Example 1.3.7 Equality of Functions

Define $f: \mathbf{R} \to \mathbf{R}$ and $g: \mathbf{R} \to \mathbf{R}$ by the following formulas:

$$f(x) = |x| \quad \text{for all } x \in \mathbf{R}.$$
$$g(x) = \sqrt{x^2} \quad \text{for all } x \in \mathbf{R}.$$

Does $f = g$?

Solution

Yes. Because the absolute value of any real number equals the square root of its square, $|x| = \sqrt{x^2}$ for all $x \in \mathbf{R}$. Hence $f = g$. ■

Test Yourself

1. Given sets A and B, a relation from A to B is ____.
2. A function F from A to B is a relation from A to B that satisfies the following two properties:
   a. for every element x of A, there is ____.
   b. for all elements x in A and y and z in B, if ____ then ____.
3. If F is a function from A to B and x is an element of A, then $F(x)$ is ____.

Exercise Set 1.3

1. Let $A = \{2, 3, 4\}$ and $B = \{6, 8, 10\}$ and define a relation R from A to B as follows: For all $(x, y) \in A \times B$,

$$(x, y) \in R \quad \text{means that} \quad \frac{y}{x} \text{ is an integer.}$$

a. Is $4\ R\ 6$? Is $4\ R\ 8$? Is $(3, 8) \in R$? Is $(2, 10) \in R$?
b. Write R as a set of ordered pairs.
c. Write the domain and co-domain of R.
d. Draw an arrow diagram for R.

2. Let $C = D = \{-3, -2, -1, 1, 2, 3\}$ and define a relation S from C to D as follows: For all $(x, y) \in C \times D$,

$$(x, y) \in S \quad \text{means that} \quad \frac{1}{x} - \frac{1}{y} \text{ is an integer.}$$

a. Is $2\ S\ 2$? Is $-1 S - 1$? Is $(3, 3) \in S$? Is $(3, -3) \in S$?
b. Write S as a set of ordered pairs.
c. Write the domain and co-domain of S.
d. Draw an arrow diagram for S.

3. Let $E = \{1, 2, 3\}$ and $F = \{-2, -1, 0\}$ and define a relation T from E to F as follows: For all $(x, y) \in E \times F$,

$$(x, y) \in T \quad \text{means that} \quad \frac{x - y}{3} \text{ is an integer.}$$

a. Is $3\ T\ 0$? Is $1 T (-1)$? Is $(2, -1) \in T$? Is $(3, -2) \in T$?
b. Write T as a set of ordered pairs.
c. Write the domain and co-domain of T.
d. Draw an arrow diagram for T.

4. Let $G = \{-2, 0, 2\}$ and $H = \{4, 6, 8\}$ and define a relation V from G to H as follows: For all $(x, y) \in G \times H$,

$$(x, y) \in V \quad \text{means that} \quad \frac{x - y}{4} \text{ is an integer.}$$

a. Is $2\ V\ 6$? Is $(-2) V (-6)$? Is $(0, 6) \in V$? Is $(2, 4) \in V$?
b. Write V as a set of ordered pairs.
c. Write the domain and co-domain of V.
d. Draw an arrow diagram for V.

5. Define a relation S from $\mathbf{R}$ to $\mathbf{R}$ as follows: For all $(x, y) \in \mathbf{R} \times \mathbf{R}$,

$$(x, y) \in S \quad \text{means that} \quad x \geq y.$$

a. Is $(2, 1) \in S$? Is $(2, 2) \in S$? Is $2\ S\ 3$? Is $(-1)\ S\ (-2)$?
b. Draw the graph of S in the Cartesian plane.

6. Define a relation R from $\mathbf{R}$ to $\mathbf{R}$ as follows: For all $(x, y) \in \mathbf{R} \times \mathbf{R}$,

$$(x, y) \in R \quad \text{means that} \quad y = x^2.$$

a. Is $(2, 4) \in R$? Is $(4, 2) \in R$? Is $(-3)\ R\ 9$? Is $9\ R\ (-3)$?
b. Draw the graph of R in the Cartesian plane.

7. Let $A = \{4, 5, 6\}$ and $B = \{5, 6, 7\}$ and define relations R, S, and T from A to B as follows: For all $(x, y) \in A \times B$,

$$(x, y) \in R \quad \text{means that} \quad x \geq y.$$
$$(x, y) \in S \quad \text{means that} \quad \frac{x - y}{2} \text{ is an integer.}$$
$$T = \{(4, 7), (6, 5), (6, 7)\}.$$

a. Draw arrow diagrams for R, S, and T.
b. Indicate whether any of the relations R, S, and T are functions.

8. Let $A = \{2, 4\}$ and $B = \{1, 3, 5\}$ and define relations U, V, and W from A to B as follows: For all $(x, y) \in A \times B$,

$$(x, y) \in U \quad \text{means that} \quad y - x > 2.$$
$$(x, y) \in V \quad \text{means that} \quad y - 1 = \frac{x}{2}.$$
$$W = \{(2, 5), (4, 1), (2, 3)\}.$$

a. Draw arrow diagrams for U, V, and W.
b. Indicate whether any of the relations U, V, and W are functions.

9. a. Find all relations from $\{0,1\}$ to $\{1\}$.
b. Find all functions from $\{0,1\}$ to $\{1\}$.
c. What fraction of the relations from $\{0,1\}$ to $\{1\}$ are functions?

10. Find four relations from $\{a, b\}$ to $\{x, y\}$ that are not functions from $\{a, b\}$ to $\{x, y\}$.

11. Define a relation P from $\mathbf{R}^+$ to $\mathbf{R}$ as follows: For all real numbers x and y with $x > 0$,

$$(x, y) \in P \quad \text{means that} \quad x = y^2.$$

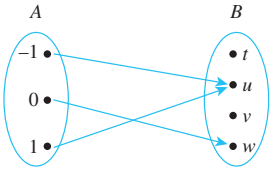
Is P a function? Explain.

12. Define a relation T from $\mathbf{R}$ to $\mathbf{R}$ as follows: For all real numbers x and y,

$$(x, y) \in T \quad \text{means that} \quad y^2 - x^2 = 1.$$

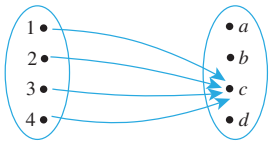
Is T a function? Explain.

13. Let $A = \{-1, 0, 1\}$ and $B = \{t, u, v, w\}$. Define a function $F: A \to B$ by the following arrow diagram:



a. Write the domain and co-domain of F.
b. Find $F(-1)$, $F(0)$, and $F(1)$.

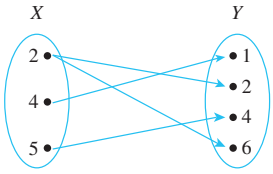
14. Let $C = \{1, 2, 3, 4\}$ and $D = \{a, b, c, d\}$. Define a function $G: C \to D$ by the following arrow diagram:



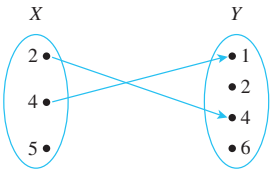
a. Write the domain and co-domain of G.
b. Find $G(1)$, $G(2)$, $G(3)$, and $G(4)$.

15. Let $X = \{2, 4, 5\}$ and $Y = \{1, 2, 4, 6\}$. Which of the following arrow diagrams determine functions from X to Y?

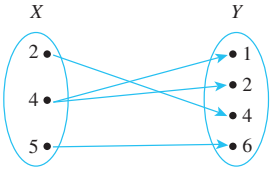
a.



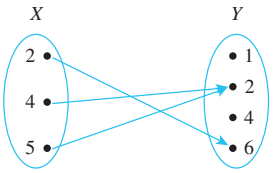
b.



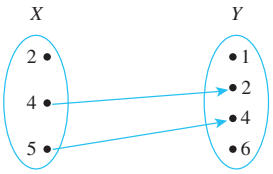
c.



d.



e.



16. Let f be the squaring function defined in Example 1.3.6. Find $f(-1)$, $f(0)$, and $f\left(\frac{1}{2}\right)$.

17. Let g be the successor function defined in Example 1.3.6. Find $g(-1000)$, $g(0)$, and $g(999)$.

18. Let h be the constant function defined in Example 1.3.6. Find $h\left(-\frac{12}{5}\right)$, $h\left(\frac{0}{1}\right)$, and $h\left(\frac{9}{17}\right)$.

19. Define functions f and g from $\mathbf{R}$ to $\mathbf{R}$ by the following formulas: For all $x \in \mathbf{R}$,

$$f(x) = 2x \quad \text{and} \quad g(x) = \frac{2x^3 + 2x}{x^2 + 1}.$$

Does $f = g$? Explain.

20. Define functions H and K from $\mathbf{R}$ to $\mathbf{R}$ by the following formulas: For all $x \in \mathbf{R}$,

$$H(x) = (x - 2)^2 \quad \text{and} \quad K(x) = (x - 1)(x - 3) + 1.$$

Does $H = K$? Explain.

Answers for Test Yourself

1. a subset of the Cartesian product $A \times B$ 2. a. an element y of B such that $(x, y) \in F$ (i.e., such that x is related to y by F) b. $(x, y) \in F$ and $(x, z) \in F$; $y = z$ 3. the unique element of B that is related to x by F