Plausible and Genuine Extensions of L′Hospital′s Rule

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A plausible extension

Roughly speaking, L'Hospital's Rule says that if $f(x)/g(x)$ is indeterminate at infinity and if x is large, then $f(x)/g(x)$ is approximately equal to $f'(x)/g'(x)$. Also, the limit comparison test says that if a_n is approximately equal to b_n then $\sum a_n$ converges if and only if $\sum b_n$ converges. The best thing that one could possibly hope for in trying to combine these two observations is the equivalence

$$\sum_{n=1}^{\infty} \frac{f'(n)}{g'(n)} \text{ converges if and only if } \sum_{n=1}^{\infty} \frac{f(n)}{g(n)} \text{ converges.} \tag{1}$$

Indeed this is true for certain classes of functions, including non-constant polynomials and exponential functions. L'Hospital's Rule as stated above is an implication, so it seems more prudent to predict that the convergence of $\sum \frac{f'(n)}{g'(n)}$ might imply the convergence of $\sum \frac{f(n)}{g(n)}$. In order to formulate a reasonable conjecture we first need the precise statement of L'Hospital's Rule [**5, 6, 7**].

We will say that two real valued functions f, g of a real variable generate an indeterminate form $0/0$ at infinity if

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0; \tag{2}$$

and an indeterminate form ∞/∞ at infinity in any of the following four cases:

$$\lim_{x\to\infty} f(x) = \pm\infty \quad \text{and} \quad \lim_{x\to\infty} g(x) = \pm\infty.$$

One standard version of L'Hospital's Rule asserts that if f, g generate the indeterminate form $0/0$ or ∞/∞ at infinity, and if

$$g'(x) \neq 0 \text{ in some neighborhood of } \infty, \tag{3}$$

then

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L \text{ implies } \lim_{x\to\infty} \frac{f(x)}{g(x)} = L \tag{4}$$

where L is an extended real number.

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PLAUSIBLE CONJECTURE. *If f, g generate the indeterminate form $0/0$ or ∞/∞ at infinity, and satisfy* (3), *and if $\sum f'(n)/g'(n)$ converges, then so does $\sum f(n)/g(n)$.*

As with L'Hospital's Rule itself, the reverse implication

$$\sum_{n=1}^{\infty} \frac{f(n)}{g(n)} \text{ converges implies } \sum_{n=1}^{\infty} \frac{f'(n)}{g'(n)} \text{ converges}$$

is not always true. For example, if we let

$$f(x) = \frac{\sin \pi x}{x} \quad \text{and} \quad g(x) = \frac{1}{x},$$

then f, g generate the indeterminate form $0/0$ at infinity. But for every integer n, $\sin \pi n = 0$ and $\cos \pi n = (-1)^n$, so

$$\sum_{n=1}^{\infty} \frac{f(n)}{g(n)} = \sum_{n=1}^{\infty} \frac{0/n}{1/n} \text{ converges,}$$

while $f'(x) = \dfrac{-\sin \pi x}{x^2} + \dfrac{\pi \cos \pi x}{x}$, $f'(n) = \dfrac{\pi\,(-1)^n}{n}$, $g'(n) = -\dfrac{1}{n^2}$, so that

$$\sum_{n=1}^{\infty} \frac{f'(n)}{g'(n)} = \sum_{n=1}^{\infty} \frac{\pi(-1)^n/n}{-1/n^2} = \sum_{n=1}^{\infty} \pi n(-1)^{n+1} \text{ diverges.}$$

Again, as in L'Hospital's Rule we need hypothesis (3), as is demonstrated by the following example, adapted from [**1**]. Let

$$\begin{aligned} f(x) &= 6\pi x + \sin 6\pi x, \\ g(x) &= e^{\sin 3\pi x} f(x). \end{aligned}$$

Then

$$\begin{aligned} \frac{f'(x)}{g'(x)} &= \frac{6\pi + 6\pi \cos 6\pi x}{3\pi(\cos 3\pi x)e^{\sin 3\pi x} f(x) + e^{\sin 3\pi x} f'(x)} \\ &= \left(\frac{2}{e^{\sin 3\pi x}}\right) \frac{\cos 6\pi x + 1}{\cos 3\pi x(6\pi x) + \cos 3\pi x(\sin 6\pi x) + 2 + 2\cos 6\pi x}. \end{aligned}$$

Hence, if n is an integer

$$\sum_{n=1}^{\infty} \frac{f'(n)}{g'(n)} = 2\sum_{n=1}^{\infty} \frac{(-1)^n}{3\pi n + 2(-1)^n},$$

which converges since its terms are alternating and decreasing to zero. (Note that we used 3π instead of π to get increasing denominators.) Nevertheless,

$$\sum_{n=1}^{\infty} \frac{f(n)}{g(n)} = \sum_{n=1}^{\infty} e^{-\sin 3\pi n} = \sum_{n=1}^{\infty} 1 = \infty.$$

Our main result is that the Plausible Conjecture is true only under certain conditions. One surprise (at least to us) is that the relationship between $\sum \frac{f(n)}{g(n)}$ and $\sum \frac{f'(n)}{g'(n)}$ is different in the $0/0$ and ∞/∞ cases, a distinction that does not occur in L'Hospital's Rule.

Two examples with $\sum \frac{f'(n)}{g'(n)}$ convergent and $\sum \frac{f(n)}{g(n)}$ divergent

EXAMPLE 1. The first example has f, g generating the indeterminate form $0/0$ at infinity, $g'(x) \neq 0$ near infinity, $\sum_{n=1}^{\infty} \frac{f'(n)}{g'(n)}$ convergent, and $\sum_{n=1}^{\infty} \frac{f(n)}{g(n)}$ divergent.

The idea is that $f(n)$ and $f'(n)$ may be chosen at will, subject only to the condition

$$\lim_{n\to\infty} f(n) = 0.$$

So we define for each natural number n,

$$f(n) = \frac{1}{n} \quad \text{and} \quad f'(n) = 0$$

and on each interval of the form $[n, n+1]$ let f be the unique cubic function satisfying these four boundary conditions:

$$f(n) = \frac{1}{n}, \quad f'(n) = 0, \quad f(n+1) = \frac{1}{n+1}, \quad f'(n+1) = 0.$$

Since every cubic with two critical points is monotone on the interval between them, we have for all x in $[n, n+1]$,

$$\frac{1}{n} \geq f(x) \geq \frac{1}{n+1},$$

so that $\lim_{x\to\infty} f(x) = 0$. On the interval $[1, 5]$, f looks like FIGURE 1.

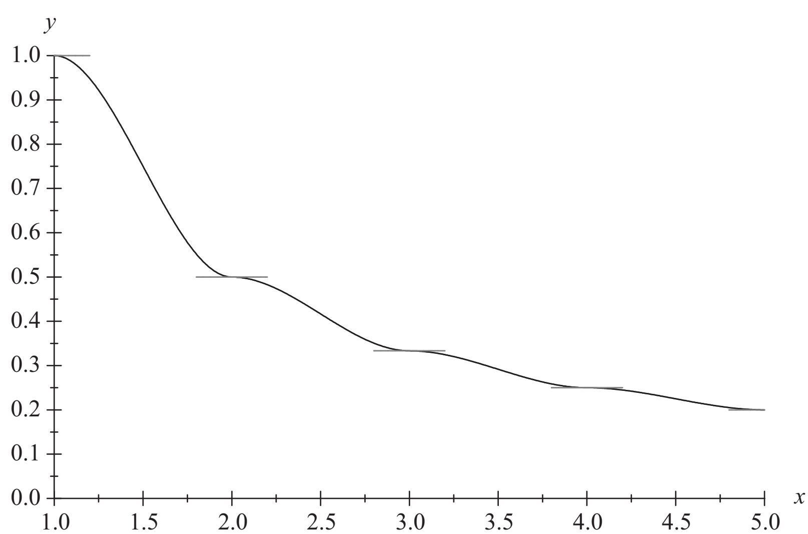

Figure 1 The gray horizontal tangent line segments show that $f'(x)$ is zero when x is an integer.

For $g(x)$, we choose $g(x) = \frac{1}{x}$ (for all $x > 0$) so that $g'(x) = -\frac{1}{x^2}$. Then we have

$$\sum \frac{f'(n)}{g'(n)} = \sum \frac{0}{-1/n^2} = \sum 0, \text{ which converges, and}$$

$$\sum \frac{f(n)}{g(n)} = \sum \frac{1/n}{1/n} = \sum 1, \text{ which diverges,}$$

and we are done.

A formula for the function $f(x)$ is

$$f(x) = \frac{1}{\lfloor x \rfloor} - \frac{(x - \lfloor x \rfloor)^2 (3 - 2(x - \lfloor x \rfloor))}{\lfloor x \rfloor (\lfloor x \rfloor + 1)}.$$

We leave as exercises for the reader to show that this example would have the same properties if $f(x)$ were replaced by $\frac{\pi}{2} - \mathrm{Si}(2\pi x)$ where $\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\, dt$.

EXAMPLE 2. The second example has f, g generating the indeterminate form ∞/∞ at infinity, $g'(x) \neq 0$ near infinity, $\frac{f'(x)}{g'(x)}$ decreasing near infinity, $\sum_{n=2}^{\infty} \frac{f'(n)}{g'(n)}$ convergent, and $\sum_{n=2}^{\infty} \frac{f(n)}{g(n)}$ divergent.

Let

$$f(x) = \ln x,$$
$$g(x) = x \ln^2 x - 2x \ln x + 2x \sim x \ln^2 x$$

so that

$$f'(x) = \frac{1}{x} \quad \text{and}$$
$$g'(x) = \ln^2 x.$$

Then

$$\sum_{n=2}^{\infty} \frac{f'(n)}{g'(n)} = \sum_{n=2}^{\infty} \frac{1}{n \ln^2 n} \text{ converges,}$$

while by the limit comparison test

$$\sum_{n=2}^{\infty} \frac{f(n)}{g(n)} \text{ diverges since } \sum_{n=2}^{\infty} \frac{\ln n}{n \ln^2 n} = \sum_{n=2}^{\infty} \frac{1}{n \ln n} \text{ diverges.}$$

Genuine extensions

The main purpose of counterexamples is to point the way to positive results. In this section we will present two of them.

The first theorem applies when the indeterminant form is $0/0$. Example 2 shows that the conclusion need not hold when $0/0$ is replaced by ∞/∞.

THEOREM 1. *Let f and g be differentiable functions on $(0, \infty)$ such that f, g generate the indeterminate form $0/0$ at infinity, $g'(x) \neq 0$ in a neighborhood of infinity, and $g(n)$ and $g'(n)$ are nonzero for all $n \in \mathbb{N}$. If*

$$\sum_{n=1}^{\infty} \sup_{x \geq n} \left| \frac{f'(x)}{g'(x)} \right| \textit{ converges,}$$

then

$$\sum_{n=1}^{\infty} \frac{f(n)}{g(n)} \textit{ converges absolutely.}$$

Proof. This is immediate once we know that from differentiability, (3), and (2) there follows this generalization of the Cauchy mean value theorem: For each $n \geq 1$, there is a $c = c_n > n$ such that

$$\left|\frac{f(n)}{g(n)}\right| = \left|\frac{f'(c)}{g'(c)}\right| \leq \sup_{x \geq n}\left|\frac{f'(x)}{g'(x)}\right|. \qquad \blacksquare$$

When $|f'(x)/g'(x)|$ is decreasing in a neighborhood of infinity, we can replace the convergence of the series of suprema by the convergence of

$$\sum_{n=1}^{\infty}\left|\frac{f'(n)}{g'(n)}\right|,$$

since in that case $\left|\frac{f'(n)}{g'(n)}\right| = \sup_{x \geq n}\left|\frac{f'(x)}{g'(x)}\right|$ for sufficiently large n.

We ask the reader to contrast Theorem 1 with Example 2. The example shows that if we change $0/0$ to ∞/∞ in the hypothesis, the desired conclusion (that $\sum f(n)/g(n)$ converges) may no longer follow. This is the distinction alluded to at the end of the first section.

Also, we can can identify a class of functions for which the original equivalence (1) holds in its entirety. This class includes polynomials, and a much wider class of functions as well.

THEOREM 2. *Let f and g be differentiable functions such that there are nonzero integers i and j and nonzero real numbers a and b and*

$$\begin{aligned} f(x) &= ax^i + o(x^i) & g(x) &= bx^j + o(x^j) \\ f'(x) &= iax^{i-1} + o(x^{i-1}) & g'(x) &= jbx^{j-1} + o(x^{j-1}). \end{aligned}$$

Then if f, g generate the indeterminate form $0/0$ or ∞/∞ at infinity and $g(n)$ and $g'(n)$ are nonzero for all $n \in \mathbb{N}$, then equivalence (1) is true.

Proof. The condition $j \geq i+2$ is necessary and sufficient for the convergence of both $\sum \frac{f(n)}{g(n)}$ and $\sum \frac{f'(n)}{g'(n)}$. $\blacksquare$

COROLLARY. *Let f and g be functions analytic in $\mathbb{C} \setminus \{0\}$ and not having an essential singularity at infinity. If f, g generate the indeterminate form $0/0$ or ∞/∞ at infinity, and $g(n)$ and $g'(n)$ are not zero for all $n \in \mathbb{N}$; then equivalence (1) is true.*

Proof. The first hypotheses means that we may write

$$f(x) = \sum_{\nu=-\infty}^{i} a_\nu x^\nu \qquad g(x) = \sum_{\nu=-\infty}^{j} b_\nu x^\nu$$

where a_i and b_j are nonzero. If $0/0$ is generated, then $i > 0$ and $j > 0$; while if ∞/∞ is generated, then $i < 0$ and $j < 0$. In either case,

$$\begin{aligned} f(x) &= a_i x^i + o(x^i) & g(x) &= b_j x^j + o(x^j) \\ f'(x) &= ia_i x^{i-1} + o(x^{i-1}) & g'(x) &= jb_j x^{j-1} + o(x^{j-1}), \end{aligned}$$

where i and j are nonzero. $\blacksquare$

Applications

One reason for the popularity of L'Hospital's Rule in calculus textbooks is the way it automates evaluating a large class of limits which might otherwise require Taylor expansion or some other special method. In the same spirit, we present five sums that are easily proved convergent by the application of Theorem 1.

(a) $\sum_{n=1}^{\infty}\left(1-n\sin\frac{1}{n}\right)$

(b) $\sum_{n=1}^{\infty} n^2\left(\cos\frac{1}{n}-1+\frac{1}{2n^2}\right)$

(c) $\sum_{n=1}^{\infty}\frac{1}{n^3\sin\frac{1}{n}}$

(d) $\sum_{n=1}^{\infty}\frac{1}{n^3\left(\ln\left(1+\frac{1}{n}\right)-1\right)}$

(e) $\sum_{n=1}^{\infty}\frac{1}{n^3\left(e^{1/n}-1\right)}$

In Application (a), the summands are $\frac{f(n)}{g(n)}$, where $f(n)=\frac{1}{n}-\sin\frac{1}{n}$ and $g(n)=\frac{1}{n}$. It is enough to study $\sum\left|\frac{f'(n)}{g'(n)}\right|$. We have

$$\left|\frac{f'(n)}{g'(n)}\right|=\left|\frac{-\frac{1}{n^2}+\frac{1}{n^2}\cos\left(\frac{1}{n}\right)}{-\frac{1}{n^2}}\right|=\left|1-\cos\frac{1}{n}\right|,$$

which the double angle formula $\cos 2\theta=1-2\sin^2\theta$ allows us to write as $2\sin^2\frac{1}{2n}$. The inequality $|\sin\theta|\le|\theta|$ then lets us bound this by $2\left(\frac{1}{2n}\right)^2=\frac{1}{2}n^{-2}$. Putting this all together, we have

$$\sum_{n=1}^{\infty}\left|\frac{f'(n)}{g'(n)}\right|\le\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

Since the series on the right hand side is convergent, by Theorem 1, $\sum_{n=1}^{\infty}\left(1-n\sin\frac{1}{n}\right)$ is absolutely convergent.

Two applications of Theorem 1 show the series of Application (b) to be absolutely convergent. In fact, an application of Theorem 1, with $f(n)=\cos\frac{1}{n}-1+\frac{1}{2n^2}$ and $g(n)=n^{-2}$, reduces Application (b) to Application (a). In each of the remaining three applications, set $f(n)=n^{-3}$. We leave working out the details of applications (b), (c), (d), and (e) as exercises for the reader.

Discrete analogues

There is a discrete version of L'Hospital's Rule called the Stolz-Cesàro Theorem. It asserts that if $\{a_n\}_{n\ge1}$ and $\{b_n\}_{n\ge1}$ are two sequences of real numbers with $\{a_n\}$, $\{b_n\}$ generating the indeterminate form $0/0$ at infinity with b_n strictly decreasing to 0 or with $\{a_n\}$, $\{b_n\}$ generating the indeterminate form ∞/∞ at infinity with b_n strictly increasing to ∞, and if $\lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n}$ exists; then $\lim_{n\to\infty}\frac{a_n}{b_n}$ also exists and has the same value. We do not know who coined the name of this very well known theorem. The ∞/∞ case is stated and proved on pages 173–175 of Stolz's 1885 book [**4**] and also on page 54 of Cesàro's 1888 article [**2**]. It appears as Problem 70 in [**3**].

A discrete analogue of what we have done above involves investigating the conjecture that $\sum\frac{a_{n+1}-a_n}{b_{n+1}-b_n}$ converges implies that $\sum\frac{a_n}{b_n}$ converges. We will give analogues of both negative examples above and of the positive results in Theorems 1 and 2. In particular, the same distinction between the $0/0$ case and the ∞/∞ case observed in the last two sections continues to hold here as well.

EXAMPLE 3. The analogue of Example 1 requires a different construction, so we give it here.

For $2^k \le n < 2^{k+1}$, let $a_n = 4^{-k}$ and $b_n = 2^{-k} + \epsilon_n$, where ϵ_n is strictly decreasing and $0 < \epsilon_n \ll 2^{-k}$. For example, $\epsilon_n = 2^{-2^n}$ will do. Then $a_n \to 0$, $b_n \to 0$, b_n is strictly decreasing, and

$$\begin{aligned}
\sum \frac{a_{n+1}-a_n}{b_{n+1}-b_n} &= \sum_{k=1}^{\infty}\left(\frac{a_{2^{k+1}-1}-a_{2^{k+1}}}{b_{2^{k+1}-1}-b_{2^{k+1}}} + \sum_{n=2^k}^{2^{k+1}-2}\frac{a_{n+1}-a_n}{b_{n+1}-b_n}\right)\\
&= \sum_{k=1}^{\infty}\left(\frac{4^{-k}-4^{-(k+1)}}{2^{-k}-2^{-(k+1)}+\left(\epsilon_{2^{k+1}-1}-\epsilon_{2^{k+1}}\right)} + \sum_{n=2^k}^{2^{k+1}-2}\frac{0}{\epsilon_n-\epsilon_{n+1}}\right)\\
&\le \sum_{k=1}^{\infty}\frac{4^{-k}-4^{-(k+1)}}{2^{-k}-2^{-(k+1)}+0}\\
&= \sum_{k=1}^{\infty}\frac{4^{-k}}{2^{-k}}\cdot\frac{1-1/4}{1-1/2} = \frac{3}{2}\sum_{k=1}^{\infty}2^{-k} < \infty,
\end{aligned}$$

so that $\sum \frac{a_{n+1}-a_n}{b_{n+1}-b_n}$ converges. On the other hand, $\sum \frac{a_n}{b_n}$ diverges, since

$$\begin{aligned}
\sum_{n=2}^{2^{N+1}-1}\frac{a_n}{b_n} &= \sum_{k=1}^{N}\left(\sum_{n=2^k}^{2^{k+1}-1}\frac{4^{-k}}{2^{-k}+\epsilon_n}\right)\\
&\ge \sum_{k=1}^{N}\left(\sum_{n=2^k}^{2^{k+1}-1}\frac{4^{-k}}{2^{-k}\cdot 2}\right)\\
&= \frac{1}{2}\sum_{k=1}^{N}2^k\frac{4^{-k}}{2^{-k}} = \frac{1}{2}\sum_{k=1}^{N}1,
\end{aligned}$$

which diverges as $N \to \infty$.

EXAMPLE 4. Our second counterexample is provided by Example 2.

Let $a_n = f(n)$ where $f(x) = \ln x$ and $b_n = g(n)$ where $g(x) = x\ln^2 x - 2x\ln x + 2x$. For every $n \in \mathbb{N}$, the Cauchy mean value theorem yields

$$\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \frac{f'(n+\theta)}{g'(n+\theta)} = \frac{1}{(n+\theta)\ln^2(n+\theta)},$$

for some $\theta = \theta(n) \in (0,1)$, so that $\sum_n \sup_{m\ge n}\left|\frac{a_{m+1}-a_m}{b_{m+1}-b_m}\right|$ converges. But as noted in the discussion of Example 2, $\sum \frac{a_n}{b_n}$ diverges. Also $b_{n+1}-b_n = g'(n+\varphi) = \ln^2(n+\varphi)$ for some $\varphi = \varphi(n) \in (0,1)$ for all $n \in \mathbb{N}$, so that $\{b_{n+1}-b_n\}$ is an increasing sequence.

Here is our first positive result.

THEOREM 3. *Let* $\{a_n\}$, $\{b_n\}$ *generate the indeterminate form* $0/0$ *as* $n \to \infty$. *Suppose* $\{b_n\}$ *is decreasing for all positive integers* n. *If*

$$\sum_{n=1}^{\infty} \sup_{m \geq n} \left| \frac{a_m - a_{m+1}}{b_m - b_{m+1}} \right| \tag{5}$$

converges, then $\sum_{n=1}^{\infty} \frac{a_n}{b_n}$ *converges absolutely.*

Proof. Since $\{b_n\}$ decreases strictly to 0, $\Delta b_n = b_n - b_{n+1} > 0$. Let $\Delta a_n = a_n - a_{n+1}$. Since $a_n \to 0$, for every n, $a_n = \sum_{m \geq n} \Delta a_m$. It is enough to show that for every n there exists an $m \geq n$ so that $|a_n/b_n| \leq |\Delta a_m / \Delta b_m|$. Suppose not. Then for some n and every $m \geq n$,

$$\frac{|a_n|}{b_n} = \left| \frac{a_n}{b_n} \right| > \left| \frac{\Delta a_m}{\Delta b_m} \right| = \frac{|\Delta a_m|}{\Delta b_m}$$

and so

$$|a_n| \, \Delta b_m > |\Delta a_m| \, b_n.$$

Sum these inequalities from m equal n to infinity to get a contradiction.

$$\begin{aligned} |a_n| \, b_n = |a_n| \sum_{n=m}^{\infty} \Delta b_m = \sum_{n=m}^{\infty} |a_n| \, \Delta b_m > \sum_{n=m}^{\infty} |\Delta a_m| \, b_n \\ \geq \left| \sum_{m=n}^{\infty} \Delta a_m \right| b_n = |a_n| \, b_n. \end{aligned}$$

∎

When $\left| \frac{a_n - a_{n+1}}{b_n - b_{n+1}} \right|$ is decreasing in a neighborhood of infinity, we can replace the convergence of the series of suprema by the convergence of

$$\sum_{n=1}^{\infty} \left| \frac{a_n - a_{n+1}}{b_n - b_{n+1}} \right|,$$

since

$$\left| \frac{a_n - a_{n+1}}{b_n - b_{n+1}} \right| = \sup_{m \geq n} \left| \frac{a_m - a_{m+1}}{b_m - b_{m+1}} \right|$$

for sufficiently large n.

The following analogue of Theorem 2 also has a simple proof that is very similar to the proof of Theorem 2.

THEOREM 4. *Let* $\{a_n\}$ *and* $\{b_n\}$ *be sequences such that there are nonzero integers* i *and* j *and nonzero real numbers* a *and* b *and*

$$\begin{aligned} a_n &= an^i + o(n^i) & b_n &= bn^j + o(n^j) \\ a_{n+1} - a_n &= ian^{i-1} + o(n^{i-1}) & b_{n+1} - b_n &= jbn^{j-1} + o(n^{j-1}). \end{aligned}$$

Then if $\{a_n\}$, $\{b_n\}$ *generate the indeterminate form* $0/0$ *or* ∞/∞ *at infinity and* b_n *and* $b_{n+1} - b_n$ *are not zero for all* $n \in \mathbb{N}$, *then the equivalence*

$$\sum_{n=1}^{\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} \text{ converges if and only if } \sum_{n=1}^{\infty} \frac{a_n}{b_n} \text{ converges}$$

is true.

COROLLARY. *Let f and g be functions analytic in $\mathbb{C} \setminus \{0\}$ and not having an essential singularity at infinity. If f, g generate the indeterminate form $0/0$ or ∞/∞ at infinity, and $g(n)$ and $g'(n)$ are not zero for all $n \in \mathbb{N}$; then the equivalence*

$$\sum_{n=1}^{\infty} \frac{f(n+1)-f(n)}{g(n+1)-g(n)} \text{ converges if and only if } \sum_{n=1}^{\infty} \frac{f(n)}{g(n)} \text{ converges.}$$

is true.

Proof. The first hypothesis mean that we may write

$$f(x) = \sum_{\nu=-\infty}^{i} a_\nu x^\nu \qquad g(x) = \sum_{\nu=-\infty}^{j} b_\nu x^\nu$$

where a_i and b_j are nonzero. If $0/0$ is generated, then $i > 0$ and $j > 0$; while if ∞/∞ is generated, then $i < 0$ and $j < 0$. In either case,

$$f(x) = a_i x^i + o\left(x^i\right) \qquad g(x) = b_j x^j + o\left(x^j\right)$$
$$f'(x) = i a_i x^{i-1} + o\left(x^{i-1}\right) \qquad g'(x) = j b_j x^{j-1} + o\left(x^{j-1}\right),$$

where i and j are nonzero. So there is a $\theta = \theta(n) \in (0, 1)$ such that

$$\begin{aligned} f(n+1) - f(n) &= f'(n+\theta) \\ &= i a_i (n+\theta)^{i-1} + o(n^{i-1}) \\ &= i a_i n^{i-1} + o(n^{i-1}). \end{aligned}$$

A similar calculation holds for g, so that we may finish by applying Theorem 4. ■

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Summary Let f and g be differentiable real valued functions. Motivated by L'Hospital's Rule, we might expect the convergence of $\sum f'(n)/g'(n)$ to imply the convergence of $\sum f(n)/g(n)$ when f, g, f', g' all have limit 0 as x tends to infinity and also when all four functions have infinite limits at infinity. We find this to be true, subject to some very mild additional conditions, when the four functions have limit zero, but not necessarily to be true in the infinity case. For limits, the discrete analogue of L'Hospital's Rule is the Stolz-Cesàro Theorem. We also find a result for series that is in the spirit of the Stolz-Cesàro Theorem.