Example 6.3.4 Deriving a Generalized Associative Law

Caution! When doing problems similar to Examples 6.3.2–6.3.4, be sure to use the set properties exactly as they are stated.

Prove that for any sets A_1, A_2, A_3, and A_4,

$$((A_1 \cup A_2) \cup A_3) \cup A_4 = A_1 \cup ((A_2 \cup A_3) \cup A_4).$$

Cite a property from Theorem 6.2.2 for every step of the proof.

Solution Let A_1, A_2, A_3, and A_4 be any sets. Then

$$((A_1 \cup A_2) \cup A_3) \cup A_4 = (A_1 \cup (A_2 \cup A_3)) \cup A_4$$

by the associative law for $\cup$ with A_1 playing the role of A, A_2 playing the role of B, and A_3 playing the role of C

$$= A_1 \cup ((A_2 \cup A_3) \cup A_4)$$

by the associative law for $\cup$ with A_1 playing the role of A, $A_2 \cup A_3$ playing the role of B, and A_4 playing the role of C. ■

Test Yourself

1. Given a proposed set identity involving set variables A, B, and C, the most common way to show that the equation does not hold in general is to find concrete sets A, B, and C that, when substituted for the set variables in the equation, _____.
2. When using the algebraic method for proving a set identity, it is important to _____ for every step.
3. When applying a property from Theorem 6.2.2, it must be used _____ as it is stated.

Exercise Set 6.3

For each of 1–4 find a counterexample to show that the statement is false. Assume all sets are subsets of a universal set U.

1. For all sets A, B, and C, $(A \cap B) \cup C = A \cap (B \cup C)$.
2. For all sets A and B, $(A \cup B)^c = A^c \cup B^c$.
3. For all sets A, B, and C, if $A \not\subseteq B$ and $B \not\subseteq C$ then $A \not\subseteq C$.
4. For all sets A, B, and C, if $B \cap C \subseteq A$ then $(A - B) \cap (A - C) = \emptyset$.

For each of 5–21 prove each statement that is true and find a counterexample for each statement that is false. Assume all sets are subsets of a universal set U.

5. For all sets A, B, and C, $A - (B - C) = (A - B) - C$.
6. For all sets A and B, $A \cap (A \cup B) = A$.
7. For all sets A, B, and C,
$$(A - B) \cap (C - B) = A - (B \cup C).$$
8. For all sets A and B, if $A^c \subseteq B$ then $A \cup B = U$.
9. For all sets A, B, and C, if $A \subseteq C$ and $B \subseteq C$ then $A \cup B \subseteq C$.
10. For all sets A and B, if $A \subseteq B$ then $A \cap B^c = \emptyset$.

H 11. For all sets A, B, and C, if $A \subseteq B$ then $A \cap (B \cap C)^c = \emptyset$.

H 12. For all sets A, B, and C,
$$A \cap (B - C) = (A \cap B) - (A \cap C).$$

13. For all sets A, B, and C,
$$A \cup (B - C) = (A \cup B) - (A \cup C).$$

H 14. For all sets A, B, and C, if $A \cap C \subseteq B \cap C$ and $A \cup C \subseteq B \cup C$, then $A \subseteq B$.

H 15. For all sets A, B, and C, if $A \cap C = B \cap C$ and $A \cup C = B \cup C$, then $A = B$.

16. For all sets A and B, if $A \cap B = \emptyset$ then $A \times B = \emptyset$.
17. For all sets A and B, if $A \subseteq B$ then $\mathscr{P}(A) \subseteq \mathscr{P}(B)$.
18. For all sets A and B, $\mathscr{P}(A \cup B) \subseteq \mathscr{P}(A) \cup \mathscr{P}(B)$.

H 19. For all sets A and B, $\mathscr{P}(A) \cup \mathscr{P}(B) \subseteq \mathscr{P}(A \cup B)$.

20. For all sets A and B, $\mathscr{P}(A \cap B) = \mathscr{P}(A) \cap \mathscr{P}(B)$.
21. For all sets A and B, $\mathscr{P}(A \times B) = \mathscr{P}(A) \times \mathscr{P}(B)$.
22. Write a negation for each of the following statements. Indicate which is true, the statement or its negation. Justify your answers.
    a. $\forall$ sets S, $\exists$ a set T such that $S \cap T = \emptyset$.
    b. $\exists$ a set S such that $\forall$ sets T, $S \cup T = \emptyset$.

H 23. Let $S = \{a, b, c\}$ and for each integer $i = 0, 1, 2, 3$, let S_i be the set of all subsets of S that have i elements. List the elements in S_0, S_1, S_2, and S_3. Is $\{S_0, S_1, S_2, S_3\}$ a partition of $\mathscr{P}(S)$?

24. Let $S = \{a, b, c\}$ and let S_a be the set of all subsets of S that contain a, let S_b be the set of all subsets of S that contain b, let S_c be the set of all subsets of S that contain c, and let $S_\emptyset$ be the set whose only element is $\emptyset$. Is $\{S_a, S_b, S_c, S_\emptyset\}$ a partition of $\mathscr{P}(S)$?