Support varieties for irreducible modules of small quantum groups

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Joint work with Daniel Nakano (UGA) and Brian Parshall (UVA).

• \( \mathfrak{g} \) finite-dimensional simple complex Lie algebra
• \( \Phi \) root system of \( \mathfrak{g} \), with highest short root \( \alpha_0 \)
• \( \rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha \) the Weyl weight
• \( h = (\rho, \alpha_0^\vee) + 1 \) the Coxeter number of \( \Phi \)
• \( \mathcal{W} \) the Weyl group of \( \Phi \)

• \( \ell \in \mathbb{N} \) odd integer with \( \ell > h \) and \( 3 \nmid \ell \) if \( \Phi \) is of type \( G_2 \)
• \( \zeta \in \mathbb{C} \) primitive \( \ell \)-th root of unity
• \( u_\zeta(\mathfrak{g}) \) small quantum group associated to \( \mathfrak{g} \), a finite-dimensional Hopf subalgebra of the Lusztig quantum group \( U_\zeta(\mathfrak{g}) \) with parameter \( \zeta \).

• \( \mathcal{W}_\ell = \mathcal{W} \ltimes \ell \mathbb{Z} \Phi \) affine Weyl group
• \( \mathcal{N} \) nullcone of \( \mathfrak{g} \), consisting of the nilpotent elements in \( \mathfrak{g} \)
Let $A$ be a Hopf algebra over an algebraically closed field $k$. Suppose $R = H^{2\bullet}(A, k)$ is finitely-generated as an algebra over $k$.

**Cohomological spectrum**

$$V_A(k) = \text{MaxSpec } H^{2\bullet}(A, k) \text{ (maximal ideal spectrum)}.$$ 

Let $M$ be a finite-dimensional $A$-module. Set $I_A(M) = \text{Ann}_R \text{Ext}^\bullet_A(M, M)$.

**Support variety of a module**

$$V_A(M) = \text{MaxSpec}(H^{2\bullet}(A, \mathbb{C})/I_A(M)), \text{ closed subvariety of } V_A(k).$$

The cases $A = kG$, the group ring of a finite group $G$, and $A = u(g)$, the restricted enveloping algebra of a $p$-restricted Lie algebra $g$, have been of interest since at least the early 1980s.
Ginzburg–Kumar (1993)

\[ H^2(\mathfrak{u}\zeta(\mathfrak{g}), \mathbb{C}) \cong \mathbb{C}[\mathcal{N}], \text{ hence } V_{\mathfrak{u}\zeta(\mathfrak{g})}(\mathbb{C}) \cong \mathcal{N}. \]

General problem that few explicit examples of support varieties of known.
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For \( \lambda \in \mathfrak{X}^+ \), have \( H^0(\lambda) \) and \( V(\lambda) \) (induced and Weyl modules for \( U_\zeta(\mathfrak{g}) \)).

Set \( \Phi_\lambda = \{ \alpha \in \Phi : (\lambda + \rho, \alpha^\vee) \equiv 0 \text{ mod } \ell \} \).

There exists \( w \in \mathcal{W} \) and a subset of simple roots \( J \) such that \( w(\Phi_\lambda) = \Phi_J \).

Let \( \mathfrak{u}_J \) be the nilradical of the standard parabolic subalgebra \( \mathfrak{p}_J \subset \mathfrak{g} \).
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\[ H^2(\mathfrak{u}_\zeta(\mathfrak{g}), \mathbb{C}) \cong \mathbb{C}[\mathcal{N}], \text{ hence } \mathcal{V}_{\mathfrak{u}_\zeta(\mathfrak{g})}(\mathbb{C}) \cong \mathcal{N}. \]

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\[ \mathcal{V}_{\mathfrak{u}_\zeta(\mathfrak{g})}(H^0(\lambda)) = \mathcal{V}_{\mathfrak{u}_\zeta(\mathfrak{g})}(V(\lambda)) = G \cdot \mathfrak{u}_J, \text{ irreducible of dimension } |\Phi| - |\Phi_J| \]
Question
What is the support variety of each irreducible $u_\zeta(g)$-module $L(\lambda)$?

No previous calculation of the support varieties for all irreducible modules of a finite-dimensional Hopf algebra (except in cases where all $V_A(L)$ equal the full cohomological spectrum, i.e., the variety of the trivial module).
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$L(\lambda) = \text{soc}_{u_\zeta(\mathfrak{g})} H^0(\lambda)$, follows via induction that $V_{u_\zeta(\mathfrak{g})}(L(\lambda)) \subseteq G \cdot u_J$.

Theorem (D–Nakano–Parshall)
Suppose $w(\Phi_\lambda) = \Phi_J$ for some $w \in W$. Then $V_{u_\zeta(\mathfrak{g})}(L(\lambda)) = G \cdot u_J$. 
Let $M$ be a finite-dimensional $U_{\zeta}(g)$-module, with $M = \bigoplus_{\lambda \in \mathcal{X}} M_\lambda$.

**Generic dimension of a weight module**

\[
\dim_t M = \sum_{\lambda \in \mathcal{X}} (\dim M_\lambda) t^{-2 \text{wht}(\lambda)} \in \mathbb{Z}[t, t^{-1}]
\]

Here $\text{wht}(\lambda) = \frac{1}{2} \sum_{\alpha \in \Phi^+} d_\alpha(\lambda, \alpha^\vee) \in \mathbb{Z}[\frac{1}{2}]$, where $d_\alpha = (\alpha, \alpha)/(\alpha_0, \alpha_0)$.
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Suppose $\zeta$ is a root of multiplicity $s$ in $\dim_t M$. Then

$$\dim V_{U_\zeta(g)}(M) \geq |\Phi| - 2s.$$
Outline of the argument for the induced modules:

“Generic” Weyl Character Formula

\[ \dim_t H^0(\mu) = D_\lambda(t)/D_0(t), \text{ where} \]

\[ D_\lambda(t) = \prod_{\alpha \in \Phi^+} (t^{d_\alpha(\lambda+\rho,\alpha^\vee)} - t^{-d_\alpha(\lambda+\rho,\alpha^\vee)}). \]

Note that \( \zeta \) is a root of \( t^{d_\alpha(\lambda+\rho,\alpha^\vee)} - t^{-d_\alpha(\lambda+\rho,\alpha^\vee)} \) if and only if \( \alpha \in \Phi^+_\lambda \).
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Then \(\zeta\) is a root of \(\dim_t H^0(\lambda)\) with multiplicity \(|\Phi^+_\lambda| = |\Phi^+_J|\), hence

\[
\dim V_{u_\zeta(g)}(H^0(\lambda)) \geq |\Phi| - 2|\Phi^+_J| = |\Phi| - |\Phi_J| = \dim G \cdot u_J.
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But \(V_{u_\zeta(g)}(H^0(\lambda)) \subseteq G \cdot u_J\) from other techniques, so by dimension comparison and irreducibility of \(G \cdot u_J\), the varieties must be equal.
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To imitate this approach for the $L(\lambda)$, we need to know their characters.
Let $\lambda \in X^+$. Choose $\lambda^- \in \overline{C_Z}$ (alcove opposite to the lowest $\ell$-alcove) and $w \in W_\ell$ of minimal length such that $\lambda = w \cdot \lambda^-$. Then

$$\dim_t L(\lambda) = \sum_{y \in W_\ell} (-1)^{\ell(w) - \ell(y)} P_{y, w}(1) \cdot \dim_t H^0(y \cdot \lambda^-).$$

Let $W_{\ell, I}$ be the standard parabolic subgroup stabilizing $\lambda^-$, and let $W^I_\ell$ be the set of minimal length right coset representatives for $W_{\ell, I}$. Then

$$\dim_t L(\lambda) = \sum_{y \in W^I_\ell} (-1)^{\ell(w) - \ell(y)} P^{I, -1}_{y, w}(1) \cdot \dim_t H^0(y \cdot \lambda^-).$$
Recall \( D_\lambda(t) = \prod_{\alpha \in \Phi^+} (t^{d_\alpha(\lambda+\rho,\alpha^\vee)} - t^{-d_\alpha(\lambda+\rho,\alpha^\vee)}) \). Then

\[
f(t) = D_0(t) \cdot \dim_t L(\lambda) = \sum_{\substack{y \in W^l_y \\mid y \cdot \lambda^- \in X^+}} (-1)^{\ell(w) - \ell(y)} P_{y,w}^{l,-1}(1) \cdot D_{y \cdot \lambda^-}(t).
\]

Set \( s = |\Phi_j^+| \). Now \( \zeta \) is a root with multiplicity \( s \) in \( f(t) \) if \( f^{(s)}(\zeta) \neq 0 \).
Recall $D_\lambda(t) = \prod_{\alpha \in \Phi^+} (t^{d_\alpha(\lambda+\rho,\alpha^\vee)} - t^{-d_\alpha(\lambda+\rho,\alpha^\vee)})$. Then

$$f(t) = D_0(t) \cdot \dim_t L(\lambda) = \sum_{y \in W^I_{\ell}, y \cdot \lambda^- \in X^+} (-1)^{\ell(w)-\ell(y)} P_{y,w}^{l,-1}(1) \cdot D_{y \cdot \lambda^-}(t).$$

Set $s = |\Phi_j^+|$. Now $\zeta$ is a root with multiplicity $s$ in $f(t)$ if $f^{(s)}(\zeta) \neq 0$.

The derivative

$$f^{(s)}(\zeta) = z \cdot \left( \sum_{y \in W^I_{\ell}, y \cdot \lambda^- \in X^+} P_{y,w}^{l,-1}(1) \prod_{\alpha \in \Phi^+_{y \cdot \lambda^-}} 2d_\alpha(y \cdot \lambda^- + \rho, \alpha^\vee) \right)$$

for some explicitly describable nonzero element $z \in \mathbb{C}$. 
Recall \( D_\lambda(t) = \prod_{\alpha \in \Phi^+} (t^{\alpha_\lambda + \rho, \alpha} - t^{-\alpha_\lambda + \rho, \alpha}) \). Then

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f(t) = D_0(t) \cdot \dim_t L(\lambda) = \sum_{y \in W^I_\ell, y \cdot \lambda^- \in X^+} (-1)^{\ell(w) - \ell(y)} P_{y,w}^{y,-1}(1) \cdot D_{y \cdot \lambda^-}(t).
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\( P_{y,w}^{y,-1}(1) \in \mathbb{N} \cup \{0\} \), and \( P_{w,w}^{w,-1}(1) = 1 \). Follows that \( f^{(s)}(\zeta) \neq 0 \).
Summary:

- $V_{u_\zeta(\mathfrak{g})}(L(\lambda)) \subseteq G \cdot u_J$
- $\dim V_{u_\zeta(\mathfrak{g})}(L(\lambda)) \geq \dim G \cdot u_J$ from differentiating the generic LCF
- By irreducibility of $G \cdot u_J$, must have $V_{u_\zeta(\mathfrak{g})}(L(\lambda)) = G \cdot u_J$. 

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Support varieties for irreducible modules
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Let $G$ be a simple simply-connected algebraic group over an algebraically closed field $k$ of characteristic $p > h$. Assume that the Lusztig character formula holds for $G$ for all restricted dominant weights. Let $\lambda \in X^+$, and suppose $\Phi_\lambda \sim \Phi_J$ for some subset of simple roots $J$. Then

$$V_{u(g)}(L(\lambda)) = G \cdot u_J.$$ 

Equivalently, $V_{G_1}(L(\lambda)) = G \cdot u_J$.

Holds for groups of type $A_1$ if $p \geq 2$, $A_2$ if $p \geq 3$, $B_2$ if $p \geq 5$, $G_2$ if $p \geq 11$, $A_3$ if $p \geq 5$, $A_4$ if $p \in \{5, 7\}$, and $p \gg 0$ in general.
Suslin, Friedlander, Bendel (1997)

Suppose $G$ admits an embedding of exponential type $G \hookrightarrow GL_n$. Then

$$V_{G_r}(k) \cong C_r(N^r) = \{(x_0, \ldots, x_{r-1}) \in N^r : [x_i, x_j] = 0 \text{ for all } i, j\}.$$

Using SFB's rank variety characterization of $V_{G_r}(M)$, Sobaje has proved:

Sobaje (2011)

Suppose $G$ is a classical group, and that $p > hc$, where $c$ is as given below. Let $\lambda = \lambda_0 + p\lambda_1 + \cdots + p^s\lambda_s$ with $\lambda_i \in X_1(T)$. Then

$$V_{G_r}(L(\lambda)) = \{(x_0, \ldots, x_{r-1}) \in C_r(N^r) : x_i \in V_{G_1}(L(\lambda_i))\}.$$

c = \left(\frac{n+1}{2}\right)^2 \text{ for } A_n, \quad \frac{n(n+1)}{2} \text{ for } B_n, \quad \frac{n^2}{2} \text{ for } C_n, \text{ and } \frac{n(n-1)}{2} \text{ for } D_n.
Results for algebraic groups

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