• Definition

Given any integer $n > 1$, the **standard factored form** of n is an expression of the form

$$n = p_1^{e_1} p_2^{e_2} p_3^{e_3} \cdots p_k^{e_k},$$

where k is a positive integer; $p_1, p_2, \ldots, p_k$ are prime numbers; $e_1, e_2, \ldots, e_k$ are positive integers; and $p_1 < p_2 < \cdots < p_k$.

Example 4.3.8 Writing Integers in Standard Factored Form

Write 3,300 in standard factored form.

Solution First find all the factors of 3,300. Then write them in ascending order:

$$3{,}300 = 100 \cdot 33 = 4 \cdot 25 \cdot 3 \cdot 11$$
$$= 2 \cdot 2 \cdot 5 \cdot 5 \cdot 3 \cdot 11 = 2^2 \cdot 3^1 \cdot 5^2 \cdot 11^1.$$

■

Example 4.3.9 Using Unique Factorization to Solve a Problem

Suppose m is an integer such that

$$8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot m = 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10.$$

Does $17 \mid m$?

Solution Since 17 is one of the prime factors of the right-hand side of the equation, it is also a prime factor of the left-hand side (by the unique factorization of integers theorem). But 17 does not equal any prime factor of 8, 7, 6, 5, 4, 3, or 2 (because it is too large). Hence 17 must occur as one of the prime factors of m, and so $17 \mid m$. ■

Test Yourself

1. To show that a nonzero integer d divides an integer n, we must show that _____.
2. To say that d divides n means the same as saying that _____ is divisible by _____.
3. If a and b are positive integers and $a \mid b$, then _____ is less than or equal to _____.
4. For all integers n and d, $d \nmid n$ if, and only if, _____.
5. If a and b are integers, the notation $a \mid b$ denotes _____ and the notation a/b denotes _____.
6. The transitivity of divisibility theorem says that for all integers a, b, and c, if _____ then _____.
7. The divisibility by a prime theorem says that every integer greater than 1 is _____.
8. The unique factorization of integers theorem says that any integer greater than 1 is either _____ or can be written as _____ in a way that is unique except possibly for the _____ in which the numbers are written.

Exercise Set 4.3

Give a reason for your answer in each of 1–13. Assume that all variables represent integers.

1. Is 52 divisible by 13?
2. Does $7 \mid 56$?
3. Does $5 \mid 0$?
4. Does 3 divide $(3k + 1)(3k + 2)(3k + 3)$?
5. Is $6m(2m + 10)$ divisible by 4?
6. Is 29 a multiple of 3?
7. Is -3 a factor of 66?
8. Is $6a(a + b)$ a multiple of $3a$?

9. Is 4 a factor of $2a \cdot 34b$?

10. Does $7 \mid 34$?

11. Does $13 \mid 73$?

12. If $n = 4k + 1$, does 8 divide $n^2 - 1$?

13. If $n = 4k + 3$, does 8 divide $n^2 - 1$?

14. Fill in the blanks in the following proof that for all integers a and b, if $a \mid b$ then $a \mid (-b)$.

Proof: Suppose a and b are any integers such that __(a)__. By definition of divisibility, there exists an integer r such that __(b)__. By substitution.

$$-b = -ar = a(-r).$$

Let $t =$ __(c)__. Then t is an integer because $t = (-1) \cdot r$, and both -1 and r are integers. Thus, by substitution, $-b = at$, where r is an integer, and so by definition of divisibility, __(d)__, as was to be shown.

Prove statements 15 and 16 directly from the definition of divisibility.

15. For all integers a, b, and c, if $a \mid b$ and $a \mid c$ then $a \mid (b + c)$.

H 16. For all integers a, b, and c, if $a \mid b$ and $a \mid c$ then $a \mid (b - c)$.

17. Consider the following statement: The negative of any multiple of 3 is a multiple of 3.
    a. Write the statement formally using a quantifier and a variable.
    b. Determine whether the statement is true or false and justify your answer.

18. Show that the following statement is false: For all integers a and b, if $3 \mid (a + b)$ then $3 \mid (a - b)$.

For each statement in 19–31, determine whether the statement is true or false. Prove the statement directly from the definitions if it is true, and give a counterexample if it is false.

H 19. For all integers a, b, and c, if a divides b then a divides bc.

20. The sum of any three consecutive integers is divisible by 3. (Two integers are **consecutive** if, and only if, one is one more than the other.)

21. The product of any two even integers is a multiple of 4.

H 22. A necessary condition for an integer to be divisible by 6 is that it be divisible by 2.

23. A sufficient condition for an integer to be divisible by 8 is that it be divisible by 16.

24. For all integers a, b, and c, if $a \mid b$ and $a \mid c$ then $a \mid (2b - 3c)$.

25. For all integers a, b, and c, if a is a factor of c then ab is a factor of c.

H 26. For all integers a, b, and c, if $ab \mid c$ then $a \mid c$ and $b \mid c$.

H 27. For all integers a, b, and c, if $a \mid (b + c)$ then $a \mid b$ or $a \mid c$.

28. For all integers a, b, and c, if $a \mid bc$ then $a \mid b$ or $a \mid c$.

29. For all integers a and b, if $a \mid b$ then $a^2 \mid b^2$.

30. For all integers a and n, if $a \mid n^2$ and $a \le n$ then $a \mid n$.

31. For all integers a and b, if $a \mid 10b$ then $a \mid 10$ or $a \mid b$.

32. A fast-food chain has a contest in which a card with numbers on it is given to each customer who makes a purchase. If some of the numbers on the card add up to 100, then the customer wins \$100. A certain customer receives a card containing the numbers

$$72, 21, 15, 36, 69, 81, 9, 27, 42, \text{and } 63.$$

Will the customer win \$100? Why or why not?

33. Is it possible to have a combination of nickels, dimes, and quarters that add up to \$4.72? Explain.

34. Is it possible to have 50 coins, made up of pennies, dimes, and quarters, that add up to \$3? Explain.

35. Two athletes run a circular track at a steady pace so that the first completes one round in 8 minutes and the second in 10 minutes. If they both start from the same spot at 4 P.M., when will be the first time they return to the start together?

36. It can be shown (see exercises 44–48) that an integer is divisible by 3 if, and only if, the sum of its digits is divisible by 3. An integer is divisible by 9 if, and only if, the sum of its digits is divisible by 9. An integer is divisible by 5 if, and only if, its right-most digit is a 5 or a 0. And an integer is divisible by 4 if, and only if, the number formed by its right-most two digits is divisible by 4. Check the following integers for divisibility by 3, 4, 5 and 9.
    a. 637,425,403,705,125
    b. 12,858,306,120,312
    c. 517,924,440,926,512
    d. 14,328,083,360,232

37. Use the unique factorization theorem to write the following integers in standard factored form.
    a. 1,176
    b. 5,733
    c. 3,675

38. Suppose that in standard factored form $a = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, where k is a positive integer; $p_1, p_2, \ldots, p_k$ are prime numbers; and $e_1, e_2, \ldots, e_k$ are positive integers.
    a. What is the standard factored form for a^2?
    b. Find the least positive integer n such that $2^5 \cdot 3 \cdot 5^2 \cdot 7^3 \cdot n$ is a perfect square. Write the resulting product as a perfect square.
    c. Find the least positive integer m such that $2^2 \cdot 3^5 \cdot 7 \cdot 11 \cdot m$ is a perfect square. Write the resulting product as a perfect square.

39. Suppose that in standard factored form $a = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, where k is a positive integer; $p_1, p_2, \ldots, p_k$ are prime numbers; and $e_1, e_2, \ldots, e_k$ are positive integers.
    a. What is the standard factored form for a^3?
    b. Find the least positive integer k such that $2^4 \cdot 3^5 \cdot 7 \cdot 11^2 \cdot k$ is a perfect cube (i.e., equals an integer to the third power). Write the resulting product as a perfect cube.

40. **a.** If a and b are integers and $12a = 25b$, does $12 \mid b$? does $25 \mid a$? Explain.
   b. If x and y are integers and $10x = 9y$, does $10 \mid y$? does $9 \mid x$? Explain.

H **41.** How many zeros are at the end of $45^8 \cdot 88^5$? Explain how you can answer this question without actually computing the number. (*Hint:* $10 = 2 \cdot 5$.)

42. If n is an integer and $n > 1$, then $n!$ is the product of n and every other positive integer that is less than n. For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$.
   a. Write 6! in standard factored form.
   b. Write 20! in standard factored form.
   c. Without computing the value of $(20!)^2$ determine how many zeros are at the end of this number when it is written in decimal form. Justify your answer.

✶43. In a certain town 2/3 of the adult men are married to 3/5 of the adult women. Assume that all marriages are monogamous (no one is married to more than one other person). Also assume that there are at least 100 adult men in the town. What is the least possible number of adult men in the town? of adult women in the town?

Definition: Given any nonnegative integer n, the **decimal representation** of n is an expression of the form

$$d_k d_{k-1} \cdots d_2 d_1 d_0,$$

where k is a nonnegative integer; $d_0, d_1, d_2, \ldots, d_k$ (called the **decimal digits** of n) are integers from 0 to 9 inclusive; $d_k \neq 0$ unless $n = 0$ and $k = 0$; and

$$n = d_k \cdot 10^k + d_{k-1} \cdot 10^{k-1} + \cdots + d_2 \cdot 10^2 + d_1 \cdot 10 + d_0.$$

(For example, $2{,}503 = 2 \cdot 10^3 + 5 \cdot 10^2 + 0 \cdot 10 + 3$.)

44. Prove that if n is any nonnegative integer whose decimal representation ends in 0, then $5 \mid n$. (*Hint:* If the decimal representation of a nonnegative integer n ends in d_0, then $n = 10m + d_0$ for some integer m.)

45. Prove that if n is any nonnegative integer whose decimal representation ends in 5, then $5 \mid n$.

46. Prove that if the decimal representation of a nonnegative integer n ends in $d_1 d_0$ and if $4 \mid (10d_1 + d_0)$, then $4 \mid n$. (*Hint:* If the decimal representation of a nonnegative integer n ends in $d_1 d_0$, then there is an integer s such that $n = 100s + 10d_1 + d_0$.)

H ✶**47.** Observe that

$$\begin{aligned}
7524 &= 7 \cdot 1000 + 5 \cdot 100 + 2 \cdot 10 + 4 \\
&= 7(999 + 1) + 5(99 + 1) + 2(9 + 1) + 4 \\
&= (7 \cdot 999 + 7) + (5 \cdot 99 + 5) + (2 \cdot 9 + 2) + 4 \\
&= (7 \cdot 999 + 5 \cdot 99 + 2 \cdot 9) + (7 + 5 + 2 + 4) \\
&= (7 \cdot 111 \cdot 9 + 5 \cdot 11 \cdot 9 + 2 \cdot 9) + (7 + 5 + 2 + 4) \\
&= (7 \cdot 111 + 5 \cdot 11 + 2) \cdot 9 + (7 + 5 + 2 + 4) \\
&= \text{(an integer divisible by 9)} \\
&\qquad + \text{(the sum of the digits of 7524).}
\end{aligned}$$

Since the sum of the digits of 7524 is divisible by 9, 7524 can be written as a sum of two integers each of which is divisible by 9. It follows from exercise 15 that 7524 is divisible by 9.

Generalize the argument given in this example to any nonnegative integer n. In other words, prove that for any nonnegative integer n, if the sum of the digits of n is divisible by 9, then n is divisible by 9.

✶48. Prove that for any nonnegative integer n, if the sum of the digits of n is divisible by 3, then n is divisible by 3.

✶49. Given a positive integer n written in decimal form, the alternating sum of the digits of n is obtained by starting with the right-most digit, subtracting the digit immediately to its left, adding the next digit to the left, subtracting the next digit, and so forth. For example, the alternating sum of the digits of 180,928 is $8 - 2 + 9 - 0 + 8 - 1 = 22$. Justify the fact that for any nonnegative integer n, if the alternating sum of the digits of n is divisible by 11, then n is divisible by 11.

Answers for Test Yourself

1. n equals d times some integer (Or: there is an integer r such that $n = dr$) 2. n; d 3. a; b 4. $\frac{n}{d}$ is not an integer 5. the sentence "a divides b"; the number obtained when a is divided by b 6. a divides b and b divides c; a divides c 7. divisible by some prime number 8. prime; a product of prime numbers; order