Equating the right-hand sides of equations (5.2.1) and (5.2.2) and dividing by $r - 1$ gives

$$S_n = \frac{r^{n+1} - 1}{r - 1}.$$

This derivation of the formula is attractive and is quite convincing. However, it is not as logically airtight as the proof by mathematical induction. To go from one step to another in the previous calculations, the argument is made that each term among those indicated by the ellipsis (. . .) has such-and-such an appearance and when these are canceled such-and-such occurs. But it is impossible actually to see each such term and each such calculation, and so the accuracy of these claims cannot be fully checked. With mathematical induction it is possible to focus exactly on what happens in the middle of the ellipsis and verify without doubt that the calculations are correct.

Test Yourself

1. Mathematical induction is a method for proving that a property defined for integers n is true for all values of n that are _____.
2. Let $P(n)$ be a property defined for integers n and consider constructing a proof by mathematical induction for the statement "$P(n)$ is true for all $n \geq a$."

   (a) In the basis step one must show that _____.

   (b) In the inductive step one supposes that _____ for some particular but arbitrarily chosen value of an integer $k \geq a$. This supposition is called the _____. One then has to show that _____.

Exercise Set 5.2

1. Use mathematical induction (and the proof of Proposition 5.2.1 as a model) to show that any amount of money of at least 14¢ can be made up using 3¢ and 8¢ coins.
2. Use mathematical induction to show that any postage of at least 12¢ can be obtained using 3¢ and 7¢ stamps.
3. For each positive integer n, let $P(n)$ be the formula
$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$
   a. Write $P(1)$. Is $P(1)$ true?
   b. Write $P(k)$.
   c. Write $P(k+1)$.
   d. In a proof by mathematical induction that the formula holds for all integers $n \geq 1$, what must be shown in the inductive step?
4. For each integer n with $n \geq 2$, let $P(n)$ be the formula
$$\sum_{i=1}^{n-1} i(i+1) = \frac{n(n-1)(n+1)}{3}.$$
   a. Write $P(2)$. Is $P(2)$ true?
   b. Write $P(k)$.
   c. Write $P(k+1)$.
   d. In a proof by mathematical induction that the formula holds for all integers $n \geq 2$, what must be shown in the inductive step?
5. Fill in the missing pieces in the following proof that
$$1 + 3 + 5 + \cdots + (2n-1) = n^2$$
   for all integers $n \geq 1$.

Proof: Let the property $P(n)$ be the equation

$$1 + 3 + 5 + \cdots + (2n-1) = n^2. \quad \leftarrow P(n)$$

Show that P(1) is true: To establish $P(1)$, we must show that when 1 is substituted in place of n, the left-hand side equals the right-hand side. But when $n = 1$, the left-hand side is the sum of all the odd integers from 1 to $2 \cdot 1 - 1$, which is the sum of the odd integers from 1 to 1, which is just 1. The right-hand side is $\underline{\text{(a)}}$, which also equals 1. So $P(1)$ is true.

Show that for all integers k ≥ 1, if P(k) is true then P(k + 1) is true: Let k be any integer with $k \geq 1$.

[Suppose $P(k)$ is true. That is:]

Suppose $1 + 3 + 5 + \cdots + (2k-1) = \underline{\text{(b)}}$. $\leftarrow P(k)$
[This is the inductive hypothesis.]

[We must show that $P(k+1)$ is true. That is:]

We must show that

$$\underline{\text{(c)}} = \underline{\text{(d)}}. \quad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$1 + 3 + 5 + \cdots + (2(k+1) - 1)$

$= 1 + 3 + 5 + \cdots + (2k+1)$ by algebra

$= [1 + 3 + 5 + \cdots + (2k-1)] + (2k+1)$ the next-to-last term is $2k - 1$ because $\underline{\text{(e)}}$

$= k^2 + (2k+1)$ by $\underline{\text{(f)}}$

$= (k+1)^2$ by algebra

which is the right-hand side of $P(k+1)$ *[as was to be shown.]*

[Since we have proved the basis step and the inductive step, we conclude that the given statement is true.]

The previous proof was annotated to help make its logical flow more obvious. In standard mathematical writing, such annotation is omitted.

Prove each statement in 6–9 using mathematical induction. Do not derive them from Theorem 5.2.2 or Theorem 5.2.3.

6. For all integers $n \geq 1$, $2+4+6+\cdots+2n = n^2+n$.

7. For all integers $n \geq 1$,

$$1+6+11+16+\cdots+(5n-4) = \frac{n(5n-3)}{2}.$$

8. For all integers $n \geq 0$, $1+2+2^2+\cdots+2^n = 2^{n+1}-1$.

9. For all integers $n \geq 3$,

$$4^3+4^4+4^5+\cdots+4^n = \frac{4(4^n-16)}{3}.$$

Prove each of the statements in 10–17 by mathematical induction.

10. $1^2+2^2+\cdots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$, for all integers $n \geq 1$.

11. $1^3+2^3+\cdots+n^3 = \left[\dfrac{n(n+1)}{2}\right]^2$, for all integers $n \geq 1$.

12. $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\cdots+\dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$, for all integers $n \geq 1$.

13. $\displaystyle\sum_{i=1}^{n-1} i(i+1) = \frac{n(n-1)(n+1)}{3}$, for all integers $n \geq 2$.

14. $\displaystyle\sum_{i=1}^{n+1} i\cdot 2^i = n\cdot 2^{n+2}+2$, for all integers $n \geq 0$.

H **15.** $\displaystyle\sum_{i=1}^{n} i(i!) = (n+1)!-1$, for all integers $n \geq 1$.

16. $\left(1-\dfrac{1}{2^2}\right)\left(1-\dfrac{1}{3^2}\right)\cdots\left(1-\dfrac{1}{n^2}\right) = \dfrac{n+1}{2n}$, for all integers $n \geq 2$.

17. $\displaystyle\prod_{i=0}^{n}\left(\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right) = \frac{1}{(2n+2)!}$, for all integers $n \geq 0$.

H ✶ **18.** If x is a real number not divisible by π, then for all integers $n \geq 1$,

$$\sin x + \sin 3x + \sin 5x + \cdots + \sin(2n-1)x = \frac{1-\cos 2nx}{2\sin x}.$$

19. (For students who have studied calculus) Use mathematical induction, the product rule from calculus, and the facts that $\dfrac{d(x)}{dx} = 1$ and that $x^{k+1} = x\cdot x^k$ to prove that for all integers $n \geq 1$, $\dfrac{d(x^n)}{dx} = nx^{n-1}$.

Use the formula for the sum of the first n integers and/or the formula for the sum of a geometric sequence to evaluate the sums in 20–29 or to write them in closed form.

20. $4+8+12+16+\cdots+200$

21. $5+10+15+20+\cdots+300$

22. $3+4+5+6+\cdots+1000$

23. $7+8+9+10+\cdots+600$

24. $1+2+3+\cdots+(k-1)$, where k is an integer and $k \geq 2$.

25. a. $1+2+2^2+\cdots+2^{25}$
b. $2+2^2+2^3+\cdots+2^{26}$

26. $3+3^2+3^3+\cdots+3^n$, where n is an integer with $n \geq 1$

27. $5^3+5^4+5^5+\cdots+5^k$, where k is any integer with $k \geq 3$.

28. $1+\dfrac{1}{2}+\dfrac{1}{2^2}+\cdots+\dfrac{1}{2^n}$, where n is a positive integer

29. $1-2+2^2-2^3+\cdots+(-1)^n 2^n$, where n is a positive integer

H **30.** Find a formula in n, a, m, and d for the sum $(a+md)+(a+(m+1)d)+(a+(m+2)d)+\cdots+(a+(m+n)d)$, where m and n are integers, $n \geq 0$, and a and d are real numbers. Justify your answer.

31. Find a formula in a, r, m, and n for the sum

$$ar^m + ar^{m+1} + ar^{m+2} + \cdots + ar^{m+n}$$

where m and n are integers, $n \geq 0$, and a and r are real numbers. Justify your answer.

32. You have two parents, four grandparents, eight great-grandparents, and so forth.
   a. If all your ancestors were distinct, what would be the total number of your ancestors for the past 40 generations (counting your parents' generation as number one)? (*Hint:* Use the formula for the sum of a geometric sequence.)
   b. Assuming that each generation represents 25 years, how long is 40 generations?
   c. The total number of people who have ever lived is approximately 10 billion, which equals 10^{10} people. Compare this fact with the answer to part (a). What do you deduce?

Find the mistakes in the proof fragments in 33–35.

H 33. **Theorem:** For any integer $n \geq 1$,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

"**Proof (by mathematical induction):** Certainly the theorem is true for $n = 1$ because $1^2 = 1$ and $\frac{1(1+1)(2 \cdot 1 + 1)}{6} = 1$. So the basis step is true.

For the inductive step, suppose that for some integer $k \geq 1$, $k^2 = \frac{k(k+1)(2k+1)}{6}$. We must show that $(k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$."

H 34. **Theorem:** For any integer $n \geq 0$,

$$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1.$$

"**Proof (by mathematical induction): Let the property** $P(n)$ be $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$.

Show that P(0) is true:

The left-hand side of $P(0)$ is $1 + 2 + 2^2 + \cdots + 2^0 = 1$ and the right-hand side is $2^{0+1} - 1 = 2 - 1 = 1$ also. So $P(0)$ is true."

H 35. **Theorem:** For any integer $n \geq 1$,

$$\sum_{i=1}^{n} i(i!) = (n+1)! - 1.$$

"**Proof (by mathematical induction): Let the property** $P(n)$ be $\sum_{i=1}^{n} i(i!) = (n+1)! - 1$.

Show that P(1) is true: When $n = 1$

$$\sum_{i=1}^{1} i(i!) = (1+1)! - 1$$

So $\quad 1(1!) = 2! - 1$

and $\quad 1 = 1$

Thus $P(1)$ is true."

✶ 36. Use Theorem 5.2.2 to prove that if m and n are any positive integers and m is odd, then $\sum_{k=0}^{m-1}(n+k)$ is divisible by m. Does the conclusion hold if m is even? Justify your answer.

H ✶ 37. Use Theorem 5.2.2 and the result of exercise 10 to prove that if p is any prime number with $p \geq 5$, then the sum of squares of any p consecutive integers is divisible by p.

Answers for Test Yourself

1. greater than or equal to some initial value 2. (a) $P(a)$ is true (b) $P(k)$ is true; inductive hypothesis; $P(k+1)$ is true

5.3 Mathematical Induction II

A good proof is one which makes us wiser. — I. Manin, *A Course in Mathematical Logic,* 1977

In natural science courses, deduction and induction are presented as alternative modes of thought—deduction being to infer a conclusion from general principles using the laws of logical reasoning, and induction being to enunciate a general principle after observing it to hold in a large number of specific instances. In this sense, then, *mathematical* induction is not inductive but deductive. Once proved by mathematical induction, a theorem is known just as certainly as if it were proved by any other mathematical method. Inductive reasoning, in the natural sciences sense, *is* used in mathematics, but only to make conjectures, not to prove them. For example, observe that

$$1 - \frac{1}{2} = \frac{1}{2}$$

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) = \frac{1}{3}$$

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right) = \frac{1}{4}$$

This pattern seems so unlikely to occur by pure chance that it is reasonable to conjecture (though it is by no means certain) that the pattern holds true in general. In a case like this, a proof by mathematical induction (which you are asked to write in exercise 1 at the end of this section) gets to the essence of why the pattern holds in general. It reveals the mathematical mechanism that necessitates the truth of each successive case from the previous one. For instance, in this example observe that if