Input: n *[a nonnegative integer]*

Algorithm Body:

$q := n, i := 0$

[Repeatedly perform the integer division of q by 2 until q becomes 0. Store successive remainders in a one-dimensional array $r[0], r[1], r[2], \ldots, r[k]$. Even if the initial-value of q equals 0, the loop should execute one time (so that $r[0]$ is computed). Thus the guard condition for the **while** *loop is $i = 0$ or $q \neq 0$.]*

while $(i = 0 \text{ or } q \neq 0)$

$r[i] := q \text{ mod } 2$

$q := q \text{ div } 2$

[$r[i]$ and q can be obtained by calling the division algorithm.]

$i := i + 1$

end while

[After execution of this step, the values of $r[0], r[1], \ldots, r[i-1]$ are all 0's and 1's, and $a = (r[i-1]r[i-2]\cdots r[2]r[1]r[0])_2$.]

Output: $r[0], r[1], r[2], \ldots, r[i-1]$ *[a sequence of integers]*

Test Yourself

Answers to Test Yourself questions are located at the end of each section.

1. The notation $\sum_{k=m}^{n} a_k$ is read "_____."
2. The expanded form of $\sum_{k=m}^{n} a_k$ is _____.
3. The value of $a_1 + a_2 + a_3 + \cdots + a_n$ when $n = 2$ is "_____."
4. The notation $\prod_{k=m}^{n} a_k$ is read "_____."
5. If n is a positive integer, then $n! =$ _____.
6. $\sum_{k=m}^{n} a_k + c\sum_{k=m}^{n} b_k =$ _____.
7. $\left(\prod_{k=m}^{n} a_k\right)\left(\prod_{k=m}^{n} b_k\right) =$ _____.

Exercise Set 5.1*

Write the first four terms of the sequences defined by the formulas in 1–6.

1. $a_k = \dfrac{k}{10+k}$, for all integers $k \geq 1$.
2. $b_j = \dfrac{5-j}{5+j}$, for all integers $j \geq 1$.
3. $c_i = \dfrac{(-1)^i}{3^i}$, for all integers $i \geq 0$.
4. $d_m = 1 + \left(\dfrac{1}{2}\right)^m$ for all integers $m \geq 0$.
5. $e_n = \left\lfloor \dfrac{n}{2} \right\rfloor \cdot 2$, for all integers $n \geq 0$.
6. $f_n = \left\lfloor \dfrac{n}{4} \right\rfloor \cdot 4$, for all integers $n \geq 1$.
7. Let $a_k = 2k + 1$ and $b_k = (k-1)^3 + k + 2$ for all integers $k \geq 0$. Show that the first three terms of these sequences are identical but that their fourth terms differ.

Compute the first fifteen terms of each of the sequences in 8 and 9, and describe the general behavior of these sequences in words. (A definition of logarithm is given in Section 7.1.)

8. $g_n = \lfloor \log_2 n \rfloor$ for all integers $n \geq 1$.
9. $h_n = n\lfloor \log_2 n \rfloor$ for all integers $n \geq 1$.

*For exercises with blue numbers or letters, solutions are given in Appendix B. The symbol ***H*** indicates that only a hint or a partial solution is given. The symbol ✶ signals that an exercise is more challenging than usual.

Find explicit formulas for sequences of the form $a_1, a_2, a_3, \ldots$ with the initial terms given in 10–16.

10. $-1, 1, -1, 1, -1, 1$

11. $0, 1, -2, 3, -4, 5$

12. $\frac{1}{4}, \frac{2}{9}, \frac{3}{16}, \frac{4}{25}, \frac{5}{36}, \frac{6}{49}$

13. $1-\frac{1}{2}, \frac{1}{2}-\frac{1}{3}, \frac{1}{3}-\frac{1}{4}, \frac{1}{4}-\frac{1}{5}, \frac{1}{5}-\frac{1}{6}, \frac{1}{6}-\frac{1}{7}$

14. $\frac{1}{3}, \frac{4}{9}, \frac{9}{27}, \frac{16}{81}, \frac{25}{243}, \frac{36}{729}$

15. $0, -\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}, \frac{6}{7}$

16. $3, 6, 12, 24, 48, 96$

✶ 17. Consider the sequence defined by $a_n = \frac{2n + (-1)^n - 1}{4}$ for all integers $n \geq 0$. Find an alternative explicit formula for a_n that uses the floor notation.

18. Let $a_0 = 2$, $a_1 = 3$, $a_2 = -2$, $a_3 = 1$, $a_4 = 0$, $a_5 = -1$, and $a_6 = -2$. Compute each of the summations and products below.

a. $\sum_{i=0}^{6} a_i$ **b.** $\sum_{i=0}^{0} a_i$ **c.** $\sum_{j=1}^{3} a_{2j}$ **d.** $\prod_{k=0}^{6} a_k$ e. $\prod_{k=2}^{2} a_k$

Compute the summations and products in 19–28.

19. $\sum_{k=1}^{5}(k+1)$

20. $\prod_{k=2}^{4} k^2$

21. $\sum_{m=0}^{3} \frac{1}{2^m}$

22. $\prod_{j=0}^{4}(-1)^j$

23. $\sum_{i=1}^{1} i(i+1)$

24. $\sum_{j=0}^{0}(j+1)\cdot 2^j$

25. $\prod_{k=2}^{2}\left(1-\frac{1}{k}\right)$

26. $\sum_{k=-1}^{1}(k^2+3)$

27. $\sum_{n=1}^{10}\left(\frac{1}{n}-\frac{1}{n+1}\right)$

28. $\prod_{i=2}^{5}\frac{i(i+2)}{(i-1)\cdot(i+1)}$

Write the summations in 29–32 in expanded form.

29. $\sum_{i=1}^{n}(-2)^i$

30. $\sum_{j=1}^{n} j(j+1)$

31. $\sum_{k=0}^{n+1}\frac{1}{k!}$

32. $\sum_{i=1}^{k+1} i(i!)$

Evaluate the summations and products in 33–36 for the indicated values of the variable.

33. $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2};\ n=1$

34. $1(1!)+2(2!)+3(3!)+\ldots+m(m!);\ m=2$

35. $\left(\frac{1}{1+1}\right)\left(\frac{2}{2+1}\right)\left(\frac{3}{3+1}\right)\cdots\left(\frac{k}{k+1}\right);\ k=3$

36. $\left(\frac{1\cdot 2}{3\cdot 4}\right)\left(\frac{4\cdot 5}{6\cdot 7}\right)\left(\frac{6\cdot 7}{8\cdot 9}\right)\cdots\left(\frac{m\cdot(m+1)}{(m+2)\cdot(m+3)}\right);\ m=1$

Rewrite 37–39 by separating off the final term.

37. $\sum_{i=1}^{k+1} i(i!)$

38. $\sum_{k=1}^{m+1} k^2$

39. $\sum_{m=1}^{n+1} m(m+1)$

Write each of 40–42 as a single summation.

40. $\sum_{i=1}^{k} i^3 + (k+1)^3$

41. $\sum_{k=1}^{m}\frac{k}{k+1}+\frac{m+1}{m+2}$

42. $\sum_{m=0}^{n}(m+1)2^m + (n+2)2^{n+1}$

Write each of 43–52 using summation or product notation.

43. $1^2-2^2+3^2-4^2+5^2-6^2+7^2$

44. $(1^3-1)-(2^3-1)+(3^3-1)-(4^3-1)+(5^3-1)$

45. $(2^2-1)\cdot(3^2-1)\cdot(4^2-1)$

46. $\frac{2}{3\cdot 4}-\frac{3}{4\cdot 5}+\frac{4}{5\cdot 6}-\frac{5}{6\cdot 7}+\frac{6}{7\cdot 8}$

47. $1-r+r^2-r^3+r^4-r^5$

48. $(1-t)\cdot(1-t^2)\cdot(1-t^3)\cdot(1-t^4)$

49. $1^3+2^3+3^3+\cdots+n^3$

50. $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots+\frac{n}{(n+1)!}$

51. $n+(n-1)+(n-2)+\cdots+1$

52. $n+\frac{n-1}{2!}+\frac{n-2}{3!}+\frac{n-3}{4!}+\cdots+\frac{1}{n!}$

Transform each of 53 and 54 by making the change of variable $i = k + 1$.

53. $\sum_{k=0}^{5} k(k-1)$

54. $\prod_{k=1}^{n}\frac{k}{k^2+4}$

Transform each of 55–58 by making the change of variable $j = i - 1$.

55. $\sum_{i=1}^{n+1}\frac{(i-1)^2}{i\cdot n}$

56. $\sum_{i=3}^{n}\frac{i}{i+n-1}$

57. $\sum_{i=1}^{n-1}\frac{i}{(n-i)^2}$

58. $\prod_{i=n}^{2n}\frac{n-i+1}{n+i}$

Write each of 59–61 as a single summation or product.

59. $3\cdot\sum_{k=1}^{n}(2k-3)+\sum_{k=1}^{n}(4-5k)$

60. $2\cdot\sum_{k=1}^{n}(3k^2+4)+5\cdot\sum_{k=1}^{n}(2k^2-1)$

61. $\left(\prod_{k=1}^{n}\frac{k}{k+1}\right)\cdot\left(\prod_{k=1}^{n}\frac{k+1}{k+2}\right)$

Compute each of 62–76. Assume the values of the variables are restricted so that the expressions are defined.

62. $\frac{4!}{3!}$

63. $\frac{6!}{8!}$

64. $\frac{4!}{0!}$

65. $\frac{n!}{(n-1)!}$

66. $\frac{(n-1)!}{(n+1)!}$

67. $\frac{n!}{(n-2)!}$

68. $\frac{((n+1)!)^2}{(n!)^2}$ **69.** $\frac{n!}{(n-k)!}$ 70. $\frac{n!}{(n-k+1)!}$

71. $\binom{5}{3}$ 72. $\binom{7}{4}$ **73.** $\binom{3}{0}$

74. $\binom{5}{5}$ **75.** $\binom{n}{n-1}$ 76. $\binom{n+1}{n-1}$

77. **a.** Prove that $n!+2$ is divisible by 2, for all integers $n \geq 2$.
b. Prove that $n!+k$ is divisible by k, for all integers $n \geq 2$ and $k = 2, 3, \ldots, n$.
H **c.** Given any integer $m \geq 2$, is it possible to find a sequence of $m-1$ consecutive positive integers none of which is prime? Explain your answer.

78. Prove that for all nonnegative integers n and r with $r+1 \leq n$, $\binom{n}{r+1} = \frac{n-r}{r+1}\binom{n}{r}$.

79. Prove that if p is a prime number and r is an integer with $0 < r < p$, then $\binom{p}{r}$ is divisible by p.

80. Suppose $a[1], a[2], a[3], \ldots, a[m]$ is a one-dimensional array and consider the following algorithm segment:

$sum := 0$
for $k := 1$ **to** m
  $sum := sum + a[k]$
next k

Fill in the blanks below so that each algorithm segment performs the same job as the one given previously.

a. $sum := 0$
for $i := 0$ **to** ______
  $sum :=$ ______
next i

b. $sum := 0$
for $j := 2$ **to** ______
  $sum :=$ ______
next j

Use repeated division by 2 to convert (by hand) the integers in 81–83 from base 10 to base 2.

81. 90 82. 98 83. 205

Make a trace table to trace the action of Algorithm 5.1.1 on the input in 84–86.

84. 23 85. 28 86. 44

87. Write an informal description of an algorithm (using repeated division by 16) to convert a nonnegative integer from decimal notation to hexadecimal notation (base 16).

Use the algorithm you developed for exercise 87 to convert the integers in 88–90 to hexadecimal notation.

88. 287 89. 693 90. 2,301

91. Write a formal version of the algorithm you developed for exercise 87.

Answers for Test Yourself

1. the summation from k equals m to n of a-sub-k 2. $a_m + a_{m+1} + a_{m+2} + \cdots + a_n$ 3. $a_1 + a_2$ 4. the product from k equals m to n of a-sub-k 5. $n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$ (*Or*: $n \cdot (n-1)!$) 6. $\sum_{k=m}^{n} (a_k + cb_k)$ 7. $\prod_{k=m}^{n} a_k b_k$

5.2 Mathematical Induction I

[Mathematical induction is] the standard proof technique in computer science.
— Anthony Ralston, 1984

Mathematical induction is one of the more recently developed techniques of proof in the history of mathematics. It is used to check conjectures about the outcomes of processes that occur repeatedly and according to definite patterns. We introduce the technique with an example.

Some people claim that the United States penny is such a small coin that it should be abolished. They point out that frequently a person who drops a penny on the ground does not even bother to pick it up. Other people argue that abolishing the penny would not give enough flexibility for pricing merchandise. What prices could still be paid with exact change if the penny were abolished and another coin worth 3¢ were introduced? The answer is that the only prices that could not be paid with exact change would be 1¢, 2¢, 4¢, and 7¢. In other words,

Any whole number of cents of at least 8¢ can be obtained using 3¢ and 5¢ coins.

More formally:

For all integers $n \geq 8$, n cents can be obtained using 3¢ and 5¢ coins.