PHY 171
Homework 3 solutions
(Due by beginning of class on Wednesday, January 18, 2012)

1. The equation of a transverse wave traveling along a very long string is given by

$$D(x,t) = (6.00 \text{ cm}) \sin\Big[(0.020\,\pi \text{ rad/cm})\,x + (4.00\,\pi \text{ rad/s})\,t\Big]$$

Solution: The equation of a traveling wave is

$$D(x,t) = D_M \sin(kx \pm \omega t)$$

where D_M is the amplitude, the wave number $k = 2\pi/\lambda$, and the angular frequency $\omega = 2\pi f$.

(a) Write down the amplitude of the wave (no explanation required).

Solution: By inspection of the given equation, the amplitude of the wave is **6.00 cm**.

(b) Calculate the wavelength of the wave. Show your calculation/explanation clearly.

Solution: By inspection of the given equation, $k = 0.020\,\pi$ rad/cm, so

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{0.020\pi} = \mathbf{100\ cm} \;\equiv \mathbf{1.00\ m}$$

(c) Calculate the frequency of the wave. Show your calculation/explanation clearly.

Solution: By inspection of the given equation, $\omega = 4.00\,\pi$ rad/s, so

$$f = \frac{\omega}{2\pi} = \frac{4.00\pi}{2\pi} = \mathbf{2.00\ Hz}$$

(d) Calculate the speed of the wave. Show your calculation/explanation clearly.

Solution: Although one could just multiply the answers in parts (b) and (c) to get the speed, I'm going to do this based on the expressions in order to illustrate the general method.

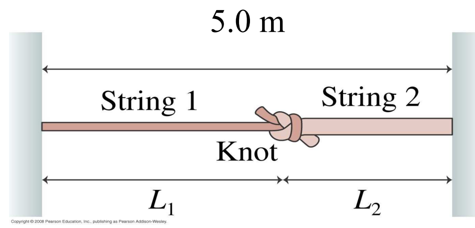
$$v = f\lambda = \left(\frac{\omega}{2\pi}\right)\left(\frac{2\pi}{k}\right) = \frac{\omega}{k} = \frac{4.00\pi}{0.020\pi} = \mathbf{200\ cm/s} \;\equiv \mathbf{2.00\ m/s}$$

(e) What is the direction of propagation of the wave? Write a sentence by way of explanation.

Solution: The direction of propagation of the way can be found simply by looking at the sign between the x and t terms in the phase (i.e., the argument of the `sine` function). A (−) sign indicates propagation along the positive x-direction. In the given equation, we have a (+) sign, so the given wave must be traveling in the **negative x-direction**.

2. Consider the the two strings connected by a knot as shown in the figure below.

String 1 has a linear density of 2.50 grams/m and string 2 has a linear density of 4.25 grams/m. A student sends pulses in both directions by quickly pulling up on the knot and then releasing it. What should the string lengths L_1 and L_2 be if the pulses are to reach the ends of the strings simultaneously?



Solution:

If the pulses are to reach the ends simultaneously, they must each take the same time to travel from the knot to the other end of their respective strings.

The time taken to travel from the knot to the end of a string can be found easily by dividing the length of the string by the speed of the pulse on the string. In other words, since the speed = distance/time, the time is just the distance/speed.

Meanwhile, the speed of a pulse on a string is just the square root of the tension in the string (F_T) divided by its mass per unit length μ (also known as the linear density).

Therefore, the time taken to travel from the knot to the end of the string of length L_1 is simply

$$t_1 = \frac{L_1}{\sqrt{F_T/\mu_1}} = \frac{L_1}{\sqrt{F_T/2.50 \times 10^{-3}}} \tag{H3.2.1}$$

Likewise, the time taken to travel from the knot to the end of the string of length L_2 is

$$t_2 = \frac{L_2}{\sqrt{F_T/\mu_2}} = \frac{L_2}{\sqrt{F_T/4.25 \times 10^{-3}}} \tag{H3.2.2}$$

Notice that I've written F_T in both expressions because both strings must be under the same tension, given the way they are connected. Also, I've converted the linear densities to kg/m to have everything in SI units (although we don't really need to do that in this problem because we are going to put t_1 and t_2 equal to each other).

Since $t_1 = t_2$ as stated in the problem, we get by equation (H3.2.1) and (H3.2.2) that

$$L_1\sqrt{2.50} = L_2\sqrt{4.25}, \text{ from which we obtain } 1.581\ L_1 = 2.062\ L_2$$

But, the two string lengths together add to 5.0 m, so $L_1 + L_2 = 5.0$, so this becomes

$$1.581\ L_1 = 2.062\,(5 - L_1)$$

After multiplying and rearranging terms, this becomes

$$\Big(1.581 + 2.062\Big)\ L_1 = 2.062\Big(5\Big) \qquad \text{so that} \qquad L_1 = \frac{2.062\Big(5\Big)}{1.581 + 2.062}$$

Therefore, we get $\boldsymbol{L_1 = 2.8}$ **m**, and $L_2 = 5 - L_1$, so $\boldsymbol{L_2 = 2.2}$ **m**.

3. When mass M is tied to the bottom of a long, thin wire suspended from the ceiling, and standing waves are set up in the wire, the second harmonic frequency of the wire is 200 Hz. Adding an additional 1.0 kg to the hanging mass (i.e., to M) increases the second harmonic frequency to 245 Hz.

(a) Find the value of M?

Solution: The second harmonic frequency of standing waves on a string is given by

$$f_2 = 2f_1 = 2\left(\frac{v}{2L}\right)$$

where L is the length of the string, and v is the speed of waves on the string, equal to the square root of the tension (F_T) divided by the mass per unit length (μ) of the string (also called the linear density).

The tension in the wire is simply the weight attached to its end, equal to Mg in the first case. Therefore, we can write

$$200 \text{ Hz} = \cancel{2}\left(\frac{v}{\cancel{2}L}\right) = \frac{1}{L}\sqrt{\frac{F_T}{\mu}} = \frac{1}{L}\sqrt{\frac{Mg}{\mu}}$$

Next, after adding 1.0 kg to the mass M, this becomes

$$245 \text{ Hz} = \frac{1}{L}\sqrt{\frac{(M+1)g}{\mu}}$$

After this, it's just algebra to solve for M. Each of you will perhaps have your own way of doing this based on how you learned to do algebra, but here is a fast way to do it — just divide the second equation by the first:

$$\frac{245}{200} = \frac{(1/L)\sqrt{(M+1)g/\mu}}{(1/L)\sqrt{Mg/\mu}} = \sqrt{\frac{M+1}{M}}$$

Squaring both sides and cross multiplying gives

$$\left(245\right)^2 M = \left(200\right)^2 (M+1)$$

Rearranging terms

$$(245)^2 M - (200)^2 M = (200)^2$$

so that

$$M = \frac{(200)^2}{(245)^2 - (200)^2} = \frac{40,000}{60,025 - 40,000} = \frac{40,000}{20,025} = 1.9975 = \mathbf{2.0\ kg}$$

(b) In order to be able to solve for M, you would have made certain assumptions in this problem. State the assumptions.

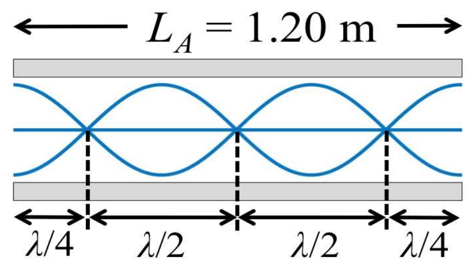
Solution: Clearly, we have considered the wire to be like a string in this problem, and applied all the considerations for standing waves on a string to standing waves in a long, thin wire. A more important assumption, though, is that we have considered the wire to be reasonably inelastic, so that it maintains the same length L after the 1 kg mass is added to M.

4. Pipe A, which is 1.20 m long and open at both ends, oscillates at its third harmonic frequency. Pipe B, which is closed at one end, oscillates at its second harmonic frequency. These frequencies happen to match (i.e., the third harmonic frequency of pipe A is the same as the second harmonic frequency of pipe B). As usual, both pipes have air inside, in which the speed of sound is 343 m/s.

(a) If an x-axis extends along the interior of pipe A, with $x = 0$ at one end, where along the axis are the displacement nodes?

Solution: Consider the figure of pipe A with its standing wave pattern shown below.

The third harmonic will have 3 nodes as shown in the figure, and the two antinodes at each end of the tube. The distance between two consecutive nodes is $\lambda/2$, whereas the distance from an antinode to its nearest node is $\lambda/4$ (all of this is shown in the figure).



Adding up the lengths shown in the figure, we get

$$\frac{\lambda}{4} + \frac{\lambda}{2} + \frac{\lambda}{2} + \frac{\lambda}{4} = 1.20 \text{ m}, \qquad \text{which gives} \quad \frac{3\lambda}{2} = 1.20 \text{ m}, \qquad \text{so that} \quad \lambda = 0.8 \text{ m}$$

If we set the zero of the x-axis at the left end of the tube, the figure above clearly shows that the nodes will be at the positions:

$$\frac{\lambda}{4} = \frac{0.8 \text{ m}}{4} = 0.2 \text{ m}, \qquad \frac{\lambda}{4}+\frac{\lambda}{2} = \frac{0.8}{4}+\frac{0.8}{2} = 0.6 \text{ m}, \qquad \frac{\lambda}{4}+\frac{\lambda}{2}+\frac{\lambda}{2} = \frac{0.8}{4}+\frac{0.8}{2}+\frac{0.8}{2} = 1.0 \text{ m}$$

that is, the nodes will be at **0.2, 0.6** and **1.0 m**, respectively.

(b) What is the length of pipe B?

Solution: Since pipe A is open at both ends, its 3^{rd} harmonic frequency is $f_{3A} = 3\left(\frac{v}{2\,L_A}\right)$, where $L_A = 1.20$ m is the length of pipe A, and $v = 343$ m/s is the speed of sound.

Meanwhile, since pipe B is ***closed at one end*** (open-closed tube), we know that it will ***support only odd harmonics***, which means its 2^{nd} harmonic frequency is $f_{3B} = 3\left(\frac{v}{4\,L_B}\right)$, where L_B is the length of pipe B.

We are told these frequencies match, $f_{3A} = f_{3B}$, so $3v/2L_A = 3v/4L_B$,
which gives $4L_B = 2L_A$, so that the length of pipe B is $L_B = L_A/2 = 1.20/2 =$ **0.6 m**.

(c) What is the lowest harmonic frequency (i.e., the fundamental frequency) of pipe A?

Solution: For pipe A, $f_1 = \frac{v}{2L_A} = \frac{343 \text{ m/s}}{2\,(1.20 \text{ m})} =$ **143 Hz**.

5. A 25 cm long wire with a linear density of 20 gram/m passes across the open end of an 85 cm long tube of air that is closed at the other end (i.e., an open-closed tube, in the language of your text). If the wire, which is fixed at both ends, vibrates at its fundamental frequency, the sound wave it generates excites the second vibrational mode of the tube of air. Find the tension in the wire. Assume that the speed of sound is 343 m/s.

   Solution: We are given the following quantities:

$$\begin{aligned} \text{Length of wire, } L_w &= 25 \text{ cm} = 0.25 \text{ m} \\ \text{Length of open-closed tube, } L_t &= 85 \text{ cm} = 0.85 \text{ m} \\ \text{Linear density of wire, } \mu &= 20 \text{ g/m} = 20 \times 10^{-3} \text{ kg/m} \\ \text{Speed of sound (for tube), } v &= 343 \text{ m/s} \end{aligned}$$

   Let us first find the second harmonic frequency in the tube, since we have all the quantities we need to do so. Once again, it is important to realize that since we have a tube closed at one end (open-closed tube), its second vibrational mode will actually be f_3, since an open-closed tube ***supports only the odd harmonics***.

   Therefore, for the open-closed tube, we have the 2nd vibrational mode

$$f_3 = 3\left(\frac{v}{4L_t}\right) = 3\left[\frac{343 \text{ m/s}}{4(0.85 \text{ m})}\right] = 302.647 \text{ Hz}$$

   where I have kept more digits than significant for the time being in order to avoid rounding off errors.

   This frequency is matched to the fundamental frequency of the wire given by $f_1 = v_w/2L_w$, where I have written v_w for the speed of waves on the wire.

   Now, since the speed of waves v_w in a wire (string) is given by the square root of the tension (F_T) in the wire divided by its linear density (μ), we can write

$$\frac{1}{2L_w}\sqrt{\frac{F_T}{\mu}} = 302.647$$

   which gives, upon squaring and cross multiplying:

$$F_T = \mu\Big[2L_w\Big(302.647\Big)\Big]^2 = \Big(20 \times 10^{-3} \text{ kg/m}\Big)\ \Big[2\Big(0.25 \text{ m}\Big)\Big(302.647\Big)\Big]^2 = \mathbf{458\ N}$$

Submit neat work, with answers or solutions clearly marked by the question number. Unstapled, untidy work will be charged a handling fee of 20% penalty. Writing only an answer without showing the steps used to get to that answer will fetch very few points, even if the answer is correct. Late homework will not be accepted.