PHY 475
Homework 2
(Due by beginning of class on Wednesday, April 18, 2012)

Submit neat work, with answers or solutions clearly marked by the question number. Unstapled, untidy work will be charged a handling fee of 20% penalty. Writing only an answer without showing the steps used to get to that answer will fetch very few points, even if the answer is correct. Late homework will not be accepted.

1. Suppose you are a two-dimensional being, living on the surface of a sphere with radius R. An object of width $ds \ll R$ is at a distance r from you (remember, all distances are measured on the surface of the sphere).

(a) What angular width $d\theta$ will you measure for the object?

(b) Examine and explain the behavior of $d\theta$ as $r \to \pi R$.

2. The Robertson-Walker (*FLRW*) metric may be written in the form

$$ds^2 = -c^2dt^2 + a(t)^2\left[dr^2 + S_\kappa(r)^2 d\Omega^2\right]$$

where $S_\kappa(r) =$

$$R_0 \sin\left(\frac{r}{R_0}\right), \text{ for } \kappa = +1; \qquad r, \text{ for } \kappa = 0; \qquad R_0 \sinh\left(\frac{r}{R_0}\right), \text{ for } \kappa = -1$$

(a) By inspection of the above equation, write down all the 16 elements (in matrix form) of $g_{\mu\nu}$ in the coordinate space (ct, r, θ, ϕ).

(b) By putting $x \equiv S_\kappa(r)$, show explicitly (i.e., for all 3 cases) that the Robertson-Walker metric takes the form

$$ds^2 = -c^2dt^2 + a(t)^2\left[\frac{dx^2}{1 - \kappa\, x^2/R_0^2} + x^2\, d\Omega^2\right]$$

You need to work only with the spatial part to demonstrate this.

3. The principle of wave-particle duality tells us that a particle with momentum p has an associated de Broglie wavelength of $\lambda = h/p$; this wavelength increases as $\lambda \propto a$, as the universe expands. The total energy density of a gas of particles can be written as $\varepsilon = nE_p$, where n is the number density of particles, and E_p is the energy per particle. For simplicity, let us assume that all the gas particles have the same mass m and momentum p. The energy per particle is then simply

$$E_p = \sqrt{(m^2c^4 + p^2c^2)}$$

(a) Compute the equation-of-state parameter w for this gas as a function of the scale factor a.

(b) Use your result in part (a) to show that $w = 1/3$ in the highly relativistic limit ($p \to \infty$).

(c) Use your result in part (a) to show that $w = 0$ in the highly non-relativistic limit ($p \to 0$).