very long. Mathematicians sometimes work for months (or longer) on difficult problems. Don't be discouraged if some problems in this book take you quite a while to solve.

Learning the skills of proof and disproof is much like learning other skills, such as those used in swimming, tennis, or playing a musical instrument. When you first start out, you may feel bewildered by all the rules, and you may not feel confident as you attempt new things. But with practice the rules become internalized and you can use them in conjunction with all your other powers—of balance, coordination, judgment, aesthetic sense—to concentrate on winning a meet, winning a match, or playing a concert successfully.

Now that you have worked through the first five sections of this chapter, return to the idea that, above all, a proof or disproof should be a convincing argument. You need to know how direct and indirect proofs and counterexamples are structured. But to use this knowledge effectively, you must use it in conjunction with your imaginative powers, your intuition, and especially your common sense.

Test Yourself

1. To prove a statement by contradiction, you suppose that _____ and you show that _____.
2. A proof by contraposition of a statement of the form "$\forall x \in D$, if $P(x)$ then $Q(x)$" is a direct proof of _____.
3. To prove a statement of the form "$\forall x \in D$, if $P(x)$ then $Q(x)$" by contraposition, you suppose that _____ and you show that _____.

Exercise Set 4.6

1. Fill in the blanks in the following proof by contradiction that there is no least positive real number.

Proof: Suppose not. That is, suppose that there is a least positive real number x. *[We must deduce* (a) *.]* Consider the number $x/2$. Since x is a positive real number, $x/2$ is also (b) . In addition, we can deduce that $x/2 < x$ by multiplying both sides of the inequality $1 < 2$ by (c) and dividing (d) . Hence $x/2$ is a positive real number that is less than the least positive real number. This is a (e) *[Thus the supposition is false, and so there is no least positive real number.]*

2. Is $\frac{1}{0}$ an irrational number? Explain.

3. Use proof by contradiction to show that for all integers n, $3n + 2$ is not divisible by 3.

4. Use proof by contradiction to show that for all integers m, $7m + 4$ is not divisible by 7.

Carefully formulate the negations of each of the statements in 5–7. Then prove each statement by contradiction.

5. There is no greatest even integer.

6. There is no greatest negative real number.

7. There is no least positive rational number.

8. Fill in the blanks for the following proof that the difference of any rational number and any irrational number is irrational.

Proof: Suppose not. That is, suppose that there exist (a) x and (b) y such that $x - y$ is rational. By definition of rational, there exist integers a, b, c, and d with $b \neq 0$ and $d \neq 0$ so that $x =$ (c) and $x - y =$ (d) . By substitution,

$$\frac{a}{b} - y = \frac{c}{d}$$

Adding y and subtracting $\frac{c}{d}$ on both sides gives

$$\begin{aligned} y &= \text{(e)} \\ &= \frac{ad}{bd} - \frac{bc}{bd} \\ &= \frac{ad - bc}{bd} \qquad \text{by algebra.} \end{aligned}$$

Now both $ad - bc$ and bd are integers because products and differences of (f) are (g) . And $bd \neq 0$ by the (h) . Hence y is a ratio of integers with a nonzero denominator, and thus y is (i) by definition of rational. We therefore have both that y is irrational and that y is rational, which is a contradiction. *[Thus the supposition is false and the statement to be proved is true.]*

9. **a.** When asked to prove that the difference of any irrational number and any rational number is irrational, a student began, "Suppose not. That is, suppose the difference of any irrational number and any rational number is rational." What is wrong with beginning the proof in this way? (*Hint*: Review the answer to exercise 11 in Section 3.2.)

b. Prove that the difference of any irrational number and any rational number is irrational.

Prove each statement in 10–17 by contradiction.

10. The square root of any irrational number is irrational.

11. The product of any nonzero rational number and any irrational number is irrational.

12. If a and b are rational numbers, $b \neq 0$, and r is an irrational number, then $a + br$ is irrational.

H **13.** For any integer n, $n^2 - 2$ is not divisible by 4.

H **14.** For all prime numbers a, b, and c, $a^2 + b^2 \neq c^2$.

H **15.** If a, b, and c are integers and $a^2 + b^2 = c^2$, then at least one of a and b is even.

H ✶**16.** For all odd integers a, b, and c, if z is a solution of $ax^2 + bx + c = 0$ then z is irrational. (In the proof, use the properties of even and odd integers that are listed in Example 4.2.3.)

17. For all integers a, if $a \text{ mod } 6 = 3$, then $a \text{ mod } 3 \neq 2$.

18. Fill in the blanks in the following proof by contraposition that for all integers n, if $5 \nmid n^2$ then $5 \nmid n$.

Proof (by contraposition): *[The contrapositive is: For all integers n, if $5 \mid n$ then $5 \mid n^2$.]* Suppose n is any integer such that $\underline{(a)}$. *[We must show that* $\underline{(b)}$.*]* By definition of divisibility, $n = \underline{(c)}$ for some integer k. By substitution, $n^2 = \underline{(d)} = 5(5k^2)$. But $5k^2$ is an integer because it is a product of integers. Hence $n^2 = 5 \cdot$(an integer), and so $\underline{(e)}$ *[as was to be shown].*

Prove the statements in 19 and 20 by contraposition.

19. If a product of two positive real numbers is greater than 100, then at least one of the numbers is greater than 10.

20. If a sum of two real numbers is less than 50, then at least one of the numbers is less than 25.

21. Consider the statement "For all integers n, if n^2 is odd then n is odd."
a. Write what you would suppose and what you would need to show to prove this statement by contradiction.
b. Write what you would suppose and what you would need to show to prove this statement by contraposition.

22. Consider the statement "For all real numbers r, if r^2 is irrational then r is irrational."
a. Write what you would suppose and what you would need to show to prove this statement by contradiction.
b. Write what you would suppose and what you would need to show to prove this statement by contraposition.

Prove each of the statements in 23–29 in two ways: (a) by contraposition and (b) by contradiction.

23. The negative of any irrational number is irrational.

24. The reciprocal of any irrational number is irrational. (The **reciprocal** of a nonzero real number x is $1/x$.)

H **25.** For all integers n, if n^2 is odd then n is odd.

26. For all integers a, b, and c, if $a \nmid bc$ then $a \nmid b$. (Recall that the symbol $\nmid$ means "does not divide.")

H **27.** For all integers m and n, if $m + n$ is even then m and n are both even or m and n are both odd.

28. For all integers m and n, if mn is even then m is even or n is even.

29. For all integers a, b, and c, if $a \mid b$ and $a \nmid c$, then $a \nmid (b + c)$. (*Hint:* To prove $p \to q \vee r$, it suffices to prove either $p \wedge \sim q \to r$ or $p \wedge \sim r \to q$. See exercise 14 in Section 2.2.)

30. The following "proof" that every integer is rational is incorrect. Find the mistake.

"**Proof (by contradiction):** Suppose not. Suppose every integer is irrational. Then the integer 1 is irrational. But $1 = 1/1$, which is rational. This is a contradiction. *[Hence the supposition is false and the theorem is true.]*"

31. **a.** Prove by contraposition: For all positive integers n, r, and s, if $rs \leq n$, then $r \leq \sqrt{n}$ or $s \leq \sqrt{n}$.
b. Prove: For all integers $n > 1$, if n is not prime, then there exists a prime number p such that $p \leq \sqrt{n}$ and n is divisible by p. (*Hints*: Use the result of part (a), Theorems 4.3.1, 4.3.3, and 4.3.4, and the transitive property of order.)
c. State the contrapositive of the result of part (b).
The results of exercise 31 provide a way to test whether an integer is prime.

Test for Primality

Given an integer $n > 1$, to test whether n is prime check to see if it is divisible by a prime number less than or equal to its square root. If it is not divisible by any of these numbers, then it is prime.

32. Use the test for primality to determine whether the following numbers are prime or not.
a. 667 **b.** 557 c. 527 d. 613

33. The sieve of Eratosthenes, named after its inventor, the Greek scholar Eratosthenes (276–194 B.C.E.), provides a way to find all prime numbers less than or equal to some fixed number n. To construct it, write out all the integers from 2 to n. Cross out all multiples of 2 except 2 itself, then all multiples of 3 except 3 itself, then all multiples of 5 except 5 itself, and so forth. Continue crossing out the

multiples of each successive prime number up to $\sqrt{n}$. The numbers that are not crossed out are all the prime numbers from 2 to n. Here is a sieve of Eratosthenes that includes the numbers from 2 to 27. The multiples of 2 are crossed out with a /, the multiples of 3 with a \, and the multiples of 5 with a —.

2 3 4 5 6 7 8 9 10 11 12 13 14
15 16 17 18 19 20 21 22 23 24 25 26 27

Use the sieve of Eratosthenes to find all prime numbers less than 100.

34. Use the test for primality and the result of exercise 33 to determine whether the following numbers are prime.
a. 9,269 **b.** 9,103 c. 8,623 d. 7,917

H ✶ **35.** Use proof by contradiction to show that every integer greater than 11 is a sum of two composite numbers.

Answers for Test Yourself

1. the statement is false; this supposition leads to a contradiction 2. the contrapositive of the statement, namely, $\forall x \in D$, if $\sim Q(x)$ then $\sim P(x)$ 3. x is any *[particular but arbitrarily chosen]* element of D for which $Q(x)$ is false; $P(x)$ is false

4.7 Indirect Argument: Two Classical Theorems

He is unworthy of the name of man who does not know that the diagonal of a square is incommensurable with its side.—Plato (ca. 428–347 B.C.E.)

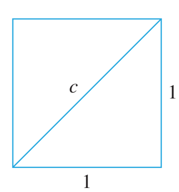
This section contains proofs of two of the most famous theorems in mathematics: that $\sqrt{2}$ is irrational and that there are infinitely many prime numbers. Both proofs are examples of indirect arguments and were well known more than 2,000 years ago, but they remain exemplary models of mathematical argument to this day.

The Irrationality of $\sqrt{2}$

When mathematics flourished at the time of the ancient Greeks, mathematicians believed that given any two line segments, say A: —— and B: ————, a certain unit of length could be found so that segment A was exactly a units long and segment B was exactly b units long. (The segments were said to be *commensurable* with respect to this special unit of length.) Then the ratio of the lengths of A and B would be in the same proportion as the ratio of the integers a and b. Symbolically:

$$\frac{\text{length } A}{\text{length } B} = \frac{a}{b}.$$

Now it is easy to find a line segment of length $\sqrt{2}$; just take the diagonal of the unit square:



By the Pythagorean theorem, $c^2 = 1^2 + 1^2 = 2$, and so $c = \sqrt{2}$. If the belief of the ancient Greeks were correct, there would be integers a and b such that

$$\frac{\text{length (diagonal)}}{\text{length (side)}} = \frac{a}{b}.$$