Appendix B
Algebra Review

In this appendix, we review basic algebra: rules for combining and simplifying expressions; fractions; exponents; factoring; quadratic equations; inequalities; and logarithms. For a more extensive treatment of basic algebra, see [Bleau; Lial; Sullivan].

Grouping

Terms with a common symbol can be combined:

$$ac + bc = (a + b)c, \qquad ac - bc = (a - b)c.$$

Technically, these equations are known as **distributive laws**.

Example B.1

$$2x + 3x = (2 + 3)x = 5x \qquad \blacksquare$$

The distributive laws, rewritten as

$$a(b + c) = ab + ac, \qquad a(b - c) = ab - ac,$$

can be used to simplify expressions.

Example B.2

$$2(x + 1) = 2x + 2 \cdot 1 = 2x + 2 \qquad \blacksquare$$

Example B.3

$$2(x + 1) + 2(x - 1) = 2x + 2 + 2x - 2 = 4x \qquad \blacksquare$$

Fractions

Formulas useful for adding, subtracting, and multiplying fractions are given as Theorem B.4.

THEOREM B.4 **COMBINING FRACTIONS**

(a) $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$

(b) $\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$

(c) $\frac{a}{c} + \frac{b}{d} = \frac{ad+bc}{cd}$

(d) $\frac{a}{c} - \frac{b}{d} = \frac{ad-bc}{cd}$

(e) $\frac{a}{c} \cdot \frac{b}{d} = \frac{ab}{cd}$ ■

EXAMPLE B.5 Using Theorem B.4(a), we obtain

$$\frac{x-1}{2} + \frac{x+1}{2} = \frac{(x-1)+(x+1)}{2} = \frac{2x}{2} = x.$$ ■

EXAMPLE B.6 Using Theorem B.4(b), we obtain

$$\frac{x-1}{2} - \frac{x+1}{2} = \frac{(x-1)-(x+1)}{2} = \frac{-2}{2} = -1.$$ ■

EXAMPLE B.7 Using Theorem B.4(c), we obtain

$$\frac{x-1}{2} + \frac{x+1}{3} = \frac{3(x-1)+2(x+1)}{2 \cdot 3} = \frac{5x-1}{6}.$$ ■

EXAMPLE B.8 Using Theorem B.4(d), we obtain

$$\frac{x-1}{2} - \frac{x+1}{3} = \frac{3(x-1)-2(x+1)}{2 \cdot 3} = \frac{x-5}{6}.$$ ■

EXAMPLE B.9 Using Theorem B.4(e), we obtain

$$\frac{2}{x} \cdot \frac{4}{y} = \frac{8}{xy}.$$ ■

Exponents

If n is a positive integer and a is a real number, we define a^n as

$$a^n = \underbrace{a \cdot a \cdots a}_{n\ a\text{'s}}.$$

If a is a nonzero real number, we define $a^0 = 1$. If n is a negative integer and a is a nonzero real number, we define a^n as

$$a^n = \frac{1}{a^{-n}}.$$

EXAMPLE B.10 If a is a real number,

$$a^4 = a \cdot a \cdot a \cdot a.$$

As a specific example,

$$2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16.$$

If a is a nonzero real number,

$$a^{-4} = \frac{1}{a^4}.$$

As a specific example,

$$2^{-4} = \frac{1}{2^4} = \frac{1}{16}.$$

■

If a is a positive real number and n is a positive integer, we define $a^{1/n}$ to be the positive number b satisfying

$$b^n = a.$$

We call b the *nth root* of a.

EXAMPLE B.11 $3^{1/4}$ to nine significant digits is 1.316074013 because $(1.316074013)^4$ is approximately 3.

■

If a is a positive real number, m is an integer, and n is a positive integer, we define

$$a^{m/n} = (a^{1/n})^m.$$

The preceding equation defines a^q for all positive real numbers a and *rational numbers* q. (Recall that a rational number is a number that is the quotient of integers.)

EXAMPLE B.12 Since $3^{1/4}$ to nine significant digits is 1.316074013,

$$3^{9/4} = (1.316074013)^9 = 11.84466612.$$

The decimal values are approximations. ■

If a is a positive real number, the definition of a^x can be extended to include all real numbers x (rational or irrational). The following theorem lists five important laws of exponents.

THEOREM B.13 **LAWS OF EXPONENTS**

Let a and b be positive real numbers, and let x and y be real numbers. Then

(a) $a^{x+y} = a^x a^y$

(b) $(a^x)^y = a^{xy}$

(c) $\dfrac{a^x}{a^y} = a^{x-y}$

(d) $a^x b^x = (ab)^x$

(e) $\dfrac{a^x}{b^x} = \left(\dfrac{a}{b}\right)^x.$ ■

EXAMPLE B.14 Let $a = 3$, $x = 2$, and $y = 4$. Then $a^x = 9$, $a^y = 81$, and $a^{x+y} = 3^{2+4} = 729$. Now

$$a^{x+y} = 729 = 9 \cdot 81 = a^x a^y,$$

which illustrates Theorem B.13(a). ■

EXAMPLE B.15 Let $a = 3$, $x = 2$, and $y = 4$. Then $a^x = 9$ and $a^{xy} = 3^8 = 6561$. Now

$$(a^x)^y = 9^4 = 6561 = a^{xy},$$

which illustrates Theorem B.13(b). ■

EXAMPLE B.16 Let $a = 3$, $x = 2$, and $y = 4$. Then $a^x = 9$, $a^y = 81$, and $a^{x-y} = 3^{-2} = 1/9$. Now

$$\frac{a^x}{a^y} = \frac{9}{81} = \frac{1}{9} = a^{x-y},$$

which illustrates Theorem B.13(c). ■

EXAMPLE B.17 Let $a = 3$, $b = 4$, and $x = 2$. Then $a^x = 9$, $b^x = 16$, and $(ab)^x = 12^2 = 144$. Now

$$a^x b^x = 9 \cdot 16 = 144 = (ab)^x,$$

which illustrates Theorem B.13(d). ■

EXAMPLE B.18 Let $a = 3$, $b = 4$, and $x = 2$. Then $a^x = 9$, $b^x = 16$, and

$$\left(\frac{a}{b}\right)^x = \left(\frac{3}{4}\right)^2 = \frac{9}{16}.$$

Now

$$\frac{a^x}{b^x} = \frac{9}{16} = \left(\frac{a}{b}\right)^x,$$

which illustrates Theorem B.13(e). ■

EXAMPLE B.19

$$2^x 2^x = 2^{x+x} = 2^{2x} = (2^2)^x = 4^x$$ ■

Factoring

We may use the equation

$$(x + b)(x + d) = x^2 + (b + d)x + bd$$

to factor an expression of the form $x^2 + c_1 x + c_2$.

EXAMPLE B.20 **FACTOR $x^2 + 3x + 2$.**
We look for integer constants in the factorization. According to the previous equation, $x^2 + 3x + 2$ factors as $(x + b)(x + d)$, where $b + d = 3$ and $bd = 2$. If $bd = 2$ and b and d are integers, the only choices for b and d are 1, 2 and $-1, -2$. We find that $b = 1$ and $d = 2$ satisfy both $b + d = 3$ and $bd = 2$. Thus

$$x^2 + 3x + 2 = (x + 1)(x + 2).$$ ■

Special cases of

$$(x + b)(x + d) = x^2 + (b + d)x + bd$$

are

$$(x + b)^2 = x^2 + 2bx + b^2$$
$$(x - b)^2 = x^2 - 2bx + b^2$$
$$(x + b)(x - b) = x^2 - b^2.$$

EXAMPLE B.21 Using the equation $(x+b)^2 = x^2 + 2bx + b^2$, we have

$$(x+9)^2 = x^2 + 18x + 81. \quad \blacksquare$$

EXAMPLE B.22 **FACTOR $x^2 - 36$.**
Since $36 = 6^2$, we have

$$x^2 - 36 = (x+6)(x-6). \quad \blacksquare$$

We may use the equation

$$(ax+b)(cx+d) = (ac)x^2 + (ad+bc)x + bd$$

to factor an expression of the form $c_0x^2 + c_1x + c_2$.

EXAMPLE B.23 **FACTOR $6x^2 - x - 2$.**
We look for integer constants in the factorization. Using the preceding notation, we must have

$$ac = 6, \qquad ad + bc = -1, \qquad bd = -2.$$

Since $ac = 6$, the possibilities for a and c are

$$1, 6 \quad 2, 3 \quad -1, -6 \quad -2, -3.$$

Since $bd = -2$, the only possibilities for b and d are $1, -2$ and $-1, 2$. Since we must also have $ad + bc = -1$, we find that $a = 2$, $b = 1$, $c = 3$, and $d = -2$ provide a solution. Therefore, the factorization is

$$6x^2 - x - 2 = (2x+1)(3x-2). \quad \blacksquare$$

EXAMPLE B.24 Show that

$$\left[\frac{n(n+1)}{2}\right]^2 + (n+1)^3 = \left[\frac{(n+1)(n+2)}{2}\right]^2.$$

We show how the left side of the equation can be rewritten as the right side of the equation. By Theorem B.13(d) and (e), we have

$$\left[\frac{n(n+1)}{2}\right]^2 + (n+1)^3 = \frac{n^2(n+1)^2}{4} + (n+1)^3.$$

Since $(n+1)^2$ is a common factor of the right side of this equation, we may write

$$\frac{n^2(n+1)^2}{4} + (n+1)^3 = (n+1)^2\left[\frac{n^2}{4} + (n+1)\right].$$

Since

$$\frac{n^2}{4} + (n+1) = \frac{n^2+4n+4}{4} = \frac{(n+2)^2}{4},$$

it follows that

$$(n+1)^2\left[\frac{n^2}{4} + (n+1)\right] = (n+1)^2\left[\frac{(n+2)^2}{4}\right] = \left[\frac{(n+1)(n+2)}{2}\right]^2. \quad \blacksquare$$

Solving a Quadratic Equation

A **quadratic equation** is an equation of the form

$$ax^2 + bx + c = 0.$$

A **solution** is a value for x that satisfies the equation.

EXAMPLE B.25 The value $x = -3$ is a solution of the quadratic equation

$$2x^2 + 2x - 12 = 0$$

because

$$2(-3)^2 + 2(-3) - 12 = 2 \cdot 9 - 6 - 12 = 18 - 18 = 0.$$ ■

If a quadratic equation can be easily factored, its solutions may be readily obtained.

EXAMPLE B.26 Solve the quadratic equation

$$3x^2 - 10x + 8 = 0.$$

We may factor $3x^2 - 10x + 8$ as

$$3x^2 - 10x + 8 = (x - 2)(3x - 4).$$

For this expression to be equal to zero, either $x - 2$ or $3x - 4$ must equal zero. If $x - 2 = 0$, we must have $x = 2$. If $3x - 4 = 0$, we must have $x = 4/3$. Thus the solutions of the given quadratic equation are

$$x = 2 \quad \text{and} \quad x = \frac{4}{3}.$$ ■

The solutions of a quadratic equation can *always* be obtained from the **quadratic formula**.

THEOREM B.27 QUADRATIC FORMULA

The solutions of

$$ax^2 + bx + c = 0$$

are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$ ■

EXAMPLE B.28 The quadratic formula gives the solutions of

$$x^2 - x - 1 = 0$$

as

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1} = \frac{1 \pm \sqrt{1 + 4}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

Thus the solutions are

$$x = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad x = \frac{1 - \sqrt{5}}{2}.$$ ■

Inequalities

If a is **less than** b, we write $a < b$. If a is **less than or equal** to b, we write $a \leq b$. If a is **greater than** b, we write $a > b$. If a is **greater than or equal** to b, we write $a \geq b$.

EXAMPLE B.29 Suppose that $a = 2, b = 8, c = 2$. We have

$$a < b, \qquad b > a, \qquad a \leq b, \qquad b \geq a, \qquad a \leq c, \qquad a \geq c.$$ ■

Important laws of inequalities are given as Theorem B.30.

THEOREM B.30 **LAWS OF INEQUALITIES**

(a) If $a < b$ and c is any number whatsoever, then $a + c < b + c$.
(b) If $a \le b$ and c is any number whatsoever, then $a + c \le b + c$.
(c) If $a > b$ and c is any number whatsoever, then $a + c > b + c$.
(d) If $a \ge b$ and c is any number whatsoever, then $a + c \ge b + c$.
(e) If $a < b$ and $c > 0$, then $ac < bc$.
(f) If $a \le b$ and $c > 0$, then $ac \le bc$.
(g) If $a < b$ and $c < 0$, then $ac > bc$.
(h) If $a \le b$ and $c < 0$, then $ac \ge bc$.
(i) If $a > b$ and $c > 0$, then $ac > bc$.
(j) If $a \ge b$ and $c > 0$, then $ac \ge bc$.
(k) If $a > b$ and $c < 0$, then $ac < bc$.
(l) If $a \ge b$ and $c < 0$, then $ac \le bc$.
(m) If $a < b$ and $b < c$, then $a < c$.
(n) If $a < b$ and $b \le c$, then $a < c$.
(o) If $a \le b$ and $b < c$, then $a < c$.
(p) If $a \le b$ and $b \le c$, then $a \le c$.
(q) If $a > b$ and $b > c$, then $a > c$.
(r) If $a > b$ and $b \ge c$, then $a > c$.
(s) If $a \ge b$ and $b > c$, then $a > c$.
(t) If $a \ge b$ and $b \ge c$, then $a \ge c$. ■

EXAMPLE B.31 Solve the inequality

$$x - 5 < 6.$$

By Theorem B.30(a), we may add 5 to both sides of the inequality to obtain the solution

$$x < 11.$$ ■

EXAMPLE B.32 Solve the inequality

$$3x + 4 < x + 10.$$

By Theorem B.30(a), we may add $-x$ to both sides of the inequality to obtain

$$2x + 4 < 10.$$

Again, by Theorem B.30(a), we may add -4 to both sides of the inequality to obtain

$$2x < 6.$$

Finally, we may use Theorem B.30(e) to multiply both sides of the inequality by $1/2$ and obtain the solution

$$x < 3.$$ ■

EXAMPLE B.33 Show that if $n > 2m$ and $m > 2p$, then $n > 4p$.

We may use Theorem B.30(i) to multiply both sides of $m > 2p$ by 2 to obtain

$$2m > 4p.$$

Since

$$n > 2m,$$

we may use Theorem B.30(q) to obtain

$$n > 4p.$$ ■

EXAMPLE B.34 Show that

$$\frac{n+2}{n+1} < \frac{4(n+1)^2}{(2n+1)^2}$$

for every positive integer n.

Since $(n+1)(2n+1)^2$ is positive, by Theorem B.30(e),

$$(n+1)(2n+1)^2 \cdot \frac{n+2}{n+1} < (n+1)(2n+1)^2 \cdot \frac{4(n+1)^2}{(2n+1)^2},$$

which can be rewritten as

$$(2n+1)^2(n+2) < (n+1)4(n+1)^2.$$

Expanding each side of the inequality, we obtain

$$4n^3 + 12n^2 + 9n + 2 < 4n^3 + 12n^2 + 12n + 4.$$

By Theorem B.30(a), we may add $-4n^3 - 12n^2 - 9n - 2$ to both sides of the inequality to obtain

$$0 < 3n + 2.$$

This last inequality is true for all positive integers n because the right side is always at least 5. Since the steps are reversible (i.e., beginning with $0 < 3n + 2$ we can obtain the original inequality using Theorem B.30), we have proved the given inequality. ■

Logarithms

Throughout this subsection, b is a positive real number not equal to 1. If x is a positive real number, the **logarithm to the base b of x** is the exponent to which b must be raised to obtain x. We denote the logarithm to the base b of x as $\log_b x$. Thus if we let $y = \log_b x$, the definition states that $b^y = x$.

EXAMPLE B.35 We have $\log_2 8 = 3$ because $2^3 = 8$. ■

EXAMPLE B.36 Given

$$2^{2^x} = n,$$

where n is a positive integer, solve for x.

Let lg denote the logarithm to the base 2. Then from the definition of logarithm,

$$2^x = \lg n.$$

Again, from the definition of logarithm,

$$x = \lg(\lg n).$$ ■

The following theorem lists important laws of logarithms.

THEOREM B.37 **LAWS OF LOGARITHMS**

Suppose that $b > 0$ and $b \neq 1$. Then

(a) $b^{\log_b x} = x$

(b) $\log_b(xy) = \log_b x + \log_b y$

(c) $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$

(d) $\log_b(x^y) = y \log_b x$

(e) If $a > 0$ and $a \neq 1$, we have $\log_a x = \frac{\log_b x}{\log_b a}$

(f) If $x > y > 0$, then $\log_b x > \log_b y$. ■

Theorem B.37(e) is known as the **change-of-base formula for logarithms**. If we know how to compute logarithms to the base b, we can perform the computation on the right side of the equation to obtain the logarithm to the base a. Theorem B.37(f) says that the logarithm function is an increasing function.

EXAMPLE B.38 Let $b = 2$ and $x = 8$. Then $\log_b x = 3$. Now

$$b^{\log_b x} = 2^3 = 8 = x,$$

which illustrates Theorem B.37(a). ■

EXAMPLE B.39 Let $b = 2$, $x = 8$, and $y = 16$. Then $\log_b x = 3$, $\log_b y = 4$, and $\log_b(xy) = \log_2 128 = 7$. Now

$$\log_b(xy) = 7 = 3 + 4 = \log_b x + \log_b y,$$

which illustrates Theorem B.37(b). ■

EXAMPLE B.40 Let $b = 2$, $x = 8$, and $y = 16$. Then $\log_b x = 3$, $\log_b y = 4$, and

$$\log_b \left(\frac{x}{y}\right) = \log_2 \frac{1}{2} = -1.$$

Now

$$\log_b \left(\frac{x}{y}\right) = -1 = \log_b x - \log_b y,$$

which illustrates Theorem B.37(c). ■

EXAMPLE B.41 Let $b = 2$, $x = 4$, and $y = 3$. Then $\log_b x = 2$ and

$$\log_b \left(x^y\right) = \log_2 64 = 6.$$

Now

$$\log_b \left(x^y\right) = 6 = 3 \cdot 2 = y \log_b x,$$

which illustrates Theorem B.37(d). ■

EXAMPLE B.42 Suppose that we have a calculator that has a logarithm key that computes logarithms to the base 10 but does not have a key that computes logarithms to the base 2. We use Theorem B.37(e) to compute $\log_2 40$.

Using our calculator, we compute

$$\log_{10} 40 = 1.602060, \qquad \log_{10} 2 = 0.301030.$$

Theorem B.37(e) now gives

$$\log_2 40 = \frac{\log_{10} 40}{\log_{10} 2} = \frac{1.602060}{0.301030} = 5.321928.$$ ■

EXAMPLE B.43 Show that if k and n are positive integers satisfying

$$2^{k-1} < n < 2^k,$$

then

$$k - 1 < \lg n < k,$$

where lg denotes the logarithm to the base 2.

By Theorem B.37(f), the logarithm function is increasing. Therefore,

$$\lg\left(2^{k-1}\right) < \lg n < \lg\left(2^k\right).$$

By Theorem B.37(d),

$$\lg\left(2^{k-1}\right) = (k-1)\lg 2.$$

Since

$$\lg 2 = \log_2 2 = 1,$$

we have

$$\lg\left(2^{k-1}\right) = (k-1)\lg 2 = k - 1.$$

Similarly,

$$\lg\left(2^k\right) = k.$$

The given inequality now follows. ■

EXERCISES

In Exercises 1–3, simplify the given expression by combining like terms.

1. $8x - 12x$

2. $8y + 3a - 4y - 9a$

3. $6(a + b) - 8(a - b)$

In Exercises 4–6, combine the given fractions.

4. $\dfrac{8x - 4b}{3} + \dfrac{7x + b}{3}$

5. $\dfrac{8x - 4b}{2} - \dfrac{7x + b}{4}$

6. $\dfrac{8x - 4b}{3} \cdot \dfrac{7x + b}{3}$

7. Show that

$$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}.$$

Use this fact to show that

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}.$$

Find the value of each expression in Exercises 8–13 without using a calculator.

8. 3^4

9. 3^{-4}

10. $(-3)^4$

11. $(-3)^{-4}$

12. 1^{10}

13. 1000^0

14. Which expressions are equal?

(a) $3^4 3^{10}$
(b) $(3^4)^{10}$
(c) 3^{14}
(d) $4^3 10^3$
(e) $2^3 20^3$
(f) 3^{40}
(g) 2187^2

15. Show that $5^n + 4 \cdot 5^n = 5^{n+1}$ for every positive integer n.

In Exercises 16–24, expand the given expression.

16. $(x + 3)(x + 5)$

17. $(x - 3)(x + 4)$

18. $(2x + 3)(3x - 4)$

19. $(x + 4)^2$

20. $(x - 4)^2$

21. $(3x + 4)^2$

22. $(x - 2)(x + 2)$

23. $(x + a)(x - a)$

24. $(2x - 3)(2x + 3)$

In Exercises 25–36, factor the given expression.

25. $x^2 + 6x + 5$

26. $x^2 - 3x - 10$

27. $x^2 + 6x + 9$

28. $x^2 - 8x + 16$

29. $x^2 - 81$

30. $x^2 - 4b^2$

31. $2x^2 + 11x + 5$

32. $6x^2 + x - 15$

33. $4x^2 - 12x + 9$

34. $4x^2 - 9$

35. $9a^2 - 4b^2$

36. $12x^2 - 50x + 50$

37. Show that

$$(n+1)! + (n+1)(n+1)! = (n+2)!$$

for every positive integer n.

38. Show that

$$\frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

for every positive integer n.

39. Show that

$$\frac{n}{2n+1} + \frac{1}{(2n+1)(2n+3)} = \frac{n+1}{2n+3}$$

for every positive integer n.

40. Show that

$$7(3 \cdot 2^{n-1} - 4 \cdot 5^{n-1}) - 10(3 \cdot 2^{n-2} - 4 \cdot 5^{n-2}) = 3 \cdot 2^n - 4 \cdot 5^n$$

for every positive integer n.

41. Simplify $2r(n-1)r^{n-1} - r^2(n-2)r^{n-2}$.

In Exercises 42–44, solve the quadratic equation.

42. $x^2 - 6x + 8 = 0$

43. $6x^2 - 7x + 2 = 0$

44. $2x^2 - 4x + 1 = 0$

In Exercises 45–47, solve the given inequality.

45. $2x + 3 \le 9$

46. $2x - 8 > 3x + 1$

47. $\frac{x-3}{6} < \frac{4x+3}{2}$

48. Show that $\sum_{i=1}^{n} i \le n^2$.

49. Show that

$$(1+ax)(1+x) \ge 1 + (a+1)x$$

for any x and $a \ge 0$.

50. Show that

$$\left(\frac{3}{2}\right)^{n-2}\left(\frac{5}{2}\right) > \left(\frac{3}{2}\right)^n$$

for every integer $n \ge 2$.

51. Show that

$$\frac{2n+1}{(n+2)n^2} > \frac{2}{(n+1)^2}$$

for every positive integer n.

52. Show that $6n^2 < 6n^2+4n+1$ for every positive integer n.

53. Show that $6n^2+4n+1 \le 11n^2$ for every positive integer n.

*Find the value of each expression in Exercises 54–58 without using a calculator (*lg *means* $\log_2$*).*

54. $\lg 64$

55. $\lg \frac{1}{128}$

56. $\lg 2$

57. $2^{\lg 10}$

58. $\lg 2^{1000}$

Given that $\lg 3 = 1.584962501$ *and* $\lg 5 = 2.321928095$, *find the value of each expression in Exercises 59–63 (*lg *means* $\log_2$*).*

59. $\lg 6$

60. $\lg 30$

61. $\lg 59049$

62. $\lg 0.6$

63. $\lg 0.0375$

Use a calculator with a logarithm key to find the value of each expression in Exercises 64–67.

64. $\log_5 47$

65. $\log_7 0.30881$

66. $\log_9 8.888^{100}$

67. $\log_{10}(\log_{10} 1054)$

In Exercises 68–70, use a calculator with a logarithm key to solve for x.

68. $5^x = 11$

69. $5^{2x}6^x = 811$

70. $5^{11^x} = 10^{100}$

71. Show that $x^{\log_b y} = y^{\log_b x}$.

8. The statement follows from the fact that such an algorithm can be modified without changing its asymptotic worst-case time to determine whether the input contains duplicates and, by Theorem 11.2.1, any algorithm that determines whether duplicates exist has worst-case time $\Omega(n \lg n)$. Duplicates exist if and only if the distance between every output pair is zero; thus, we need only check one pair to determine whether there are duplicates or not.

9. Let L be the vertical line through p. By the choice of p, no points of S lie to the right of L. If p is the only point of S on L, p is a hull point. If other points of S lie on L, they all lie below p. In this case, if we rotate L clockwise slightly about p, L will contain only p and all other points of S will be to the left of L. Again we conclude that p is a hull point.

10. Let L be the line segment joining p and q. Let L' be the line through p perpendicular to L. There can be no other point r of S on L' or on the side of L' opposite q, for if there were such a point r, the distance from r to q would exceed the distance from p to q, which is impossible. Thus p is a hull point. Similarly, q is a hull point.

11. The points [sorted with respect to (1,2)] are (1,2), (11,3), (8,4), (14,7), (5,4), (11,7), (17,10), (7,6), (8,7), (12,10), (8,9), (5,9), (3,7), (3,11), (1,5), (1,9). The following table shows each triple that is examined in the while loop, whether it makes a left turn, and the action taken with respect to the triple:

Triple	*Left Turn?*	*Discard Middle Point?*
(1,2), (11,3), (8,4)	Yes	No
(11,3), (8,4), (14,7)	No	Yes
(1,2), (11,3), (14,7)	Yes	No
(11,3), (14,7), (5,4)	Yes	No
(14,7), (5,4), (11,7)	No	Yes
(11,3), (14,7), (11,7)	Yes	No
(14,7), (11,7), (17,10)	No	Yes
(11,3), (14,7), (17,10)	No	Yes
(1,2), (11,3), (17,10)	Yes	No
(11,3), (17,10), (7,6)	Yes	No
(17,10), (7,6), (8,7)	No	Yes
(11,3), (17,10), (8,7)	Yes	No
(17,10), (8,7), (12,10)	No	Yes
(11,3), (17,10), (12,10)	Yes	No
(17,10), (12,10), (8,9)	Yes	No
(12,10), (8,9), (5,9)	No	Yes
(17,10), (12,10), (5,9)	Yes	No
(12,10), (5,9), (3,7)	Yes	No
(5,9), (3,7), (3,11)	No	Yes
(12,10), (5,9), (3,11)	No	Yes
(17,10), (12,10), (3,11)	No	Yes
(11,3), (17,10), (3,11)	Yes	No
(17,10), (3,11), (1,5)	Yes	No
(3,11), (1,5), (1,9)	No	Yes
(17,10), (3,11), (1,9)	Yes	No

The convex hull is (1,2), (11,3), (17,10), (3,11), (1,9).

12. Run the part of Graham's Algorithm that follows the sort on the remaining points.

Appendix A

1. $\begin{pmatrix} 2+a & 4+b & 1+c \\ 6+d & 9+e & 3+f \\ 1+g & -1+h & 6+i \end{pmatrix}$

2. $\begin{pmatrix} 5 & 7 & 7 \\ -7 & 10 & -1 \end{pmatrix}$

5. $\begin{pmatrix} 3 & 18 & 27 \\ 0 & 12 & -6 \end{pmatrix}$

8. $\begin{pmatrix} -2 & -35 & -56 \\ -7 & -18 & 13 \end{pmatrix}$

9. $\begin{pmatrix} 18 & 10 \\ 14 & -6 \\ 23 & 1 \end{pmatrix}$

12. (-4)

14. (a) 2×3, 3×3, 3×2

(b) $AB = \begin{pmatrix} 33 & 18 & 47 \\ 8 & 9 & 43 \end{pmatrix}$

$$AC = \begin{pmatrix} 16 & 56 \\ 14 & 63 \end{pmatrix}$$

$$CA = \begin{pmatrix} 4 & 18 & 38 \\ 0 & 0 & 0 \\ 2 & 17 & 75 \end{pmatrix}$$

$$AB^2 = \begin{pmatrix} 177 & 215 & 531 \\ 80 & 93 & 323 \end{pmatrix}$$

$$BC = \begin{pmatrix} 18 & 65 \\ 34 & 25 \\ 12 & 54 \end{pmatrix}$$

17. Let $A = (b_{ij})$, $I_n = (a_{jk})$, $AI_n = (c_{ik})$. Then

$$c_{ik} = \sum_{j=1}^{n} b_{ij}a_{jk} = b_{ik}a_{kk} = b_{ik}.$$

Therefore, $AI_n = A$. Similarly, $I_nA = A$.

20. The solution is $X = A^{-1}C$.

Appendix B

1. $-4x$

4. $\dfrac{15x - 3b}{3} = 5x - b$

7. $\dfrac{1}{n} - \dfrac{1}{n+1} = \dfrac{n+1-n}{n(n+1)} = \dfrac{1}{n(n+1)}$

We may use this equation to compute $\sum_{i=1}^{n} \frac{1}{i(i+1)}$ as follows:

$$\sum_{i=1}^{n} \frac{1}{i(i+1)}$$
$$= \sum_{i=1}^{n} \frac{1}{i} - \frac{1}{i+1}$$
$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right)$$
$$+ \left(\frac{1}{n} - \frac{1}{n+1}\right)$$
$$= 1 - \frac{1}{n+1} = \frac{n+1-1}{n+1} = \frac{n}{n+1}.$$

8. 81

11. 1/81

14. (a), (c), and (g) are equal. (b) and (f) are equal. (d) and (e) are equal.

16. $x^2 + 8x + 15$

19. $x^2 + 8x + 16$

22. $x^2 - 4$

25. $(x + 5)(x + 1)$

28. $(x - 4)^2$

31. $(2x + 1)(x + 5)$

34. $(2x + 3)(2x - 3)$

37. $(n + 1)! + (n + 1)(n + 1)! = (n + 1)![1 + (n + 1)] = (n + 1)!(n + 2) = (n + 2)!$

40.
$$\begin{aligned} & 7(3 \cdot 2^{n-1} - 4 \cdot 5^{n-1}) - 10(3 \cdot 2^{n-2} - 4 \cdot 5^{n-2}) \\ &= 2^{n-2}(7 \cdot 3 \cdot 2 - 10 \cdot 3) + 5^{n-2}(-7 \cdot 4 \cdot 5 + 10 \cdot 4) \\ &= 2^{n-2} \cdot 12 + 5^{n-2}(-100) \\ &= 2^{n-2}(2^2 \cdot 3) - 5^{n-2}(5^2 \cdot 4) \\ &= 3 \cdot 2^n - 4 \cdot 5^n \end{aligned}$$

42. Factoring gives $(x - 4)(x - 2) = 0$, which has solutions $x = 4, 2$.

45. $2x \leq 6, x \leq 3$

48. $i \leq n$ for $i = 1, \ldots, n$. Summing these inequalities, we obtain

$$\sum_{i=1}^{n} i \leq n \cdot n = n^2.$$

51. Multiply by $(n + 2)n^2(n + 1)^2$ to get

$$(2n + 1)(n + 1)^2 > 2(n + 2)n^2$$

or

$$2n^3 + 5n^2 + 4n + 1 > 2n^3 + 4n^2$$

or

$$n^2 + 4n + 1 > 0,$$

which is true if $n \geq 1$.

54. 6

57. 10

59. 2.584962501

62. −0.736965594

64. 2.392231208

67. 0.480415248

68. 1.489896102

71. Let $u = \log_b y$ and $v = \log_b x$. By definition, $b^u = y$ and $b^v = x$. Now

$$x^{\log_b y} = x^u = (b^v)^u = b^{vu} = (b^u)^v = y^v = y^{\log_b x}.$$