Formula Sheet for Part I of the comprehensive exam

Applied and Computational Mathematics

Numerical Analysis

Sensitivity of root finding: Assume that r is a root of $f(x)$ and $r+\Delta r$ is a roof of $f(x)+\varepsilon g(x)$.

Then $\Delta r \approx -\dfrac{\varepsilon g(r)}{f'(r)}$ if $\varepsilon << f'(r)$. **Error magnification factor** = $\dfrac{|g(r)|}{|rf'(r)|}$.

Quadratic convergence of Newton's method: If r is a simple root of $f(x)$, then Newton's method is locally and quadratically convergent to r. The error e_i at step i satisfies the relation

$$\lim_{i\to\infty}\frac{e_{i+1}}{e_i^2} = M \quad \text{where } M = \frac{f''(r)}{2f'(r)}.$$

Operation count for Gaussian elimination: the elimination step for a system of n equations in n variables can be completed in about $2n^3/3$ operations, while the back-substitution step can be completed in about n^2 operations.

Error magnification and condition number: Let x_a be an approximate solution of the linear system $Ax=b$. The vector $r = Ax_a - b$ is called the residual.

- The backward error is the norm of the residual $\|b - Ax_a\|$ and the forward error is $\|x - x_a\|$.
- The relative backward error is $\|r\|/\|b\|$ and the relative forward error is $\|x - x_a\|/\|x\|$.
- The error magnification factor is the ratio of the relative forward error to relative backward error. The condition number of a square matrix, $cond(A)$, is the maximum possible error magnification factor for solving $Ax=b$, over all righ-hand sides b; $cond(A) = \|A\|\cdot\|A^{-1}\|$.

(The norm considered is usually the infinity (or maximum) norm.)

Iterative Methods: The system *Ax=b* can be solved iteratively starting with an initial vector x_0 using one of the following methods:

- Jacobi: $x_{k+1} = D^{-1}(b-(L+U)x_k)$
- Gauss-Seidel: $x_{k+1} = D^{-1}(b - Ux_k - Lx_{k+1})$
- SOR: $x_{k+1} = (\omega L + D)^{-1}[(1-\omega)Dx_k - \omega Ux_k] + \omega(D+\omega L)^{-1}b$

Here, D denotes the main diagonal of A, L is the lower triangle of A and U is the upper triangle.

Multivariate Newton's Method: A nonlinear system of equations *F(x)=0* can be solved iteratively starting with an initial vector x_0 and $x_{k+1} = x_k - (DF(x_k))^{-1}F(x_k)$.

Interpolation error formula: Assume that *P(x)* is the interpolating polynomial fitting the n points $(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)$. The interpolation error is

$$f(x) - P(x) = \frac{(x - x_1)(x - x_2)...(x - x_n)}{n!} f^{(n)}(c)$$

where c lies between the smallest and largest of the numbers $x, x_1, x_2, ..., x_n$.

Normal equations for least squares: Given the inconsistent system *Ax=b*, one solves the normal system $A^T Ax = A^T b$ for the least squares solution $\bar{x}$ that minimizes the Euclidean norm of the residual $r = b - Ax$.

Numerical Differentiation

- Two point forward-difference: $f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{h}{2} f''(c)$
- Three point centered-difference: $f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2}{6} f'''(c)$

Numerical Integration

- Trapezoid rule: $\int_a^b f(x)dx = \frac{h}{2}(f(a) + f(b)) - \frac{h^3}{12} f''(c)$ where $h = b - a$ and $c \in [a, b]$.
- Simpson's rule: $\int_a^b f(x)dx = \frac{h}{3}(f(a) + 4f((a+b)/2) + f(b)) - \frac{h^5}{90} f^{(4)}(c)$
  where $h = (b - a)/2$ and $c \in [a, b]$.
- Midpoint rule: $\int_a^b f(x)dx = hf\left(\frac{a+b}{2}\right) + \frac{h^3}{24} f''(c)$ where $h = b - a$ and $c \in [a, b]$.
- Composite Trapezoid rule: $\int_a^b f(x)dx = \frac{h}{2}\left[f(a) + f(b) + 2\sum_{i=1}^{m-1} f(x_i)\right] - \frac{(b-a)h^2}{12} f''(c)$
  where $h = (b - a)/m$ and $c \in [a, b]$.
- Composite Simpson's rule:
  $$\int_a^b f(x)dx = \frac{h}{3}\left[f(a) + f(b) + 4\sum_{i=1}^{m} f(x_{2i-1}) + 2\sum_{i=1}^{m-1} f(x_{2i})\right] - \frac{(b-a)h^4}{180} f^{(4)}(c)$$
  where $h = (b - a)/(2m)$ and $c \in [a, b]$.
- Composite Midpoint rule: $\int_a^b f(x)dx = h\sum_{i=1}^{m} f(w_i) + \frac{(b-a)h^2}{24} f''(c)$ where
  $h = (b - a)/m$ and $c \in [a, b]$. The w_i are the midpoints of the m subinterval of [a,b].

Splines: A set of cubic splines $S_1(x), S_2(x), ..., S_{n-1}(x)$ for the data set $(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)$ is called:

- natural if $S_1''(x) = 0$ and $S_{n-1}''(x) = 0$
- parabolically terminated if $S_1(x)$ and $S_{n-1}(x)$ have degree at most 2
- not-a-knot if $S_1'''(x_2) = S_2'''(x_2)$ and $S_{n-2}'''(x_{n-1}) = S_{n-1}'''(x_{n-1})$

Power Iteration Method: To find the dominant eigenvalue and corresponding eigenvector, begin with an initial vector x_0; each iteration consists of normalizing the current vector and multiplying by matrix A. The Rayleigh quotient is used to approximate the eigenvalue.

For k=1,2,3... $u_{k-1} = x_{k-1} / \|x_{k-1}\|_2$; $x_k = Au_{k-1}$; $\lambda_k = u_{k-1}^T A u_{k-1}$

The Power Iteration method converges linearly to an eigenvectors associated to the dominant eigenvalue λ_1 with rate $S = |\lambda_2 / \lambda_1|$.

Singular Value Decomposition: Let *A* be an *mxn* matrix. There exist two orthonormal bases $\{v_1, ..., v_n\}$ of R^n and $\{u_1, ..., u_m\}$ of R^m, and real numbers $s_1 \geq s_2 \geq ... \geq s_n \geq 0$ such that:

- $Av_i = s_i u_i$ for $1 \leq i \leq \min\{m, n\}$
- the columns of $V = [v_1 | ... | v_n]$ are the orthonormal vectors of $A^T A$
- the columns of $U = [u_1 | ... | u_m]$ are the orthonormal vectors of AA^T
- s_i are the roots of the eigenvalues of AA^T (or $A^T A$).
- $A = USV^T$, where *S* is the diagonal matrix consisting of the s_i entries