Tiling rectangles and deficient rectangles with trominoes

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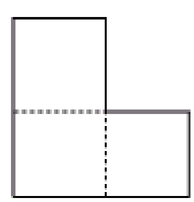
ABSTRACT. A (right) tromino is a 2×2 square with a 1×1 corner removed. Let m and n be integers ≥ 2. An $m \times n$ rectangle can be tiled with trominoes whenver 3 divides its area mn, except when one of m and n is 3 and the other is odd. An $m \times n$ rectangle with a 1×1 corner removed can be tiled with trominoes exactly when 3 divides its area $mn - 1$.An $m \times n$ rectangle with $m \leq n$ is tilable no matter which one of its 1×1 squares is removed provided 3 divides $mn - 1$ except when $m = 2$ and $n \geq 3$ or $m = 5$.The exceptional rectangles with a 1×1 square missing that cannot be tiled are classified by specifying which tiles must be removed. For every rectangle there is a pair of 1×1 squares whose removal produces an untilable shape.

1. Results

We will call the 2×2 three-cornered square



a **tromino**. The area of a tromino is 3, so only shapes whose area is a multiple of 3 can be tiled by trominoes. From now on we will simply say tiled to mean tiled by trominoes. The integers m and n will always be greater than or equal to 2.

THEOREM 1. *An $m \times n$ rectangle can always be tiled by trominoes if 3 divides its area mn, except when one dimension is 3 and the other is odd.*

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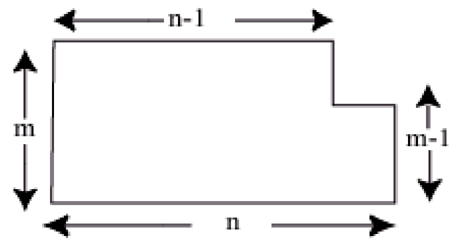
An $m \times n$ **three-cornered rectangle** is an $m \times n$ rectangle with a 1×1 corner square removed.



Note that the area of an $m \times n$ three-cornered rectangle is $mn - 1$.

THEOREM 2. *An $m \times n$ three-cornered rectangle can be tiled with trominoes exactly when 3 divides its area.*

Let

$$S = \{(m, n) : \text{ the three-cornered } m \times n \text{ rectangle can be tiled with trominoes}\}.$$

Our result is that a pair (m, n) always is in S except when it obviously is not.

To formulate this claim precisely, note that if mn is congruent to 0 or 2 modulo 3, then the area of the $m \times n$ three-cornered rectangle is not congruent to 0 modulo 3 and so that three-cornered rectangle can not be tiled by trominoes, since the area of any region tiled by trominoes must be an integral multiple of 3. So the only $m \times n$ three-cornered rectangles that could possibly be tiled by trominoes are those for which mn is congruent to 1. In other words, the only $m \times n$ three-cornered rectangles that could possibly be tiled by trominoes are those for which $m \equiv n \equiv 1 \bmod 3$ or $m \equiv n \equiv 2 \bmod 3$, and, indeed, all those three-cornered rectangles can be tiled.

Call a rectangle with one 1×1 square missing a **deficient rectangle**. The question of whether a deficient rectangle can be tiled with trominoes is clearly equivalent to the question of whether the full rectangle consisting of the disjoint union of the deficient rectangle and the 1×1 square can be tiled by a set of trominoes and a single 1×1 tile called a **monomino**, with the monomino covering the missing square.

THEOREM 3. *An $m \times n$ deficient rectangle with $m \leq n$ is tilable no matter which tile was removed to create the deficiency provided 3 divides its area $mn - 1$ except when $m = 2$ and $n \geq 5$ or when $m = 5$.*

It is disappointing that Theorem 3 is not as pretty as Theorem 2, but we will try to compensate for this by classifying all the exceptional cases. Suppose that 3 divides $mn - 1$. If $m = 2$, we must have $n = 2 + 3k, k = 0, 1, ...$, while if $m = 5$, we must have $n = 5 + 3k, k = 0, 1, 2, ...$ Also notice that the $(m, n) = (2, 2)$ case is *not* an exception. Denote the square in row i and column j by (i, j).

THEOREM 4. *Let R be a deficient $m \times n$ rectangle which cannot be tiled even though 3 divides $mn - 1$. If $m = n = 5$, there are 9 good (a 1×1 square is* ***good*** *if its removal from a full* $m \times n$ *rectangle produces a deficient rectangle that can be tiled) and* 16 *bad squares. If* $m = 5$ *and* $n = 5 + 3k, k = 1, 2, ...,$ *then there are only* 2 *bad squares, namely* $(2, 3)$ *and its reflection* $(4 + 3k, 3)$*. If* $m = 2$ *and* $n = 2 + 3k, k = 1, 2, ...,$ *then exactly the squares* $(3j, 1)$ *and* $(3j, 2), j = 1, 2, ..., k$ *are bad.*

The following proposition disallows the possibility of making the natural definition of deficiency of order $k, k \geq 2$ and then finding a direct extension of Theorem 3 for higher deficiencies.

PROPOSITION 1. *No rectangle has the property that no matter which two* 1×1 *tiles are removed, the remaining shape of area* $mn - 2$ *can be tiled.*

For if the squares $(2, 1)$ and $(2, 2)$ are removed, then the square $(1, 1)$ can not be covered by a tromino.

REMARK 1. *Trominoes were introduced in* [**G**]*. The proof in* [**G**] *of the special cases of Theorem 2 for dyadic squares,* $m = n = 2^k$ *provides a very beautiful example of mathematical induction. This proof that every* $2^k \times 2^k$ *deficient square can be tiled is based on subdividing a* $2^{n+1} \times 2^{n+1}$ *square into* 4 *congruent quadrants and then placing a single tromino at the center in such a way that it has one square in each of the three quadrants not containing the removed square. The reader is strongly encouraged to work this out.*[**G**]*, page 4 of* [**G1**]*, page 45 of* [**J**]*, problem 2.3.38 of* [**Z**] *Chu and Johnsonbaugh extended Golomb's work to the special cases of Theorem 2 for squares in general,* $m = n$*.*[**CJ**] *Theorems 1, 3, and 4 and Proposition 1 answer questions that they posed.*

REMARK 2. *As an application of Theorems 1 and 2, we consider the practical question of tiling as much as possible of any* $m \times n$ *rectangle, where* m *and* n *both exceed* 3*. There are 3 cases depending on the value of* mn *modulo* 3*. If* $mn \equiv 0$*, tile the entire rectangle with Theorem 1. If* $mn \equiv 1$*, remove a single corner tile and then use Theorem 2 to tile the rest of the rectangle. If* $mn \equiv 2$*, we must remove* 2 *tiles. It turns out that if a corner tile and a boundary tile adjacent to it are both removed, what remains can always be tiled. This can be proved by methods very similar to those used to prove Theorems 1 to 4. We will leave its proof as an exercise.*

REMARK 3. *When author Ash first heard the statement of Golomb's* $m = n = 2^k$ *result, he thought that deficiency meant that the removed square had to be removed from the corner. This led him first to the tiling of the* $(5, 5)$ *three-cornered square and eventually to Theorem 2 above. His wife Alison has occasionally pointed out that he doesn't always listen as carefully as he should. Well, at least once in his life this character flaw has proved beneficial.*

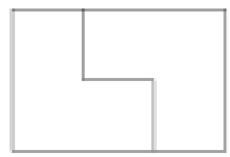
REMARK 4. *The proof that the* 16 *bad squares are indeed bad in the* 5×5 *case of Theorem 4 is especially recommended to the casual reader.*

PROBLEM 1. *Deal with the general case of deficiency two by proving an analogue of Theorem 4 for the case of an* $m \times n$ *rectangle where* $m \equiv n \equiv 2 \bmod 3$*. Slightly less generally, exactly when can such a rectangle be covered by one domino and* $(mn - 2)/3$ *trominoes?*

We will discuss this problem at the end of the paper.

2. Proofs

One very simple fact that we will need is that 2×3 rectangles can be tiled by trominoes.



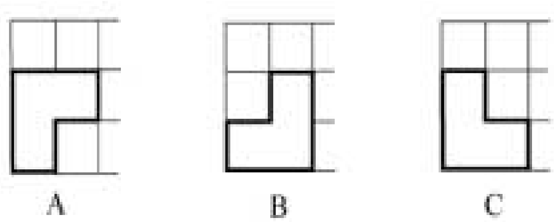
(2.1)

Denote an $i \times j$ rectangle by $R(i,j)$. We will indicate decompositions into nonoverlapping subrectangles by means of an additive notation. For example, a $3i \times 2j$ rectangle can be decomposed into ij 3×2 subrectangles and we write this fact as $R(3i,2j) = \sum_{\mu=1}^{i} \sum_{\nu=1}^{j} R(3,2) = ijR(3,2)$. It follows from this and the tiling (2.1) that

(2.2) any $3i \times 2j$ or $2i \times 3j$ rectangle can be tiled.

From now on, any rectangle decomposed into a combination of 3×2 subrectangles, 2×3 subrectangles and trominoes will be considered as successfully tiled by trominoes. We begin with the proof of Theorem 1.

Proof. We start by eliminating the exceptional cases. Assume that for some natural number k, $R = R(3,2k+1)$ has been tiled. We will show that this implies the tiling of $R(3,1)$, which is obviously impossible . Some tromino must cover square $(1,1)$. Here are the three possible ways that it can do that.



Orientation A is immediately ruled out, since square $(3,1)$ can not be tiled. But in cases B and C, the tiling must tile the leftmost 3×2 subrectangle of R, so that the original tiling is also a tiling of the rightmost $3 \times [(2k+1)-2]$ subrectangle. Repeating this argument $k-1$ more times leads to a tiling of the rightmost 3×1 subrectangle, which is the desired contradiction.

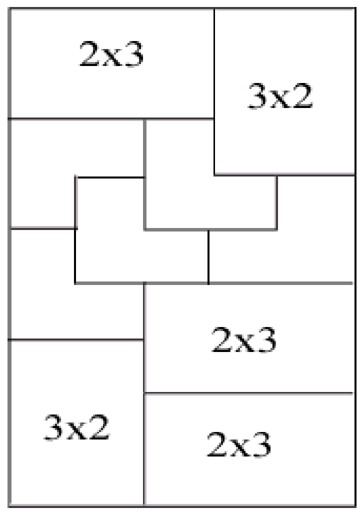
We may suppose that 3 divides $m = 3i$. For the case of n even, see (2.2). So we may assume that $n = 2j+1, j \geq 1$ and, since we are avoiding the exceptional cases, that $i \geq 2$. So we may write either $3i = 6k$ or $3i = 3 + 6k$ for some natural number k. In the former case we have

(2.3) $$R(6k,2j+1) = R(6k,2(j-1)) + R(6k,3)$$

and we invoke (2.2) to tile both subrectangles. In the latter case, since we are avoiding the exceptional cases we must have $j \geq 2$ so that we may write

$$\begin{aligned} R(3+6k, 2j+1) &= R(9, 2j+1) + R(6(k-1), 2j+1) \\ &= R(9,5) + R(9, 2(j-2)) + R(6(k-1), 2j+1). \end{aligned}$$

The first rectangle is tiled like this



the second rectangle is tiled using (2.2), and the third is just like the case (2.3) above. $\square$

For the proof of Theorem 2 we will denote the three-cornered rectangle $R(m,n) \backslash \{(m,n)\}$ by $R(m,n)^-$.

Proof. Let $m \leq n$. As mentioned above, the necessary condition that 3 divide the area of $R^- = R(m,n)^-$ splits into the cases $m \equiv n \equiv 1 \bmod 3$ and $m \equiv n \equiv 2 \bmod 3$. In the former case, we have either $R(7,7)^-$, whose tiling appears in the paper [**CJ**], or else we can write

$$\begin{aligned} R^- &= R(3j+1, 3k+1)^- = R(3(j-1)+4, 3(k-1)+4)^- \\ &= R(3(j-1), 3(k-1)) + R(4, 3(k-1)) + R(3(j-1), 4) + R(4,4)^-, \end{aligned}$$

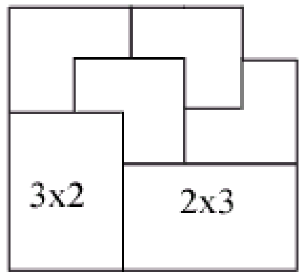
where $(j,k) \neq (2,2)$. The first three rectangles are tilable by Theorem 1, while the last term is a $2^2 \times 2^2$ square of deficiency 1 that is tilable by the original theorem of Golomb mentioned above.

In the latter case, we must tile $R^- = R(3j+2, 3k+2)^-$ where $0 \leq j \leq k$. If $j \neq 1$, we have

$$R^- = R(3j, 3k) + R(3j, 2) + R(2, 3k) + R(2,2)^-.$$

The first three terms are tiled by Theorem 1, while the last term actually *is* a tromino. Let $j = 1$. Either k is odd, $3k+2 = 6\ell+5$; or else k is even, $3k+2 = 6\ell+8$.

Correspondingly, either $R^{-} = R(5, 6\ell + 5)^{-} = R(5, 6\ell) + R(5, 5)^{-}$ where the first term is tiled with Theorem 1 and $R(5, 5)^{-}$ is tiled like this



(2.4) ;

or else $R^{-} = R(5, 6\ell + 8)^{-} = R(5, 6\ell) + R(5, 8)^{-}$ where again the first term is tiled with Theorem 1 and we also have $R(5, 8)^{-} = R(5, 6) + R(3, 2) + R(2, 2)^{-}$, the first two terms being tiled by Theorem 1, while the last term is a tromino. $\square$

Proof of Theorem 3. We eliminate the exceptional cases. If $m = 2$ and $n \geq 5$, then $R(2, n) \setminus \{(3, 1)\}$ can not be tiled because some tromino T must cover the square $(3, 2)$ and the untiled subregion lying to the left of $T + \{(3, 2)\}$ has area either 2 or 4 and so cannot be tiled. If $m = 5$, then $R(5, n) \setminus \{(2, 3)\}$ cannot be tiled because some tromino T must cover the square $(1, 3)$. If T lies above $(2, 3)$ the square $(1, 5)$ cannot be reached, otherwise the square $(1, 1)$ cannot be reached.

Now let $R(m, n)^{-}$ denote any $m \times n$ rectangle of deficiency 1.The "outlier" $R(2, 2)^{-}$ is tiled with one tromino. The method of proof is to proceed inductively after treating the cases $m = 4, 7, 8, 10$, and 11 individually. If $m \geq 13$, then $m - 6 > 6$ so that we may slice a full rectangle of height 6 off of either the top or the bottom of $R(m, n)^{-}$, i.e., $R(m, n)^{-} = R(m - 6, n)^{-} + R(6, n)$. Since the last term is tilable by Theorem 1, this first reduces the cases $m \in [13, 17]$ to the cases $m \in [7, 11]$, then the cases $m \in [19, 25]$ to the cases $m \in [13, 17]$, and so on.

If $m = 4$, write $R(4, 3k + 1)^{-} = R(4, 3(k - 1)) + R(4, 4)^{-}$. Apply Theorem 1 to the first term and the Golomb $m = n = 2^{k}$ Theorem to the second. Next $R(7, n)^{-} = R(3, n) + R(4, n)^{-}$ reduces the $m = 7$ case to the $m = 4$ case when n is even; while if n is odd, 6 divides $n - 7$ and the identity $R(7, n)^{-} = R(7, 6(\frac{n-7}{6})) + R(7, 7)^{-}$ is tiled by Theorem 1 and the Chu-Johnsonbaugh Theorem. If $m = 10$, $R(10, n)^{-} = R(7, 10)^{-} + R(3, 10)$ so that Theorem 1 provides a reduction to the $m = 7$ case. If $m = 8$, $R(8, n)^{-} = R(8, 8 + 3k)^{-} = R(8, 3k) + R(8, 8)^{-}$ is tiled by Theorem 1 and the Golomb $m = n = 2^{k}$ Theorem. Finally if $m = 11$, $R(11, 11)^{-}$ is tiled by the Chu-Johnsonbaugh Theorem, while the identity $R(11, 11 + 3k)^{-} = R(11, 6) + R(11, 3k - 6)^{-}$ with the first term tiled by Theorem 1 inductively reduces the cases $k \in \{1, 2\}$ to $k \in \{-1, 0\}$ (notice that if $k = -1$, $R(11, 8)^{-} = R(8, 11)^{-}$ is tilable by the $m = 8$ case), then the cases $k \in \{3, 4\}$ to $k \in \{1, 2\}$, and so on. $\square$

It remains only to analyze the exceptional cases, so we pass to the proof of Theorem 4.

Proof. The exceptional rectangles occur when

I. $(m,n) = (2,5+3k)\,, k = 0,1,2,...,$ or

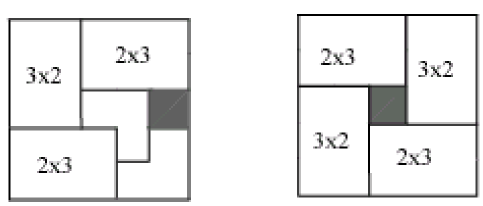
II. $(m,n) = (5,5+3k)\,, k = 0,1,2,....$

Case I. Fix k. The bad squares are exactly those of the form $(3j,y)$ where $j = 1,2,...,k+1$ and $y = 1$ or 2. By symmetry we may assume that $y = 1$. First we show that $R\,(2,5+3k) \setminus \{(3j,1)\}$ cannot be tiled. Assume the opposite and let T be the tromino covering the square $(3j,2)$. Then to the left of $T + \{(3j,1)\}$ lies either the rectangle $R\,(2,3j-1)$ or the rectangle $R\,(2,3j-2)$, neither of which can be tiled. To see that the remaining squares are good, use the decompositions $R\,(2,5+3k) \setminus \{(3j+1,1)\} = R\,(2,3j) + S + R\,(2,3\,(k-j+1))$ where S is the tromino covering $(3j+1,2)$ and $(3j+2,1)$, and $R\,(2,5+3k) \setminus \{(3j+2,1)\} = R\,(2,3j) + T + R\,(2,3\,(k-j+1))$ where T is the tromino covering $(3j+2,2)$ and $(3j+1,1)$.

Case II. The case $(5,5)$ is exceptional. Mark each of the nine squares

$$\left\{ \begin{array}{ccc} (1,5)\,, & (3,5)\,, & (5,5)\,, \\ (1,3)\,, & (3,3)\,, & (5,3)\,, \\ (1,1)\,, & (3,1)\,, & (5,1)\,, \end{array} \right\}$$

and assume that one of the 16 unmarked squares has been removed from $R\,(5,5)$ to form R^-. Then a proposed tiling of R^- must contain one tromino for each of the 9 marked squares, so that tiling must have area at least $9 \cdot 3 = 27$, which is absurd since the area of R^- is 24. Thus all 16 of the unmarked squares are bad. The tiling (2.4) above shows that $(5,5)$ is good, while these two tilings show $(5,3)$ and $(3,3)$ to be good.

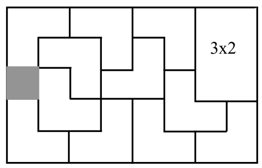


(2.5)

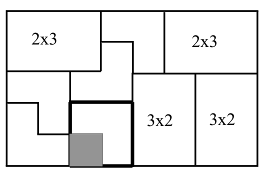
Symmetry considerations show that the remaining six marked tiles are also good. Thus all nine marked tiles are good.

At the beginning of the proof of Theorem 3 we showed that the square $(2,3)$ is bad in the $5 \times (5+3k)$ case. Symmetrically, $(4+3k,3)$ is also bad there. It remains to show that for $k \geq 1$, all the remaining squares of $R\,(5,5+3k)$ are good. Assume that all the cases $R\,(5,8)^-$ and $R\,(5,11)^-$ have been done and that any square removed from now on is not $(2,3)$. Let the square (i,j) be removed from $R\,(5,14)$. Symmetry allows the assumption $j \leq 7$. If $(i,j) \neq (7,3)$, then the decomposition of the resulting $R\,(5,14)^-$ into an $R\,(5,8)^-$ on the left and an $R\,(5,6)$ on the right allows a tiling, while $R\,(5,14) \setminus \{(7,3)\}$ is tiled by decomposing it into an $R\,(5,6)$ on the left and an $R\,(5,8)^-$ on the right. The cases of $R\,(5,n)\,, n \geq 17$ will be treated inductively. Symmetry allows us to consider only $R\,(5,n) \setminus \{(i,j)\}$ where $j \leq n/2 < n-8$ and where all but 2 tiles of $R\,(5,n-6)$ are good. Now decompose into $R(5, n-6)^{-}$ on the left and $R(5,6)$ on the right. Since $j \neq (n-6)-1$ the first term may be tiled, while the second is tiled with Theorem 1.

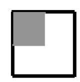
The cases $R(5,8)$. By symmetry we may assume $i \leq 4$ and $j \leq 3$. Since $(2,3)$ is bad, we have 11 cases to show good. If $i \in \{1,2,4\}$ and $j \leq 2$, then (i,j) is a good square of $R(2,8)$, so the decomposition of $R(5,8)^{-}$ into a full upper rectangle $R(3,8)$ and a lower $R(2,8)^{-}$ works in all six of these cases. There remain the five cases $(i,j) = (1,3), (3,1), (3,2), (3,3)$, and $(4,3)$. These are done in ad hoc fashion in this figure.

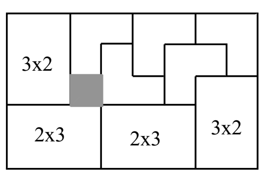


(1,3)

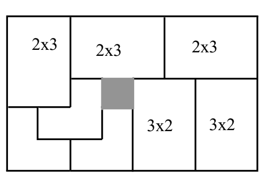


(3,1)

Replace the dark outlined square by  to tile the (3,2) case.

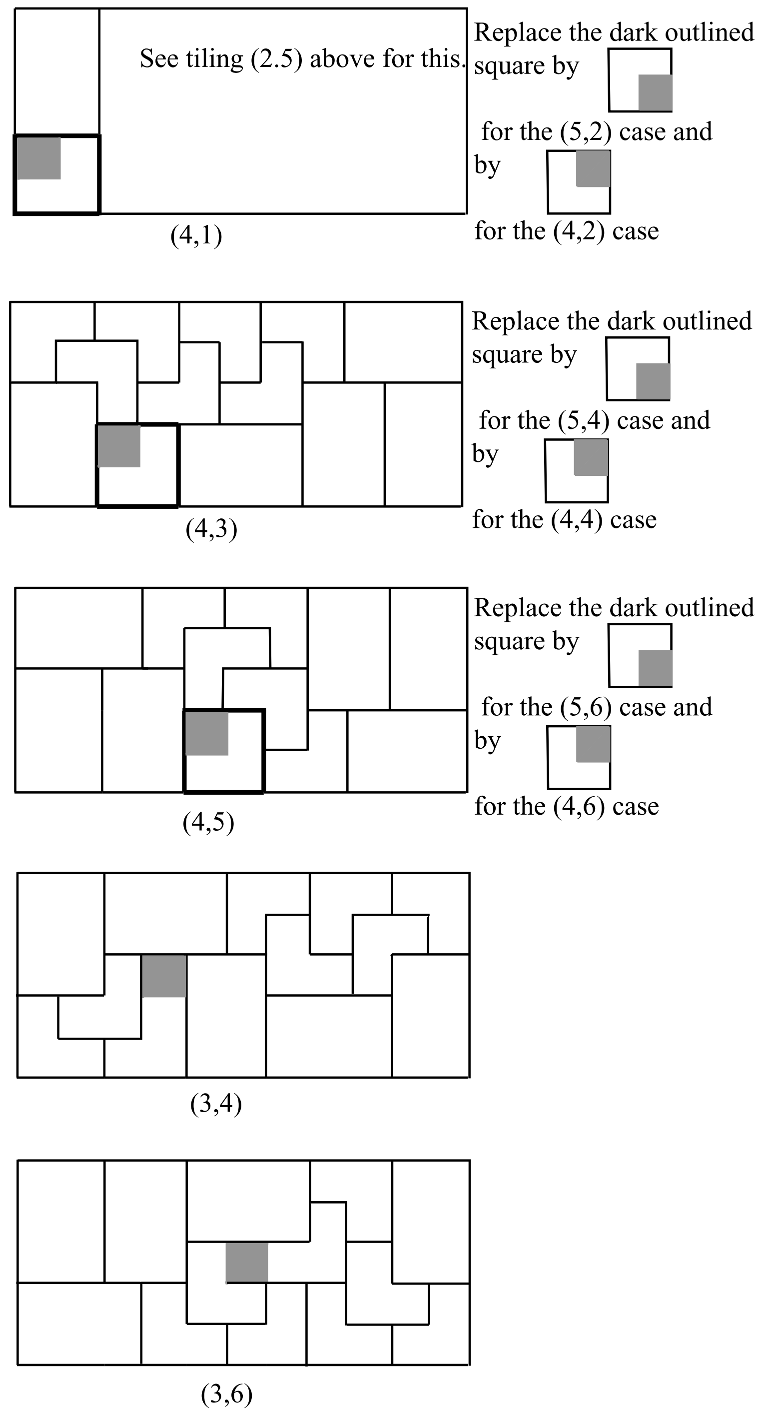


(3,3)



(4,3)

The cases $R(5,11)$. By symmetry we may assume $i \leq 6$ and $j \leq 3$. Since $(2,3)$ is bad, we have 17 cases to show good. If i and j are both odd, (i,j) is a good square of $R(5,5)$, so the decomposition of $R(5,11)^{-}$ into a left $R(5,5)^{-}$ and a full right $R(5,6)$ works for these 6 cases. The remaining 11 cases are done in ad hoc fashion in this figure.



□

At the end of the first section we raised the question of exactly when can an $m \times n$ rectangle where $mn \equiv 2 \bmod 3$ be covered by $(mn-2)/3$ trominoes when two squares are removed. On the negative side, as we pointed out in the proof of Proposition 1, if square $(2,1)$ and square $(2,2)$ are removed, then no tromino can cover square $(1,1)$. On the positive side, in Remark 2 it is pointed out that such a tiling is always possible if the the removed squares are in the corner of the rectangle. Now consider the 5×7 case. As in the 5×5 case (see the proof of Theorem 4), mark each of the 12 squares that have both coordinates odd and assume that two of the 23 unmarked squares have been removed from $R(5,7)$ to form $R^{=}$. Then a proposed tiling of $R^{=}$ must contain one tromino for each of the 12 marked squares, so that tiling must have area at least $12 \cdot 3 = 36$, which is absurd since the area of $R^{=}$ is 33. This reasoning disqualifies $\binom{23}{2} = 253$ pairs. Similar reasoning identifies a large number of bad pairs for $R(5,13), ..., R(5,7+6k), ...$

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