Test Yourself

1. Mathematical induction differs from the kind of induction used in the natural sciences because it is actually a form of ____ reasoning.

2. Mathematical induction can be used to _____ conjectures that have been made using inductive reasoning.

Exercise Set 5.3

1. Based on the discussion of the product $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{n}\right)$ at the beginning of this section, conjecture a formula for general n. Prove your conjecture by mathematical induction.

2. Experiment with computing values of the product $\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{n}\right)$ for small values of n to conjecture a formula for this product for general n. Prove your conjecture by mathematical induction.

3. Observe that

$$\frac{1}{1\cdot 3}=\frac{1}{3}$$
$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}=\frac{2}{5}$$
$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}=\frac{3}{7}$$
$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\frac{1}{7\cdot 9}=\frac{4}{9}$$

Guess a general formula and prove it by mathematical induction.

H **4.** Observe that

$$1=1,$$
$$1-4=-(1+2),$$
$$1-4+9=1+2+3,$$
$$1-4+9-16=-(1+2+3+4),$$
$$1-4+9-16+25=1+2+3+4+5.$$

Guess a general formula and prove it by mathematical induction.

5. Evaluate the sum $\displaystyle\sum_{k=1}^{n}\frac{k}{(k+1)!}$ for $n=1,2,3,4,$ and 5. Make a conjecture about a formula for this sum for general n, and prove your conjecture by mathematical induction.

6. For each positive integer n, let $P(n)$ be the property

$$5^n-1 \text{ is divisible by } 4.$$

a. Write $P(0)$. Is $P(0)$ true?
b. Write $P(k)$.
c. Write $P(k+1)$.
d. In a proof by mathematical induction that this divisibility property holds for all integers $n\ge 0$, what must be shown in the inductive step?

7. For each positive integer n, let $P(n)$ be the property

$$2^n<(n+1)!.$$

a. Write $P(2)$. Is $P(2)$ true?
b. Write $P(k)$.
c. Write $P(k+1)$.
d. In a proof by mathematical induction that this inequality holds for all integers $n\ge 2$, what must be shown in the inductive step?

Prove each statement in 8–23 by mathematical induction.

8. 5^n-1 is divisible by 4, for each integer $n\ge 0$.

9. 7^n-1 is divisible by 6, for each integer $n\ge 0$.

10. n^3-7n+3 is divisible by 3, for each integer $n\ge 0$.

11. $3^{2n}-1$ is divisible by 8, for each integer $n\ge 0$.

12. For any integer $n\ge 0$, 7^n-2^n is divisible by 5.

H **13.** For any integer $n\ge 0$, x^n-y^n is divisible by $x-y$, where x and y are any integers with $x\ne y$.

H **14.** n^3-n is divisible by 6, for each integer $n\ge 0$.

15. $n(n^2+5)$ is divisible by 6, for each integer $n\ge 0$.

16. $2^n<(n+1)!$, for all integers $n\ge 2$.

17. $1+3n\le 4^n$, for every integer $n\ge 0$.

18. $5^n+9<6^n$, for all integers $n\ge 2$.

19. $n^2<2^n$, for all integers $n\ge 5$.

20. $2^n<(n+2)!$, for all integers $n\ge 0$.

21. $\sqrt{n}<\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}$, for all integers $n\ge 2$.

22. $1+nx\le(1+x)^n$, for all real numbers $x>-1$ and integers $n\ge 2$.

23. a. $n^3>2n+1$, for all integers $n\ge 2$.
b. $n!>n^2$, for all integers $n\ge 4$.

24. A sequence $a_1, a_2, a_3, \ldots$ is defined by letting $a_1=3$ and $a_k=7a_{k-1}$ for all integers $k\ge 2$. Show that $a_n=3\cdot 7^{n-1}$ for all integers $n\ge 1$.

25. A sequence $b_0, b_1, b_2, \ldots$ is defined by letting $b_0=5$ and $b_k=4+b_{k-1}$ for all integers $k\ge 1$. Show that $b_n>4n$ for all integers $n\ge 0$.

26. A sequence $c_0, c_1, c_2, \ldots$ is defined by letting $c_0 = 3$ and $c_k = (c_{k-1})^2$ for all integers $k \geq 1$. Show that $c_n = 3^{2^n}$ for all integers $n \geq 0$.

27. A sequence $d_1, d_2, d_3, \ldots$ is defined by letting $d_1 = 2$ and $d_k = \dfrac{d_{k-1}}{k}$ for all integers $k \geq 2$. Show that for all integers $n \geq 1$, $d_n = \dfrac{2}{n!}$.

28. Prove that for all integers $n \geq 1$,

$$\frac{1}{3} = \frac{1+3}{5+7} = \frac{1+3+5}{7+9+11} = \cdots = \frac{1+3+\cdots+(2n-1)}{(2n+1)+\cdots+(4n-1)}.$$

29. As each of a group of businesspeople arrives at a meeting, each shakes hands with all the other people present. Use mathematical induction to show that if n people come to the meeting then $[n(n-1)]/2$ handshakes occur.

In order for a proof by mathematical induction to be valid, the basis statement must be true for $n = a$ and the argument of the inductive step must be correct for every integer $k \geq a$. In 30 and 31 find the mistakes in the "proofs" by mathematical induction.

30. "**Theorem:**" For any integer $n \geq 1$, all the numbers in a set of n numbers are equal to each other.

"**Proof (by mathematical induction):** It is obviously true that all the numbers in a set consisting of just one number are equal to each other, so the basis step is true. For the inductive step, let $A = \{a_1, a_2, \ldots, a_k, a_{k+1}\}$ be any set of $k+1$ numbers. Form two subsets each of size k:

$$B = \{a_1, a_2, a_3, \ldots, a_k\} \quad \text{and}$$
$$C = \{a_1, a_3, a_4, \ldots, a_{k+1}\}.$$

(B consists of all the numbers in A except a_{k+1}, and C consists of all the numbers in A except a_2.) By inductive hypothesis, all the numbers in B equal a_1 and all the numbers in C equal a_1 (since both sets have only k numbers). But every number in A is in B or C, so all the numbers in A equal a_1; hence all are equal to each other."

H 31. "**Theorem:**" For all integers $n \geq 1$, $3^n - 2$ is even.

"**Proof (by mathematical induction):** Suppose the theorem is true for an integer k, where $k \geq 1$. That is, suppose that $3^k - 2$ is even. We must show that $3^{k+1} - 2$ is even. But

$$3^{k+1} - 2 = 3^k \cdot 3 - 2 = 3^k(1+2) - 2$$
$$= (3^k - 2) + 3^k \cdot 2.$$

Now $3^k - 2$ is even by inductive hypothesis and $3^k \cdot 2$ is even by inspection. Hence the sum of the two quantities is even (by Theorem 4.1.1). It follows that $3^{k+1} - 2$ is even, which is what we needed to show."

H 32. Some 5×5 checkerboards with one square removed can be completely covered by L-shaped trominoes, whereas other 5×5 checkerboards cannot. Find examples of both kinds of checkerboards. Justify your answers.

33. Consider a 4×6 checkerboard. Draw a covering of the board by L-shaped trominoes.

H 34. a. Use mathematical induction to prove that any checkerboard with dimensions $3 \times 2n$ can be completely covered with L-shaped trominoes for any integer $n \geq 1$.

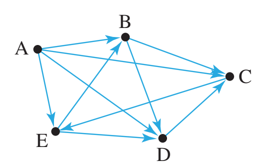
b. Let n be any integer greater than or equal to 1. Use the result of part (a) to prove by mathematical induction that for all integers m, any checkerboard with dimensions $2m \times 3n$ can be completely covered with L-shaped trominoes.

35. Let m and n be any integers that are greater than or equal to 1.

a. Prove that a necessary condition for an $m \times n$ checkerboard to be completely coverable by L-shaped trominoes is that mn be divisible by 3.

H b. Prove that having mn be divisible by 3 is not a sufficient condition for an $m \times n$ checkerboard to be completely coverable by L-shaped trominoes.

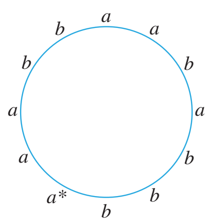
36. In a round-robin tournament each team plays every other team exactly once. If the teams are labeled $T_1, T_2, \ldots, T_n$, then the outcome of such a tournament can be represented by a drawing, called a *directed graph,* in which the teams are represented as dots and an arrow is drawn from one dot to another if, and only if, the team represented by the first dot beats the team represented by the second dot. For example, the directed graph below shows one outcome of a round-robin tournament involving five teams, A, B, C, D, and E.



Use mathematical induction to show that in any round-robin tournament involving n teams, where $n \geq 2$, it is possible to label the teams $T_1, T_2, \ldots, T_n$ so that T_i beats T_{i+1} for all $i = 1, 2, \ldots, n-1$. (For instance, one such labeling in the example above is $T_1 = A, T_2 = B, T_3 = C, T_4 = E, T_5 = D$.) (*Hint:* Given $k+1$ teams, pick one—say T'—and apply the inductive hypothesis to the remaining teams to obtain an ordering $T_1, T_2, \ldots, T_k$. Consider three cases: T' beats T_1, T' loses to the first m teams (where $1 \leq m \leq k-1$) and beats the $(m+1)$st team, and T' loses to all the other teams.)

H ✶ 37. On the outside rim of a circular disk the integers from 1 through 30 are painted in random order. Show that no matter what this order is, there must be three successive integers whose sum is at least 45.

H **38.** Suppose that n a's and n b's are distributed around the outside of a circle. Use mathematical induction to prove that for all integers $n \geq 1$, given any such arrangement, it is possible to find a starting point so that if one travels around the circle in a clockwise direction, the number of a's one has passed is never less than the number of b's one has passed. For example, in the diagram shown below, one could start at the a with an asterisk.



39. For a polygon to be **convex** means that all of its interior angles are less than 180 degrees. Use mathematical induction to prove that for all integers $n \geq 3$, the angles of any n-sided convex polygon add up to $180(n-2)$ degrees.

40. a. Prove that in an 8×8 checkerboard with alternating black and white squares, if the squares in the top right and bottom left corners are removed the remaining board cannot be covered with dominoes. (*Hint*: Mathematical induction is not needed for this proof.)

    b. Use mathematical induction to prove that for all integers n, if a $2n \times 2n$ checkerboard with alternating black and white squares has one white square and one black square removed anywhere on the board, the remaining squares can be covered with dominoes.

Answers for Test Yourself

1. deductive 2. prove

5.4 Strong Mathematical Induction and the Well-Ordering Principle for the Integers

Mathematics takes us still further from what is human into the region of absolute necessity, to which not only the actual world, but every possible world, must conform.
— Bertrand Russell, 1902

Strong mathematical induction is similar to ordinary mathematical induction in that it is a technique for establishing the truth of a sequence of statements about integers. Also, a proof by strong mathematical induction consists of a basis step and an inductive step. However, the basis step may contain proofs for several initial values, and in the inductive step the truth of the predicate $P(n)$ is assumed not just for one value of n but for *all* values through k, and then the truth of $P(k+1)$ is proved.

Principle of Strong Mathematical Induction

Let $P(n)$ be a property that is defined for integers n, and let a and b be fixed integers with $a \leq b$. Suppose the following two statements are true:

1. $P(a)$, $P(a+1)$, ..., and $P(b)$ are all true. **(basis step)**
2. For any integer $k \geq b$, if $P(i)$ is true for all integers i from a through k, then $P(k+1)$ is true. **(inductive step)**

Then the statement

$$\text{for all integers } n \geq a,\ P(n)$$

is true. (The supposition that $P(i)$ is true for all integers i from a through k is called the **inductive hypothesis.** Another way to state the inductive hypothesis is to say that $P(a)$, $P(a+1)$, ..., $P(k)$ are all true.)