Comparing algebraic and finite group cohomology

Christopher Drupieski
Department of Mathematics
University of Georgia

Southeastern Lie Theory Workshop on Finite and Algebraic Groups

June 1, 2011
# University of Georgia VIGRE Algebra Group

## Faculty
- Brian Boe
- Jon Carlson
- Leonard Chastkofsky
- Daniel Nakano
- Lisa Townsley

## Postdoctoral Fellows
- Christopher Drupieski
- Niles Johnson

## Graduate Students
- Brian Bonsignore
- Theresa Brons
- Wenjing Li
- Phong Thanh Luu
- Tiago Macedo
- Nham Ngo
- Brandon Samples
- Andrew Talian
- Benjamin Wyser
- $G$ - simple, simply-connected algebraic group over $\overline{\mathbb{F}}_p$
- $G(\mathbb{F}_q)$ - finite subgroup of $\mathbb{F}_q$-rational points in $G$, $q = p^r$
- $B$ - Borel subgroup of $G$
- $U$ - unipotent radical of $B$
- $G_r$ - Frobenius kernel of $G$
- $L(\lambda)$ - irreducible $G$-module of highest weight $\lambda$
- $V(\lambda)$ - Weyl module of highest weight $\lambda$
- $H^0(\lambda) = \text{ind}^G_B(\lambda)$ - induced module

e.g.,
- $G = SL_n(\overline{\mathbb{F}}_p)$
- $G(\mathbb{F}_q) = SL_n(\mathbb{F}_q)$
- $B$ - lower triangular invertible matrices
- $U$ - lower triangular unipotent matrices
Problem

Compute

\[ H^1(G(\mathbb{F}_q), L(\lambda)) \quad \text{and} \quad H^2(G(\mathbb{F}_q), L(\lambda)) \]

for \( \lambda \) small, say, less than or equal to a fundamental dominant weight.

Cline, Parshall, Scott (1975, 1977), Jones (1975)

Let \( \lambda \) be a minimal nonzero dominant weight. Then

\[ \dim H^1(G(\mathbb{F}_q), L(\lambda)) \leq \begin{cases} 2 & \text{if } p = 2 \\ 1 & \text{if } p \neq 2 \end{cases} \]

Our goal: Compare \( H^i(G(\mathbb{F}_q), L(\lambda)) \) to \( H^i(G, L(\lambda)) \).
Commutative square of restriction maps:

\[
\begin{array}{ccc}
H^i(G, V) & \sim & H^i(B, V) \\
\downarrow & & \downarrow \\
H^i(G(F_q), V) & \sim & H^i(B(F_q), V).
\end{array}
\]

Cline, Parshall, Scott, van der Kallen (1977)

Let \( V \) be a finite-dimensional rational \( G \)-module, and let \( i \in \mathbb{N} \). Then for all sufficiently large \( e \) and \( q \), the restriction map is an isomorphism

\[
H^i(G, V^{(e)}) \sim H^i(G(F_q), V^{(e)}).
\]

Avoid twists and \( q \gg 0 \) by more direct appeal to the left column.
Consider the functor \( \text{ind}_G^{G(\mathbb{F}_q)}(-) \). There exists a short exact sequence

\[
0 \to k \to \text{ind}_G^{G(\mathbb{F}_q)}(k) \to N \to 0.
\]

Let \( M \) be a rational \( G \)-module. Then there exists a short exact sequence

\[
0 \to M \to \text{ind}_G^{G(\mathbb{F}_q)}(M) \to M \otimes N \to 0.
\]

Using \( \text{Ext}^n_G(k, \text{ind}_G^{G(\mathbb{F}_q)}(M)) \cong \text{Ext}^n_G(\mathbb{F}_q)(k, M) \), we get:

**Long exact sequence for restriction**

\[
0 \to \text{Hom}_G(k, M) \to \text{Hom}_G(\mathbb{F}_q)(k, M) \to \text{Hom}_G(k, M \otimes N) \\
\to \text{Ext}^1_G(k, M) \to \text{Ext}^1_G(\mathbb{F}_q)(k, M) \to \text{Ext}^1_G(k, M \otimes N) \\
\to \text{Ext}^2_G(k, M) \to \text{Ext}^2_G(\mathbb{F}_q)(k, M) \to \text{Ext}^2_G(k, M \otimes N) \\
\]
Restriction Isomorphism Theorem

Let $\lambda$ be less than or equal to a fundamental dominant weight. Suppose $p$ and $q$ are as below. Then $\text{Ext}^i_G(k, L(\lambda)) \cong \text{Ext}^i_{G(\mathbb{F}_q)}(k, L(\lambda))$ for $i \leq 2$.

<table>
<thead>
<tr>
<th>Type</th>
<th>Conditions on $p$ and $q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_n$</td>
<td>$p$ odd, $q &gt; 3$</td>
</tr>
<tr>
<td>$B_n$</td>
<td>$p &gt; 3$ ($q &gt; 5$ if $n \leq 3$)</td>
</tr>
<tr>
<td>$C_n$</td>
<td>$p &gt; 3$, $q &gt; 5$</td>
</tr>
<tr>
<td>$D_n$</td>
<td>$p$ odd, $q &gt; 3$</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$p &gt; 3$</td>
</tr>
<tr>
<td>$E_7$</td>
<td>$p &gt; 3$, $q &gt; 5$</td>
</tr>
<tr>
<td>$E_8$</td>
<td>$p &gt; 5$</td>
</tr>
<tr>
<td>$F_4$</td>
<td>$p &gt; 3$, $q &gt; 5$</td>
</tr>
<tr>
<td>$G_2$</td>
<td>$p &gt; 5$</td>
</tr>
</tbody>
</table>
Bendel, Nakano, Pillen (2010)

$\text{ind}^G_G(\mathbb{F}_q) (k)$ admits a filtration by $G$-submodules with sections of the form

$$H^0(\mu) \otimes H^0(\mu^*)(r) \quad \mu \in X(T)_+.$$ 

Corollary: $N = \text{coker}(k \rightarrow \text{ind}^G_G(\mathbb{F}_q) (k))$ admits such a filtration with $\mu \neq 0$.

Then $\text{Ext}^i_G(k, L(\lambda) \otimes N) = 0$ if it is zero for each section, i.e., if for $\mu \neq 0$,

$$\text{Ext}^i_G(k, L(\lambda) \otimes H^0(\mu) \otimes H^0(\mu^*)(r))$$

$$\cong \text{Ext}^i_G(V(\mu)(r), L(\lambda) \otimes H^0(\mu)) = 0.$$
Analyze the spectral sequences

\[ E_{2}^{i,j} = \text{Ext}_{G/Gr}^{i} (V(\mu)^{(r)}, \text{Ext}_{Gr}^{j} (k, L(\lambda) \otimes H^{0}(\mu))) \]

\[ \Rightarrow \text{Ext}_{G}^{i+j} (V(\mu)^{(r)}, L(\lambda) \otimes H^{0}(\mu)) \]

and \[ E_{2}^{i,j} = R^{i} \text{ind}_{B/Br}^{G/Gr} \text{Ext}_{Br}^{j} (k, L(\lambda) \otimes \mu) \Rightarrow \text{Ext}_{Gr}^{i+j} (k, L(\lambda) \otimes H^{0}(\mu)). \]

**Critical calculation**

Let \( \lambda \in X(T)_{+} \) be less than or equal to a fundamental dominant weight, and let \( p \) and \( q \) be as above. Then there exists \( I \subseteq \Delta \) such that

\[ \text{Ext}_{Ur}^{1} (k, L(\lambda)) \cong \bigoplus_{\alpha \in I} -s_{\alpha} \cdot \lambda^{*} \oplus \bigoplus_{\sigma \uparrow \lambda} (-\sigma)^{\oplus m_{\sigma}} \]

where \( m_{\sigma} = \dim \text{Ext}_{G}^{1} (L(\lambda^{*}), H^{0}(\sigma)) \).
First Cohomology Main Theorem

Let $\lambda \in X(T)_+$ be a fundamental dominant weight. Assume $q > 3$ and

$p > 2$ if $\Phi$ has type $A_n, D_n$;
$p > 3$ if $\Phi$ has type $B_n, C_n, E_6, E_7, F_4, G_2$;
$p > 5$ if $\Phi$ has type $E_8$.

Then $\dim H^1(G(\mathbb{F}_q), L(\lambda)) = \dim H^1(G, L(\lambda)) \leq 1$.

Space is one-dimensional in the following cases:

- $\Phi$ has type $E_7$, $p = 7$, and $\lambda = \omega_6$; and
- $\Phi$ has type $C_n$, $n \geq 3$, and $\lambda = \omega_j$ with $\frac{j}{2}$ a nonzero term in the $p$-adic expansion of $n + 1$, but not the last term in the expansion.

Reasons for vanishing: Linkage principle for $G$, $\text{Ext}^1_G(V(0), H^0(\lambda)) = 0$. 
Second Cohomology Main Theorem

Let $\lambda \in X(T)_+$ be less than or equal to a fundamental dominant weight. Let $p > 7$. Then $\operatorname{Ext}^2_{G(F_q)}(k, L(\lambda)) \cong \operatorname{Ext}^2_G(k, L(\lambda)) = 0$, except possibly in the cases

- $\Phi = E_8$, $p = 31$, and $\lambda \in \{\omega_6 + \omega_8, \omega_7 + \omega_8\}$
- $\Phi = C_n$, $n \geq 3$, and $\lambda = \omega_j$ with $j$ even
Adamovich described combinatorially the submodule structure of Weyl modules in Type C having fundamental highest weight. We use this and $\text{Ext}^2_{C_n}(k, L(\omega_j)) \cong \text{Ext}^1_{C_n}(\text{rad}_G V(\omega_j), k)$ to make computations.
Values of $n$ and $j$ for which $H^2(Sp_{2n}, L(\omega_j)) \neq 0$, $p = 3$.

In each case, $H^2$ is 1-dimensional.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$j$</th>
<th>$n$</th>
<th>$j$</th>
<th>$n$</th>
<th>$j$</th>
<th>$n$</th>
<th>$j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>6</td>
<td>15</td>
<td>6, 8</td>
<td>24</td>
<td>6, 8, 18</td>
<td>33</td>
<td>6, 8, 18</td>
</tr>
<tr>
<td>7</td>
<td>6</td>
<td>16</td>
<td>6, 10</td>
<td>25</td>
<td>6, 10, 18</td>
<td>34</td>
<td>6, 10, 18</td>
</tr>
<tr>
<td>8</td>
<td></td>
<td>17</td>
<td></td>
<td>26</td>
<td></td>
<td>35</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>6</td>
<td>18</td>
<td>6, 14</td>
<td>27</td>
<td>6, 14</td>
<td>36</td>
<td>6, 14</td>
</tr>
<tr>
<td>10</td>
<td>6</td>
<td>19</td>
<td>6, 16</td>
<td>28</td>
<td>6, 16</td>
<td>37</td>
<td>6, 16</td>
</tr>
<tr>
<td>11</td>
<td></td>
<td>20</td>
<td>18</td>
<td>29</td>
<td>18</td>
<td>38</td>
<td>18</td>
</tr>
<tr>
<td>12</td>
<td>6</td>
<td>21</td>
<td>6, 18</td>
<td>30</td>
<td>6, 18</td>
<td>39</td>
<td>6, 18, 20</td>
</tr>
<tr>
<td>13</td>
<td>6</td>
<td>22</td>
<td>6, 18</td>
<td>31</td>
<td>6, 18</td>
<td>40</td>
<td>6, 18, 22</td>
</tr>
<tr>
<td>14</td>
<td></td>
<td>23</td>
<td>18</td>
<td>32</td>
<td>18</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For $n = 12$, we have also $H^1(Sp_{2n}, L(\omega_6)) \neq 0$ (parity vanishing violated).
Values of $n$ and $j$ for which $H^2(Sp_{2n}, L(\omega_j)) \neq 0$: $p = 5$.

In each case, $H^2$ is 1-dimensional.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$j$</th>
<th>$n$</th>
<th>$j$</th>
<th>$n$</th>
<th>$j$</th>
<th>$n$</th>
<th>$j$</th>
<th>$n$</th>
<th>$j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>10</td>
<td>20</td>
<td>10</td>
<td>30</td>
<td>10</td>
<td>40</td>
<td>10, 22</td>
<td>50</td>
<td>10, 42</td>
</tr>
<tr>
<td>11</td>
<td>10</td>
<td>21</td>
<td>10</td>
<td>31</td>
<td>10</td>
<td>41</td>
<td>10, 24</td>
<td>51</td>
<td>10, 44</td>
</tr>
<tr>
<td>12</td>
<td>10</td>
<td>22</td>
<td>10</td>
<td>32</td>
<td>10</td>
<td>42</td>
<td>10, 26</td>
<td>52</td>
<td>10, 46</td>
</tr>
<tr>
<td>13</td>
<td>10</td>
<td>23</td>
<td>10</td>
<td>33</td>
<td>10</td>
<td>43</td>
<td>10, 28</td>
<td>53</td>
<td>10, 48</td>
</tr>
<tr>
<td>14</td>
<td>24</td>
<td>34</td>
<td></td>
<td>44</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>10</td>
<td>25</td>
<td>10</td>
<td>35</td>
<td>10, 12</td>
<td>45</td>
<td>10, 32</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>10</td>
<td>26</td>
<td>10</td>
<td>36</td>
<td>10, 14</td>
<td>46</td>
<td>10, 34</td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>10</td>
<td>27</td>
<td>10</td>
<td>37</td>
<td>10, 16</td>
<td>47</td>
<td>10, 36</td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>10</td>
<td>28</td>
<td>10</td>
<td>38</td>
<td>10, 18</td>
<td>48</td>
<td>10, 38</td>
<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>29</td>
<td>39</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For $n = 30$, we also have $H^1(Sp_{2n}, L(\omega_{10})) \neq 0$ (parity vanishing violated).