(ABC), (BCD) imply (ACD)

Now make the following permutation of the letters A, B, C, D:

$$\begin{array}{cccc} A & B & C & D \\ \downarrow & \downarrow & \downarrow & \downarrow \\ D & C & B & A \end{array}$$

With this relabeling, the preceding becomes a proof that

(DCB), (CBA) imply (DBA)

or equivalently (by II,1),

(ABC), (BCD) imply (ABD)

which yields the other half of the theorem's conclusion.

Theorem B-6

Let A, B, C, and D be distinct points of a line. If (ABC) and (ACD), then (BCD) and (ABD).

This can be proved with similar techniques, as can the following more general result, which was listed as an axiom in Hilbert's original 1899 edition, but was subsequently found by E. H. Moore to depend on the other axioms.

Theorem B-7

Given any four distinct points on a line, they can be labeled A, B, C, and D in such a way that (ABC), (ABD), (BCD), and (ACD).

Definition B-8

For two distinct points A and B, the ray $\overrightarrow{AB}$ is the set consisting of the points of $\overline{AB}$ together with all points C such that (ABC). We say that A is the initial point of $\overrightarrow{AB}$ or that $\overrightarrow{AB}$ emanates from A.

Definition B-9

Let A, B, and C be three distinct points on a line a. We say that B and C lie on opposite sides of A if (BAC). We say that B and C lie on the same side of A if (BAC) is false, i.e., if either (ACB) or (ABC).