Exercise Set 4.8

Find the value of z when each of the algorithm segments in 1 and 2 is executed.

1. $i := 2$
if $(i > 3$ or $i \leq 0)$
  then $z := 1$
  else $z := 0$

2. $i := 3$
if $(i \leq 3$ or $i > 6)$
  then $z := 2$
  else $z := 0$

3. Consider the following algorithm segment:

if $x \cdot y > 0$ **then do** $y := 3 \cdot x$
  $x := x + 1$ **end do**
$z := x \cdot y$

Find the value of z if prior to execution x and y have the values given below.
a. $x = 2, y = 3$ b. $x = 1, y = 1$

Find the values of a and e after execution of the loops in 4 and 5:

4. $a := 2$
for $i := 1$ **to** 2
  $a := \frac{a}{2} + \frac{1}{a}$
next i

5. $e := 0, f := 2$
for $j := 1$ **to** 4
  $f := f \cdot j$
  $e := e + \frac{1}{f}$
next j

Make a trace table to trace the action of Algorithm 4.8.1 for the input variables given in 6 and 7.

6. $a = 26, d = 7$ 7. $a = 59, d = 13$

8. The following algorithm segment makes change; given an amount of money A between 1¢ and 99¢, it determines a breakdown of A into quarters (q), dimes (d), nickels (n), and pennies (p).

$q := A$ *div* 25
$A := A$ *mod* 25
$d := A$ *div* 10
$A := A$ *mod* 10
$n := A$ *div* 5
$p := A$ *mod* 5

a. Trace this algorithm segment for $A = 69$.
b. Trace this algorithm segment for $A = 87$.

Find the greatest common divisor of each of the pairs of integers in 9–12. (Use any method you wish.)

9. 27 and 72 **10.** 5 and 9
11. 7 and 21 12. 48 and 54

Use the Euclidean algorithm to hand-calculate the greatest common divisors of each of the pairs of integers in 13–16.

13. 1,188 and 385 **14.** 509 and 1,177
15. 832 and 10,933 16. 4,131 and 2,431

Make a trace table to trace the action of Algorithm 4.8.2 for the input variables given in 17 and 18.

17. 1,001 and 871 18. 5,859 and 1,232

H **19.** Prove that for all positive integers a and b, $a \mid b$ if, and only if, $\gcd(a, b) = a$. (Note that to prove "A if, and only if, B," you need to prove "if A then B" and "if B then A.")

20. a. Prove that if a and b are integers, not both zero, and $d = \gcd(a, b)$, then a/d and b/d are integers with no common divisor that is greater than one.
b. Write an algorithm that accepts the numerator and denominator of a fraction as input and produces as output the numerator and denominator of that fraction written in lowest terms. (The algorithm may call upon the Euclidean algorithm as needed.)

21. Complete the proof of Lemma 4.8.2 by proving the following: If a and b are any integers with $b \neq 0$ and q and r are any integers such that

$$a = bq + r.$$

then $\quad \gcd(b, r) \leq \gcd(a, b).$

H 22. **a.** Prove: If a and d are positive integers and q and r are integers such that $a = dq + r$ and $0 < r < d$, then

$$-a = d(-(q + 1)) + (d - r)$$

and $\quad 0 < d - r < d.$

b. Indicate how to modify Algorithm 4.8.1 to allow for the input a to be negative.

23. **a.** Prove that if a, d, q, and r are integers such that $a = dq + r$ and $0 \leq r < d$, then

$$q = \lfloor a/d \rfloor \quad \text{and} \quad r = a - \lfloor a/d \rfloor \cdot d.$$

b. In a computer language with a built-in-floor function, *div* and *mod* can be calculated as follows:

$$a \textit{ div } d = \lfloor a/d \rfloor \quad \text{and} \quad a \textit{ mod } d = a - \lfloor a/d \rfloor \cdot d.$$

Rewrite the steps of Algorithm 4.8.2 for a computer language with a built-in floor function but without *div* and *mod*.

24. An alternative to the Euclidean algorithm uses subtraction rather than division to compute greatest common divisors. (After all, division is repeated subtraction.) It is based on the following lemma:

Lemma 4.8.3

If $a \geq b > 0$, then $\gcd(a, b) = \gcd(b, a - b)$.

Algorithm 4.8.3 Computing gcd's by Subtraction

[Given two positive integers A and B, variables a and b are set equal to A and B. Then a repetitive process begins. If $a \neq 0$, and $b \neq 0$, then the larger of a and b is set equal to $a - b$ (if $a \geq b$) or to $b - a$ (if $a < b$), and the smaller of a and b is left unchanged. This process is repeated over and over until eventually a or b becomes 0. By Lemma 4.8.3, after each repetition of the process,

$$\gcd(A, B) = \gcd(a, b).$$

After the last repetition,

$$\gcd(A, B) = \gcd(a, 0) \quad \textit{or} \quad \gcd(A, B) = \gcd(0, b)$$

depending on whether a or b is nonzero. But by Lemma 4.8.1,

$$\gcd(a, 0) = a \quad \textit{and} \quad \gcd(0, b) = b.$$

Hence, after the last repetition,

$$\gcd(A, B) = a \textit{ if } a \neq 0 \quad \textit{or} \quad \gcd(A, B) = b \textit{ if } b \neq 0.]$$

Input: A, B *[positive integers]*

Algorithm Body:

$a := A, b := B$

while ($a \neq 0$ and $b \neq 0$)

  if $a \geq b$ **then** $a := a - b$

    else $b := b - a$

end while

  if $a = 0$ **then** $\gcd := b$

  else $\gcd := a$

  [After execution of the **if-then-else** *statement,* $\gcd = \gcd(A, B)$*.]*

Output: gcd *[a positive integer]*

a. Prove Lemma 4.8.3.

b. Trace the execution of Algorithm 4.8.3 for $A = 630$ and $B = 336$.

c. Trace the execution of Algorithm 4.8.3 for $A = 768$ and $B = 348$.

Exercises 25–29 refer to the following definition.

Definition: The **least common multiple** of two nonzero integers a and b, denoted $\mathbf{lcm}(a, b)$, is the positive integer c such that

a. $a \mid c$ and $b \mid c$

b. for all positive integers m, if $a \mid m$ and $b \mid m$, then $c \leq m$.

25. Find
 a. $\text{lcm}(12, 18)$ b. $\text{lcm}(2^2 \cdot 3 \cdot 5, 2^3 \cdot 3^2)$
 c. $\text{lcm}(2800, 6125)$

26. Prove that for all positive integers a and b, $\gcd(a, b) = \text{lcm}(a, b)$ if, and only if, $a = b$.

27. Prove that for all positive integers a and b, $a \mid b$ if, and only if, $\text{lcm}(a, b) = b$.

28. Prove that for all integers a and b, $\gcd(a, b) \mid \text{lcm}(a, b)$.

H 29. Prove that for all positive integers a and b, $\gcd(a, b) \cdot \text{lcm}(a, b) = ab$.

Answers for Test Yourself

1. the expression e is evaluated (using the current values of all the variables in the expression), and this value is placed in the memory location corresponding to x (replacing any previous contents of the location) 2. statement s_1 is executed; statement s_2 is executed 3. all statements in the body of the loop are executed in order and then execution moves back to the beginning of the loop and the process repeats; execution passes to the next algorithm statement following the loop 4. the statements in the body of the loop are executed in order, *variable* is increased by 1, and execution returns to the top of the loop; execution passes to the next algorithm statement following the loop 5. integers q and r with the property that $n = dq + r$ and $0 \leq r < d$ 6. d divides both a and b; if c is a common divisor of both a and b, then $c \leq d$ 7. r 8. $\gcd(b, r)$ 9. the greatest common divisor of A and B (*Or*: gcd(A,B))

CHAPTER 5

SEQUENCES, MATHEMATICAL INDUCTION, AND RECURSION

One of the most important tasks of mathematics is to discover and characterize regular patterns, such as those associated with processes that are repeated. The main mathematical structure used in the study of repeated processes is the *sequence,* and the main mathematical tool used to verify conjectures about sequences is *mathematical induction*. In this chapter we introduce the notation and terminology of sequences, show how to use both ordinary and strong mathematical induction to prove properties about them, illustrate the various ways recursively defined sequences arise, describe a method for obtaining an explicit formula for a recursively defined sequence, and explain how to verify the correctness of such a formula. We also discuss a principle—the well-ordering principle for the integers—that is logically equivalent to the two forms of mathematical induction, and we show how to adapt mathematical induction to prove the correctness of computer algorithms. In the final section we discuss more general recursive definitions, such as the one used for the careful formulation of the concept of Boolean expression, and the idea of recursive function.

5.1 Sequences

A mathematician, like a painter or poet, is a maker of patterns.
— G. H. Hardy, *A Mathematician's Apology,* 1940

Imagine that a person decides to count his ancestors. He has two parents, four grandparents, eight great-grandparents, and so forth, These numbers can be written in a row as

$$2, 4, 8, 16, 32, 64, 128, \ldots$$

The symbol "..." is called an *ellipsis*. It is shorthand for "and so forth."

To express the pattern of the numbers, suppose that each is labeled by an integer giving its position in the row.

Position in the row	1	2	3	4	5	6	7 . . .
Number of ancestors	2	4	8	16	32	64	128 . . .

The number corresponding to position 1 is 2, which equals 2^1. The number corresponding to position 2 is 4, which equals 2^2. For positions 3, 4, 5, 6, and 7, the corresponding