf{Z} \to 2\mathbf{Z}$ by the rule $h(n) = 2n$, for all integers n. Is h onto? Prove or give a counterexample.

***H* 11.** a. Define $g: \mathbf{Z} \to \mathbf{Z}$ by the rule $g(n) = 4n - 5$, for all integers n.

(i) Is g one-to-one? Prove or give a counterexample.

(ii) Is g onto? Prove or give a counterexample.

b. Define $G: \mathbf{R} \to \mathbf{R}$ by the rule $G(x) = 4x - 5$ for all real numbers x. Is G onto? Prove or give a counterexample.

12. a. Define $F: \mathbf{Z} \to \mathbf{Z}$ by the rule $F(n) = 2 - 3n$, for all integers n.

(i) *Is* F one-to-one? Prove or give a counterexample.

(ii) Is F onto? Prove or give a counterexample.

b. Define $G: \mathbf{R} \to \mathbf{R}$ by the rule $G(x) = 2 - 3x$ for all real numbers x. Is G onto? Prove or give a counterexample.

13. **a.** Define $H: \mathbf{R} \to \mathbf{R}$ by the rule $H(x) = x^2$, for all real numbers x.

(i) Is H one-to-one? Prove or give a counterexample.

(ii) Is H onto? Prove or give a counterexample.

b. Define $K: \mathbf{R}^{nonneg} \to \mathbf{R}^{nonneg}$ by the rule $K(x) = x^2$, for all nonnegative real numbers x. Is K onto? Prove or give a counterexample.

14. Explain the mistake in the following "proof."

Theorem: The function $f: \mathbf{Z} \to \mathbf{Z}$ defined by the formula $f(n) = 4n + 3$, for all integers n, is one-to-one.

"**Proof:** Suppose any integer n is given. Then by definition of f, there is only one possible value for $f(n)$, namely, $4n + 3$. Hence f is one-to-one."

In each of 15–18 a function f is defined on a set of real numbers. Determine whether or not f is one-to-one and justify your answer.

15. $f(x) = \dfrac{x+1}{x}$, for all real numbers $x \neq 0$

16. $f(x) = \dfrac{x}{x^2+1}$, for all real numbers x

17. $f(x) = \dfrac{3x-1}{x}$, for all real numbers $x \neq 0$

18. $f(x) = \dfrac{x+1}{x-1}$, for all real numbers $x \neq 1$

19. Referring to Example 7.2.3, assume that records with the following social security numbers are to be placed in sequence into Table 7.2.1. Find the position into which each record is placed.

a. 417-30-2072 b. 364-98-1703 c. 283-09-0787

20. Define Floor: $\mathbf{R} \to \mathbf{Z}$ by the formula $\text{Floor}(x) = \lfloor x \rfloor$, for all real numbers x.
a. Is Floor one-to-one? Prove or give a counterexample.
b. Is Floor onto? Prove or give a counterexample.

21. Let S be the set of all strings of 0's and 1's, and define $l: S \to \mathbf{Z}^{nonneg}$ by

$$l(s) = \text{the length of } s, \quad \text{for all strings } s \text{ in } S.$$

a. Is l one-to-one? Prove or give a counterexample.
b. Is l onto? Prove or give a counterexample.

22. Let S be the set of all strings of 0's and 1's, and define $D: S \to \mathbf{Z}$ as follows: For all $s \in S$,

$$D(s) = \text{the number of 1's in } s \text{ minus the number of 0's in } s.$$

a. Is D one-to-one? Prove or give a counterexample.
b. Is D onto? Prove or give a counterexample.

23. Define $F: \mathscr{P}(\{a, b, c\}) \to \mathbf{Z}$ as follows: For all A in $\mathscr{P}(\{a, b, c\})$,

$$F(A) = \text{the number of elements in } A.$$

a. Is F one-to-one? Prove or give a counterexample.
b. Is F onto? Prove or give a counterexample.

24. Let S be the set of all strings of a's and b's, and define $N: S \to \mathbf{Z}$ by

$$N(s) = \text{the number of } a\text{'s in } s, \quad \text{for all } s \in S.$$

a. Is N one-to-one? Prove or give a counterexample.
b. Is N onto? Prove or give a counterexample.

25. Let S be the set of all strings in a's and b's, and define $C: S \to S$ by

$$C(s) = as, \quad \text{for all } s \in S.$$

(C is called **concatenation** by a on the left.)
a. Is C one-to-one? Prove or give a counterexample.
b. Is C onto? Prove or give a counterexample.

26. Define $S: \mathbf{Z}^+ - \mathbf{Z}^+$ by the rule: For all integers n, $S(n) = $ the sum of the positive divisors of n.
a. Is S one-to-one? Prove or give a counterexample.
b. Is S onto? Prove or give a counterexample.

H 27. Let D be the set of all finite subsets of positive integers, and define $T: \mathbf{Z}^+ \to D$ by the rule: For all integers n, $T(n) = $ the set of all of the positive divisors of n.
a. Is T one-to-one? Prove or give a counterexample.
b. Is T onto? Prove or give a counterexample.

28. Define $G: \mathbf{R} \times \mathbf{R} \to \mathbf{R} \times \mathbf{R}$ as follows: $G(x, y) = (2y, -x)$ for all $(x, y) \in \mathbf{R} \times \mathbf{R}$.
a. Is G one-to-one? Prove or give a counterexample.
b. Is G onto? Prove or give a counterexample.

29. Define $H: \mathbf{R} \times \mathbf{R} \to \mathbf{R} \times \mathbf{R}$ as follows: $H(x, y) = (x + 1, 2 - y)$ for all $(x, y) \in \mathbf{R} \times \mathbf{R}$.
a. Is H one-to-one? Prove or give a counterexample.
b. Is H onto? Prove or give a counterexample.

30. Define $J: \mathbf{Q} \times \mathbf{Q} \to \mathbf{R}$ by the rule $J(r, s) = r + \sqrt{2}s$ for all $(r, s) \in \mathbf{Q} \times \mathbf{Q}$.
a. Is J one-to-one? Prove or give a counterexample.
b. Is J onto? Prove or give a counterexample.

✶ 31. Define $F: \mathbf{Z}^+ \times \mathbf{Z}^+ \to \mathbf{Z}^+$ and $G: \mathbf{Z}^+ \times \mathbf{Z}^+ \to \mathbf{Z}^+$ as follows: For all $(n, m) \in \mathbf{Z}^+ \times \mathbf{Z}^+$,

$$F(n, m) = 3^n 5^m \quad \text{and} \quad G(n, m) = 3^n 6^m.$$

H a. Is F one-to-one? Prove or give a counterexample.
b. Is G one-to-one? Prove or give a counterexample.

32. a. Is $\log_8 27 = \log_2 3$? Why or why not?
b. Is $\log_{16} 9 = \log_4 3$? Why or why not?

The properties of logarithm established in 33–35 are used in Sections 11.4 and 11.5.

33. Prove that for all positive real numbers b, x, and y with $b \neq 1$,

$$\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y.$$

34. Prove that for all positive real numbers b, x, and y with $b \neq 1$,

$$\log_b(xy) = \log_b x + \log_b y.$$

H 35. Prove that for all real numbers a, b, and x with b and x positive and $b \neq 1$,

$$\log_b(x^a) = a \log_b x.$$

Exercises 36 and 37 use the following definition: If $f: \mathbf{R} \to \mathbf{R}$ and $g: \mathbf{R} \to \mathbf{R}$ are functions, then the function $(f + g): \mathbf{R} \to \mathbf{R}$ is defined by the formula $(f + g)(x) = f(x) + g(x)$ for all real numbers x.

36. If $f: \mathbf{R} \to \mathbf{R}$ and $g: \mathbf{R} \to \mathbf{R}$ are both one-to-one, is $f + g$ also one-to-one? Justify your answer.

37. If $f: \mathbf{R} \to \mathbf{R}$ and $g: \mathbf{R} \to \mathbf{R}$ are both onto, is $f + g$ also onto? Justify your answer.

Exercises 38 and 39 use the following definition: If $f: \mathbf{R} \to \mathbf{R}$ is a function and c is a nonzero real number, the function $(c \cdot f): \mathbf{R} \to \mathbf{R}$ is defined by the formula $(c \cdot f)(x) = c \cdot f(x)$ for all real numbers x.

38. Let $f: \mathbf{R} \to \mathbf{R}$ be a function and c a nonzero real number. If f is one-to-one, is $c \cdot f$ also one-to-one? Justify your answer.

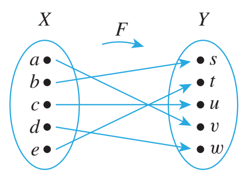
39. Let $f: \mathbf{R} \to \mathbf{R}$ be a function and c a nonzero real number. If f is onto, is $c \cdot f$ also onto? Justify your answer.

H 40. Suppose $F: X \to Y$ is one-to-one.
a. Prove that for all subsets $A \subseteq X$, $F^{-1}(F(A)) = A$.
b. Prove that for all subsets A_1 and A_2 in X, $F(A_1 \cap A_2) = F(A_1) \cap F(A_2)$.

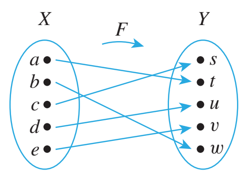
41. Suppose $F: X \to Y$ is onto. Prove that for all subsets $B \subseteq Y$, $F(F^{-1}(B)) = B$.

Let $X = \{a, b, c, d, e\}$ and $Y = \{s, t, u, v, w\}$. In each of 42 and 43 a one-to-one correspondence $F: X \to Y$ is defined by an arrow diagram. In each case draw an arrow diagram for F^{-1}.

42.



43.



In 44–55 indicate which of the functions in the referenced exercise are one-to-one correspondences. For each function that is a one-to-one correspondence, find the inverse function.

44. Exercise 10a

45. Exercise 10b

46. Exercise 11a

47. Exercise 11b

48. Exercise 12a

49. Exercise 12b

50. Exercise 21

51. Exercise 22

52. Exercise 15 with the co-domain taken to be the set of all real numbers not equal to 1.

H **53.** Exercise 16 with the co-domain taken to be the set of all real numbers.

54. Exercise 17 with the co-domain taken to be the set of all real numbers not equal to 3.

55. Exercise 18 with the co-domain taken to be the set of all real numbers not equal to 1.

56. In Example 7.2.8 a one-to-one correspondence was defined from the power set of $\{a, b\}$ to the set of all strings of 0's and 1's that have length 2. Thus the elements of these two sets can be matched up exactly, and so the two sets have the same number of elements.
   a. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set with n elements. Use Example 7.2.8 as a model to define a one-to-one correspondence from $\mathscr{P}(X)$, the set of all subsets of X, to the set of all strings of 0's and 1's that have length n.
   b. Use the one-to-one correspondence of part (a) to deduce that a set with n elements has 2^n subsets. (This provides an alternative proof of Theorem 6.3.1.)

H **57.** Write a computer algorithm to check whether a function from one finite set to another is one-to-one. Assume the existence of an independent algorithm to compute values of the function.

H **58.** Write a computer algorithm to check whether a function from one finite set to another is onto. Assume the existence of an independent algorithm to compute values of the function.

Answers for Test Yourself

1. for all x_1 and x_2 in X, if $F(x_1) = F(x_2)$ then $x_1 = x_2$ 2. there exist elements x_1 and x_2 in X such that $F(x_1) = F(x_2)$ and $x_1 \neq x_2$ 3. for all y in Y, there exists at least one element x in X such that $f(x) = y$ 4. there exists an element y in Y such that for all elements x in X, $f(x) \neq y$ 5. logically equivalent ways of expressing what it means for a function H to be one-to-one (The second is the contrapositive of the first.) 6. x_1 and x_2 are any *[particular but arbitrarily chosen]* elements in X with the property that $F(x_1) = F(x_2)$; $x_1 = x_2$ 7. y is any *[particular but arbitrarily chosen]* element in Y; there exists at least one element x in X such that $F(x) = y$ 8. show that there are concrete elements x_1 and x_2 in X with the property that $F(x_1) = F(x_2)$ and $x_1 \neq x_2$ 9. show that there is a concrete element y in Y with the property that $F(x) \neq y$ for any element x in X 10. function from X to Y; both one-to-one and onto 11. the unique element x in X such that $F(x) = y$ (in other words, $F^{-1}(y)$ is the unique preimage of y in X)

7.3 Composition of Functions

It is no paradox to say that in our most theoretical moods we may be nearest to our most practical applications. — Alfred North Whitehead

Consider two functions, the successor function and the squaring function, defined from **Z** (the set of integers) to **Z**, and imagine that each is represented by a machine. If the two machines are hooked up so that the output from the successor function is used as input