Cohomology rings of infinitesimal unipotent algebraic and quantum groups

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Joint work with Daniel Nakano and Nham Ngo.

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Standard notation:

- \( k \) an algebraically closed field of characteristic \( p \),
- \( G \) a simple, simply-connected algebraic group (e.g., \( G = SL_n(\overline{\mathbb{F}_p}) \)),
- \( T \subset G \) a maximal torus,
- \( \Phi \) the root system of \( T \) in \( G \),
- \( h \) the Coxeter number of \( \Phi \),
- \( B \subset G \) a Borel subgroup corresponding to \( \Phi^+ \),
- \( U \subset B \) the unipotent radical of \( B \),
- \( U_1 \subset B_1 \) the first Frobenius kernels of \( U \) and \( B \).
- \( n = \text{Lie}(U) \), the nilradical of \( b = \text{Lie}(B) \).
Problem

What is the ring structure of the cohomology ring $H^\bullet(U_1, k)$?

Equivalently: What is the ring structure of $H^\bullet(u(n), k)$?

$u(n)$ = restricted enveloping algebra of $n$. 
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Equivalently: What is the ring structure of $H^\bullet(u(n), k)$?

$u(n) =$ restricted enveloping algebra of $n$. 
Theorem (Friedlander–Parshall, 1986)

Suppose \( p > h \). Then there exists a filtration on \( H^\bullet(U_1, k) \) such that

\[
gr H^\bullet(U_1, k) \cong S^\bullet(n^*)^{(1)} \otimes H^\bullet(n, k).
\]

\( H^\bullet(n, k) \) = ordinary Lie algebra cohomology.
\( S^\bullet(n^*)^{(1)} \) = polynomial ring generated in degree two.
Filtration is by polynomials of higher degree.
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In $H^\bullet(U_1, k)$:

$$(a \otimes b)(c \otimes d) = (ac \otimes bd) + \text{ terms with higher degree polynomial part.}$$
Main tools used to prove the ring isomorphism:
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1. F-P Spectral sequence:

\[ E_2^{2i,j} = S^i(n^*)^{(1)} \otimes H^j(n, k) \Rightarrow H^{2i+j}(U_1, k). \]
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1. F-P Spectral sequence:

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2. Explicit computation of \( H^\bullet(n, k) \) as a \( B \)-module: If \( p > h \), then

\[ H^\bullet(n, k) = \bigoplus_{w \in W} w \cdot 0. \quad \text{(Kostant’s Theorem)} \]
Problem

*Can we “ungrade” the ring isomorphism, i.e., is the vector space isomorphism* $\mathbb{H}^\bullet(U_1, k) \cong S^\bullet(n^*)(1) \otimes \mathbb{H}^\bullet(n, k)$ *also a ring isomorphism?*
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$H^\bullet(B_1, k) = H^\bullet(U_1, k)^{T_1} \cong S^\bullet(n^*)(1)$ is already a polynomial subalgebra.


Suppose $G = SL_n$ and $p > h = n + 1$. Then, as a ring,

$H^\bullet(U_1, k) \cong S^\bullet(n^*)(1) \otimes H^\bullet(n, k)$. 
Theorem (DNN)

Suppose $p > 2(h - 1)$. Then, as a ring,

$$H^\bullet(U_1, k) \cong S^\bullet(n^*)^{(1)} \otimes H^\bullet(n, k).$$
Proof.

Look at the weight of $x_1 x_2$, for $x_i \in H^\bullet(U_1, k)_{w_i \cdot 0} \cong H^\bullet(n, k)_{w_i \cdot 0}$. 

Either $x_1 x_2 = 0$ in $H^\bullet(U_1, k)$, or $x_1 x_2$ has $T$-weight $w_1 \cdot 0 + w_2 \cdot 0 = w_3 \cdot 0 + \sigma$ for some $w_3 \in W$ and $\sigma \in N_{\Phi}$. 

If $\sigma = 0$, then $x_1 x_2 \in H^\bullet(U_1, k)_{w_3 \cdot 0} \subset H^\bullet(n, k)_{w_i \cdot 0}$. 

Suppose $\sigma \neq 0$. Then for some $y, y', w'_1, w'_2 \in W$, $y(w'_1 \cdot 0) + y'(w'_2 \cdot 0) = p \tilde{\sigma} \in X(T) + \cap Z_{\Phi}$. 

(1) Now $p \tilde{\sigma} \leq 2 \rho + 2 \rho$. 

Then $2p \leq p(\tilde{\sigma}, \alpha \vee 0) \leq 4(\rho, \alpha \vee 0) = 4(h - 1)$. 

So $p > 2(h - 1)$ implies $\sigma = 0$. 

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Proof.

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- Suppose $\sigma \neq 0$. Then for some $y, y', w'_1, w'_2 \in W$,

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Christopher Drupieski (UGA)  
Cohomology rings  
April 11, 2010  9 / 16
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\[\square\]
Problem

What happens for $h < p < 2(h - 1)$?
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Example (Type $B_2$, $p = 5$)

Let $\alpha, \beta$ be simple with $\alpha$ long. Note that $h = 4 < p < 6 = 2(h - 1)$.

$$s_\beta s_\alpha \cdot 0 + s_\beta s_\alpha \cdot 0 = s_\alpha s_\beta \cdot 0 + 5(-\beta)$$

Corresponds to squaring an element in $H^2(n, \mathbb{F}_5)$ of weight $s_\beta s_\alpha \cdot 0$. 
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Corresponds to squaring an element in \( H^2(n, \mathbb{F}_5) \) of weight \( s_\beta s_\alpha \cdot 0 \).

Though all elements of \( H^\bullet(n, \mathbb{F}_5) \) square to zero, we have verified using MAGMA that this vector does NOT square to zero in \( H^\bullet(U_1, \mathbb{F}_5) \). So the ring isomorphism need not hold for \( h < p < 2(h - 1) \).
Can we generalize the cohomology ring calculation to quantum groups (i.e., quantized enveloping algebras) at a root of unity?
Problem
Can we generalize the cohomology ring calculation to quantum groups (i.e., quantized enveloping algebras) at a root of unity?

Can we also generalize to quantum groups another calculation of F-P:

Theorem (Friedlander–Parshall, 1986)
Suppose \( \lambda \in C_\mathbb{Z} \). Then, as a graded \( T \)-module and as a \( H^\bullet(B_1, k) \)-module,

\[
H^\bullet(U_1, L(\lambda)) \cong S^\bullet(n^*)^{(1)} \otimes H^\bullet(u, L(\lambda)).
\]
Let $q$ be an indeterminate. Set $\mathfrak{g} = \text{Lie}(G)$.

**Definition**

The quantized enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ is a $\mathbb{C}(q)$-algebra defined by generators and relations similar to those defining the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. 
Let $\zeta \in \mathbb{C}$ be a primitive $\ell$-th root of unity, $A = \mathbb{Z}[q, q^{-1}]$.

<table>
<thead>
<tr>
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Christopher Drupieski (UGA)  
Cohomology rings  
April 11, 2010  
13 / 16
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- If $p > h$, $u(n) \cong \mathcal{U}(n) // Z$, where $Z$ is generated by $\{ E^{p}_{\alpha} : \alpha \in \Phi^+ \}$. 
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- If $p > h$, $u(n) \cong \mathcal{U}(n) / \mathbb{Z}$, where $\mathbb{Z}$ is generated by $\{E^p_\alpha : \alpha \in \Phi^+\}$.
- $u_\zeta(n) \cong \mathcal{U}_\zeta(n) / \mathbb{Z}$, where $\mathbb{Z}$ is generated by $\{E^\ell_\alpha : \alpha \in \Phi^+\}$.
We do have the LHS spectral sequence:

\[ E^{i, j} = H^i(u_\lambda(n), H^j(Z, L_\lambda(\lambda))) \Rightarrow H^{i+j}(U_\lambda(n), L_\lambda(\lambda)) \]

Have a computation for the target:

Theorem (UGA VIGRE Algebra Group, 2008)

Suppose \( \ell > h \) and \( \lambda \in C_{\mathbb{Z}} \). Then

\[ H^\bullet(U_\lambda(n), L_\lambda(\lambda)) \sim = \bigoplus_{w \in W} w \cdot \lambda. \]

The Borel subalgebras \( u_\lambda(b) \sim = u_0 \otimes u_\lambda(n) \) and \( U_\lambda(b) \sim = u_0 \otimes U_\lambda(n) \) are Hopf algebras, as is \( Z \subset U_\lambda(b) \).
No analogue of the F-P spectral sequence for quantum groups. We do have the LHS spectral sequence:

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*Suppose \( \ell > h \) and \( \lambda \in C_Z \). Then \( H^\bullet(U^\zeta(n), L^\zeta(\lambda)) \cong \bigoplus_{w \in W} w \cdot \lambda. \)

\( U^\zeta(n) \) and \( u^\zeta(n) \) are not Hopf algebras, so we don’t automatically have nice product structures on the LHS spectral sequence.
1. No analogue of the F-P spectral sequence for quantum groups. We do have the LHS spectral sequence:

\[ E_2^{i,j}(L^\lambda) = H^i(u^\lambda(n), H^j(Z, L^\lambda)) \Rightarrow H^{i+j}(U^\lambda(n), L^\lambda) \]

2. Have a computation for the target:

**Theorem (UGA VIGRE Algebra Group, 2008)**

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3. \( U^\lambda(n) \) and \( u^\lambda(n) \) are not Hopf algebras, so we don’t automatically have nice product structures on the LHS spectral sequence.

4. The Borel subalgebras \( u^\lambda(b) \cong u^0_\lambda \otimes u^\lambda(n) \) and \( U^\lambda(b) \cong u^0_\lambda \otimes U^\lambda(n) \) are Hopf algebras, as is \( Z \subset U^\lambda(b) \).
Work one $u_\zeta^0$-weight space at a time.

$$\bigoplus_{\mu \in X} H^\bullet(u_\zeta(n), L_\zeta^{\lambda}(\mu))_{w \cdot \lambda + \ell \mu} \cong H^\bullet(u_\zeta(b), L_\zeta^{\lambda} \otimes -w \cdot \lambda).$$

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Work one $u^0_\zeta$-weight space at a time.

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LHS Spectral sequence for the Borel subalgebras:

$$E_2^{i,j} = H^i(u^\zeta(b), H^j(Z, L^\zeta(\lambda) \otimes -w \cdot \lambda) \Rightarrow H^{i+j}(\mathcal{U}_\zeta(b), L^\zeta(\lambda) \otimes -w \cdot \lambda).$$

This LHS spectral sequence is compatible with cup products.
Theorem (DNN)

Suppose \( \ell \) is odd, coprime to 3 if \( \Phi \) has type \( G_2 \), and \( \ell > 2(h - 1) \). Then there exists a ring isomorphism

\[
H^\bullet(u_\zeta(n), \mathbb{C}) \cong H^\bullet(u_\zeta(b), \mathbb{C}) \otimes H^\bullet(U_\zeta(n), \mathbb{C}) \\
\cong S^\bullet(n^*)(1) \otimes H^\bullet(U_\zeta(n), \mathbb{C}).
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Theorem (DNN)

Suppose \( \ell \) is odd, coprime to 3 if \( \Phi \) has type \( G_2 \). Suppose \( \lambda \in \mathbb{C}^\mathbb{Z} \). Then

\[
H^\bullet(u_\zeta(n), L_\zeta(\lambda)) \cong H^\bullet(u_\zeta(b), \mathbb{C}) \otimes H^\bullet(U_\zeta(n), L^\zeta(\lambda)) \\
\cong S^\bullet(n^*)^{(1)} \otimes H^\bullet(U_\zeta(n), L^\zeta(\lambda))
\]

as a weight module and as a module for \( H^\bullet(u_\zeta(b), \mathbb{C}) \).