Example 6.1.15 Tracing Algorithm 6.1.1

Trace the action of Algorithm 6.1.1 on the variables i, j, *found*, and *answer* for $m = 3$, $n = 4$, and sets A and B represented as the arrays $a[1] = u$, $a[2] = v$, $a[3] = w$, $b[1] = w$, $b[2] = x$, $b[3] = y$, and $b[4] = u$.

Solution

i	1					2					3
j	1	2	3	4	5	1	2	3	4	5	
found	no			yes		no					
answer	$A \subseteq B$									$A \not\subseteq B$	

■

In the exercises at the end of this section, you are asked to write an algorithm to check whether a given element is in a given set. To do this, you can represent the set as a one-dimensional array and compare the given element with successive elements of the array to determine whether the two elements are equal. If they are, then the element is in the set; if the given element does not equal any element of the array, then the element is not in the set.

Test Yourself

Answers to Test Yourself questions are located at the end of each section.

1. The notation $A \subseteq B$ is read "_____" and means that _____.
2. To use an element argument for proving that a set X is a subset of a set Y, you suppose that _____ and show that _____.
3. To disprove that a set X is a subset of a set Y, you show that there is _____.
4. An element x is in $A \cup B$ if, and only if, _____.
5. An element x is in $A \cap B$ if, and only if, _____.
6. An element x is in $B - A$ if, and only if, _____.
7. An element x is in A^c if, and only if, _____.
8. The empty set is a set with _____.
9. The power set of a set A is _____.
10. Sets A and B are disjoint if, and only if, _____.
11. A collection of nonempty sets $A_1, A_2, A_3, \ldots$ is a partition of a set A if, and only if, _____.
12. Given sets $A_1, A_2, \ldots, A_n$, the Cartesian product $A_1 \times A_2 \times \ldots \times A_n$ is _____.

Exercise Set 6.1*

1. In each of (a)–(f), answer the following questions: Is $A \subseteq B$? Is $B \subseteq A$? Is either A or B a proper subset of the other?
   a. $A = \{2, \{2\}, (\sqrt{2})^2\}$, $B = \{2, \{2\}, \{\{2\}\}\}$
   b. $A = \{3, \sqrt{5^2 - 4^2}, 24 \, mod \, 7\}$, $B = \{8 \, mod \, 5\}$
   c. $A = \{\{1, 2\}, \{2, 3\}\}$, $B = \{1, 2, 3\}$
   d. $A = \{a, b, c\}$, $B = \{\{a\}, \{b\}, \{c\}\}$
   e. $A = \{\sqrt{16}, \{4\}\}$, $B = \{4\}$
   f. $A = \{x \in \mathbf{R} \mid \cos x \in \mathbf{Z}\}$, $B = \{x \in \mathbf{R} \mid \sin x \in \mathbf{Z}\}$

2. Complete the proof from Example 6.1.3: Prove that $B \subseteq A$ where

$$A = \{m \in \mathbf{Z} \mid m = 2a \text{ for some integer } a\}$$

and

$$B = \{n \in \mathbf{Z} \mid n = 2b - 2 \text{ for some integer } b\}$$

* For exercises with blue numbers or letters, solutions are given in Appendix B. The symbol ***H*** indicates that only a hint or a partial solution is given. The symbol ✶ signals that an exercise is more challenging than usual.

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3. Let sets R, S, and T be defined as follows:
$$R = \{x \in \mathbf{Z} \mid x \text{ is divisible by } 2\}$$
$$S = \{y \in \mathbf{Z} \mid y \text{ is divisible by } 3\}$$
$$T = \{z \in \mathbf{Z} \mid z \text{ is divisible by } 6\}$$
a. Is $R \subseteq T$? Explain.
b. Is $T \subseteq R$? Explain.
c. Is $T \subseteq S$? Explain.

4. Let $A = \{n \in \mathbf{Z} \mid n = 5r \text{ for some integer } r\}$ and $B = \{m \in \mathbf{Z} \mid m = 20s \text{ for some integer } s\}$.
a. Is $A \subseteq B$? Explain. b. Is $B \subseteq A$? Explain.

5. Let $C = \{n \in \mathbf{Z} \mid n = 6r - 5 \text{ for some integer } r\}$ and $D = \{m \in \mathbf{Z} \mid m = 3s + 1 \text{ for some integer } s\}$.
Prove or disprove each of the following statements.
a. $C \subseteq D$ b. $D \subseteq C$

6. Let $A = \{x \in \mathbf{Z} \mid x = 5a + 2 \text{ for some integer } a\}$, $B = \{y \in \mathbf{Z} \mid y = 10b - 3 \text{ for some integer } b\}$, and $C = \{z \in \mathbf{Z} \mid z = 10c + 7 \text{ for some integer } c\}$.
Prove or disprove each of the following statements.
a. $A \subseteq B$ b. $B \subseteq A$ ***H*** **c.** $B = C$

7. Let $A = \{x \in \mathbf{Z} \mid x = 6a + 4 \text{ for some integer } a\}$, $B = \{y \in \mathbf{Z} \mid y = 18b - 2 \text{ for some integer } b\}$, and $C = \{z \in \mathbf{Z} \mid z = 18c + 16 \text{ for some integer } c\}$.
Prove or disprove each of the following statements.
a. $A \subseteq B$ b. $B \subseteq A$ c. $B = C$

8. Write in words how to read each of the following out loud. Then write the shorthand notation for each set.
a. $\{x \in U \mid x \in A \text{ and } x \in B\}$
b. $\{x \in U \mid x \in A \text{ or } x \in B\}$
c. $\{x \in U \mid x \in A \text{ and } x \notin B\}$
d. $\{x \in U \mid x \notin A\}$

9. Complete the following sentences without using the symbols $\cup$, $\cap$, or $-$.
a. $x \notin A \cup B$ if, and only if, _____.
b. $x \notin A \cap B$ if, and only if, _____.
c. $x \notin A - B$ if, and only if, _____.

10. Let $A = \{1, 3, 5, 7, 9\}$, $B = \{3, 6, 9\}$, and $C = \{2, 4, 6, 8\}$. Find each of the following:
a. $A \cup B$ **b.** $A \cap B$ **c.** $A \cup C$ **d.** $A \cap C$
e. $A - B$ f. $B - A$ g. $B \cup C$ h. $B \cap C$

11. Let the universal set be the set $\mathbf{R}$ of all real numbers and let $A = \{x \in \mathbf{R} \mid 0 < x \le 2\}$, $B = \{x \in \mathbf{R} \mid 1 \le x < 4\}$, and $C = \{x \in \mathbf{R} \mid 3 \le x < 9\}$. Find each of the following:
a. $A \cup B$ b. $A \cap B$ c. A^c d. $A \cup C$
e. $A \cap C$ f. B^c g. $A^c \cap B^c$
h. $A^c \cup B^c$ i. $(A \cap B)^c$ j. $(A \cup B)^c$

12. Let the universal set be the set $\mathbf{R}$ of all real numbers and let $A = \{x \in \mathbf{R} \mid -3 \le x \le 0\}$, $B = \{x \in \mathbf{R} \mid -1 < x < 2\}$, and $C = \{x \in \mathbf{R} \mid 6 < x \le 8\}$. Find each of the following:
a. $A \cup B$ b. $A \cap B$ c. A^c d. $A \cup C$
e. $A \cap C$ f. B^c g. $A^c \cap B^c$
h. $A^c \cup B^c$ i. $(A \cap B)^c$ j. $(A \cup B)^c$

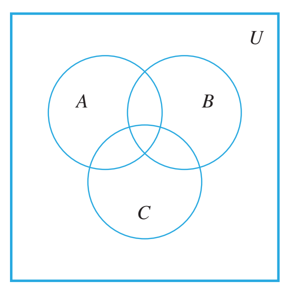
13. Indicate which of the following relationships are true and which are false:
a. $\mathbf{Z}^+ \subseteq \mathbf{Q}$ **b.** $\mathbf{R}^- \subseteq \mathbf{Q}$
c. $\mathbf{Q} \subseteq \mathbf{Z}$ **d.** $\mathbf{Z}^- \cup \mathbf{Z}^+ = \mathbf{Z}$
e. $\mathbf{Z}^- \cap \mathbf{Z}^+ = \emptyset$ f. $\mathbf{Q} \cap \mathbf{R} = \mathbf{Q}$
g. $\mathbf{Q} \cup \mathbf{Z} = \mathbf{Q}$ h. $\mathbf{Z}^+ \cap \mathbf{R} = \mathbf{Z}^+$
i. $\mathbf{Z} \cup \mathbf{Q} = \mathbf{Z}$

14. In each of the following, draw a Venn diagram for sets A, B, and C that satisfy the given conditions:
a. $A \subseteq B$; $C \subseteq B$; $A \cap C = \emptyset$
b. $C \subseteq A$; $B \cap C = \emptyset$

15. Draw Venn diagrams to describe sets A, B, and C that satisfy the given conditions.
a. $A \cap B = \emptyset$, $A \subseteq C$, $C \cap B \ne \emptyset$
b. $A \subseteq B$, $C \subseteq B$, $A \cap C \ne \emptyset$
c. $A \cap B \ne \emptyset$, $B \cap C \ne \emptyset$, $A \cap C = \emptyset$, $A \not\subseteq B$, $C \not\subseteq B$

16. Let $A = \{a, b, c\}$, $B = \{b, c, d\}$, and $C = \{b, c, e\}$.
a. Find $A \cup (B \cap C)$, $(A \cup B) \cap C$, and $(A \cup B) \cap (A \cup C)$. Which of these sets are equal?
b. Find $A \cap (B \cup C)$, $(A \cap B) \cup C$, and $(A \cap B) \cup (A \cap C)$. Which of these sets are equal?
c. Find $(A - B) - C$ and $A - (B - C)$. Are these sets equal?

17. Consider the Venn diagram shown below. For each of (a)–(f), copy the diagram and shade the region corresponding to the indicated set.
a. $A \cap B$ b. $B \cup C$ c. A^c
d. $A - (B \cup C)$ e. $(A \cup B)^c$ f. $A^c \cap B^c$



18. **a.** Is the number 0 in $\emptyset$? Why? **b.** Is $\emptyset = \{\emptyset\}$? Why?
c. Is $\emptyset \in \{\emptyset\}$? Why? d. Is $\emptyset \in \emptyset$? Why?

19. Let $A_i = \{i, i^2\}$ for all integers $i = 1, 2, 3, 4$.
a. $A_1 \cup A_2 \cup A_3 \cup A_4 = ?$
b. $A_1 \cap A_2 \cap A_3 \cap A_4 = ?$
c. Are A_1, A_2, A_3, and A_4 mutually disjoint? Explain.

20. Let $B_i = \{x \in \mathbf{R} \mid 0 \le x \le i\}$ for all integers $i = 1, 2, 3, 4$.
a. $B_1 \cup B_2 \cup B_3 \cup B_4 = ?$
b. $B_1 \cap B_2 \cap B_3 \cap B_4 = ?$
c. Are B_1, B_2, B_3, and B_4 mutually disjoint? Explain.

21. Let $C_i = \{i, -i\}$ for all nonnegative integers i.
a. $\bigcup_{i=0}^{4} C_i = ?$ b. $\bigcap_{i=0}^{4} C_i = ?$

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c. Are $C_0, C_1, C_2, \ldots$ mutually disjoint? Explain.

d. $\bigcup_{i=0}^{n} C_i = ?$ e. $\bigcap_{i=0}^{n} C_i = ?$

f. $\bigcup_{i=0}^{\infty} C_i = ?$ g. $\bigcap_{i=0}^{\infty} C_i = ?$

22. Let $D_i = \{x \in \mathbf{R} \mid -i \le x \le i\} = [-i, i]$ for all nonnegative integers i.

a. $\bigcup_{i=0}^{4} D_i = ?$ b. $\bigcap_{i=0}^{4} D_i = ?$

c. Are $D_0, D_1, D_2, \ldots$ mutually disjoint? Explain.

d. $\bigcup_{i=0}^{n} D_i = ?$ e. $\bigcap_{i=0}^{n} D_i = ?$

f. $\bigcup_{i=0}^{\infty} D_i = ?$ g. $\bigcap_{i=0}^{\infty} D_i = ?$

23. Let $V_i = \left\{x \in \mathbf{R} \mid -\frac{1}{i} \le x \le \frac{1}{i}\right\} = \left[-\frac{1}{i}, \frac{1}{i}\right]$ for all positive integers i.

a. $\bigcup_{i=1}^{4} V_i = ?$ b. $\bigcap_{i=1}^{4} V_i = ?$

c. Are $V_1, V_2, V_3, \ldots$ mutually disjoint? Explain.

d. $\bigcup_{i=1}^{n} V_i = ?$ e. $\bigcap_{i=1}^{n} V_i = ?$

f. $\bigcup_{i=1}^{\infty} V_i = ?$ g. $\bigcap_{i=1}^{\infty} V_i = ?$

24. Let $W_i = \{x \in \mathbf{R} \mid x > i\} = (i, \infty)$ for all nonnegative integers i.

a. $\bigcup_{i=0}^{4} W_i = ?$ b. $\bigcap_{i=0}^{4} W_i = ?$

c. Are $W_0, W_1, W_2, \ldots$ mutually disjoint? Explain.

d. $\bigcup_{i=0}^{n} W_i = ?$ e. $\bigcap_{i=0}^{n} W_i = ?$

f. $\bigcup_{i=0}^{\infty} W_i = ?$ g. $\bigcap_{i=0}^{\infty} W_i = ?$

25. Let $R_i = \left\{x \in \mathbf{R} \mid 1 \le x \le 1 + \frac{1}{i}\right\} = \left[1,\ 1 + \frac{1}{i}\right]$ for all positive integers i.

a. $\bigcup_{i=1}^{4} R_i = ?$ b. $\bigcap_{i=1}^{4} R_i = ?$

c. Are $R_1, R_2, R_3, \ldots$ mutually disjoint? Explain.

d. $\bigcup_{i=1}^{n} R_i = ?$ e. $\bigcap_{i=1}^{n} R_i = ?$

f. $\bigcup_{i=1}^{\infty} R_i = ?$ g. $\bigcap_{i=1}^{\infty} R_i = ?$

26. Let $S_i = \left\{x \in \mathbf{R} \mid 1 < x < 1 + \frac{1}{i}\right\} = \left(1,\ 1 + \frac{1}{i}\right)$ for all positive integers i.

a. $\bigcup_{i=1}^{4} S_i = ?$ b. $\bigcap_{i=1}^{4} S_i = ?$

c. Are $S_1, S_2, S_3, \ldots$ mutually disjoint? Explain.

d. $\bigcup_{i=1}^{n} S_i = ?$ e. $\bigcap_{i=1}^{n} S_i = ?$

f. $\bigcup_{i=1}^{\infty} S_i = ?$ g. $\bigcap_{i=1}^{\infty} S_i = ?$

27. **a.** Is $\{\{a, d, e\}, \{b, c\}, \{d, f\}\}$ a partition of $\{a, b, c, d, e, f\}$?

b. Is $\{\{w, x, v\}, \{u, y, q\}, \{p, z\}\}$ a partition of $\{p, q, u, v, w, x, y, z\}$?

c. Is $\{\{5, 4\}, \{7, 2\}, \{1, 3, 4\}, \{6, 8\}\}$ a partition of $\{1, 2, 3, 4, 5, 6, 7, 8\}$?

d. Is $\{\{3, 7, 8\}, \{2, 9\}, \{1, 4, 5\}\}$ a partition of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$?

e. Is $\{\{1, 5\}, \{4, 7\}, \{2, 8, 6, 3\}\}$ a partition of $\{1, 2, 3, 4, 5, 6, 7, 8\}$?

28. Let E be the set of all even integers and O the set of all odd integers. Is $\{E, O\}$ a partition of $\mathbf{Z}$, the set of all integers? Explain your answer.

29. Let $\mathbf{R}$ be the set of all real numbers. Is $\{\mathbf{R}^+, \mathbf{R}^-, \{0\}\}$ a partition of $\mathbf{R}$? Explain your answer.

30. Let $\mathbf{Z}$ be the set of all integers and let

$$A_0 = \{n \in \mathbf{Z} \mid n = 4k, \text{ for some integer } k\},$$
$$A_1 = \{n \in \mathbf{Z} \mid n = 4k + 1, \text{ for some integer } k\},$$
$$A_2 = \{n \in \mathbf{Z} \mid n = 4k + 2, \text{ for some integer } k\}, \text{ and}$$
$$A_3 = \{n \in \mathbf{Z} \mid n = 4k + 3, \text{ for some integer } k\}.$$

Is $\{A_0, A_1, A_2, A_3\}$ a partition of $\mathbf{Z}$? Explain your answer.

31. Suppose $A = \{1, 2\}$ and $B = \{2, 3\}$. Find each of the following:

a. $\mathscr{P}(A \cap B)$ b. $\mathscr{P}(A)$

c. $\mathscr{P}(A \cup B)$ d. $\mathscr{P}(A \times B)$

32. **a.** Suppose $A = \{1\}$ and $B = \{u, v\}$. Find $\mathscr{P}(A \times B)$.

b. Suppose $X = \{a, b\}$ and $Y = \{x, y\}$. Find $\mathscr{P}(X \times Y)$.

33. a. Find $\mathscr{P}(\emptyset)$. **b.** Find $\mathscr{P}(\mathscr{P}(\emptyset))$.

c. Find $\mathscr{P}(\mathscr{P}(\mathscr{P}(\emptyset)))$.

34. Let $A_1 = \{1, 2, 3\}$, $A_2 = \{u, v\}$, and $A_3 = \{m, n\}$. Find each of the following sets:

a. $A_1 \times (A_2 \times A_3)$ b. $(A_1 \times A_2) \times A_3$

c. $A_1 \times A_2 \times A_3$

35. Let $A = \{a, b\}$, $B = \{1, 2\}$, and $C = \{2, 3\}$. Find each of the following sets.

a. $A \times (B \cup C)$ **b.** $(A \times B) \cup (A \times C)$

c. $A \times (B \cap C)$ d. $(A \times B) \cap (A \times C)$

36. Trace the action of Algorithm 6.1.1 on the variables i, j, *found*, and *answer* for $m = 3, n = 3$, and sets A and B represented as the arrays $a[1] = u, a[2] = v, a[3] = w$, $b[1] = w, b[2] = u$, and $b[3] = v$.

37. Trace the action of Algorithm 6.1.1 on the variables i, j, *found*, and *answer* for $m = 4, n = 4$, and sets A and B represented as the arrays $a[1] = u, a[2] = v, a[3] = w$, $a[4] = x, b[1] = r, b[2] = u, b[3] = y, b[4] = z$.

38. Write an algorithm to determine whether a given element x belongs to a given set, which is represented as an array $a[1], a[2], \ldots, a[n]$.

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