PHY 475
Homework 5 solutions
(Due by noon on Thursday, May 31, 2012)

1. Answer the following questions.

(a) Show that the Doppler effect implies that an observer moving with a nonrelativistic velocity $\vec{v}$ through an isotropic CMB would see a temperature dipole anisotropy of

$$\frac{\delta T}{T}(\theta) = \left(\frac{v}{c}\right) \cos\theta$$

where θ is the angle from the direction of motion.

Solution:

The non-relativistic Doppler effect is given by

$$f' = f \left[1 - \frac{\hat{n} \cdot \vec{v}}{c}\right]$$

where $\vec{v}$ is the relative velocity between source and observer, and $\hat{n}$ is a unit vector along the direction of motion.

Since θ is the angle from the direction of motion, we have $\hat{n} \cdot \vec{v} = v \cos\theta$, so that the above equation becomes

$$f' = f \left[1 - \left(\frac{v}{c}\right) \cos\theta\right]$$

Now, since temperature T scales as the inverse of the scale factor a, and $a \propto \lambda$, whereas $\lambda \propto 1/f$, we get $\delta T/T = \delta f/f$.

Therefore,

$$\frac{\delta T}{T} = \frac{\delta f}{f} = \frac{f - f'}{f} = \left(\frac{v}{c}\right) \cos\theta \qquad \longrightarrow \qquad \text{proved}$$

(b) Smarty P. Antz tells you that recombination took place when the mean energy per CMB photon fell below the ionization energy of hydrogen (13.6 eV). A quick calculation shows you that this would correspond to a temperature of 60,000 K. Yet we know that recombination took place long afterward, when the Universe had cooled down to 3800 K. Why was this the case (i.e., what is wrong with Smarty P. Antz's reasoning)?

Solution: Even though the mean energy per CMB photon is below the ionization energy, there are still a large number of photons in the high energy tail of the distribution, more so because there are almost a billion photons for every baryon. This high energy tail of photons continues to break up any hydrogen atoms that form, until the Universe has cooled to a much lower temperature.

2. When the Universe was fully ionized, photons interacted primarily with electrons via Thomson scattering, for which the cross-section is $\sigma_e = 6.65 \times 10^{-29}$ m^2.

(a) Find the average distance traveled by a photon between collisions, also known as the mean free path: $\lambda_{\text{mfp}} = 1/n_e\sigma_e$, at the time of radiation-matter equality when $a_{\text{rm}} \approx 3 \times 10^{-4}$.

Solution:

When fully ionized, $n_e \approx n_{\text{bary}}$. But $n_{\text{bary}} \propto 1/a^3$, so we get $n_{\text{bary}} = n_{\text{bary}}/a^3$, assuming $a_0 = 1$.

Therefore, with $a = a_{\text{rm}}$ at the time of radiation-matter equality, we get

$$\lambda_{\text{mfp}} = \frac{1}{\left(n_{\text{bary}}/a_{\text{rm}}^3\right)\sigma_e} = \frac{a_{\text{rm}}^3}{n_{\text{bary}}\,\sigma_e} = \frac{(3 \times 10^{-4})^3}{(0.22 \text{ m}^{-3})(6.65 \times 10^{-29} \text{ m}^2)} = \boxed{\mathbf{1.8 \times 10^{18} \text{ m}}}$$

(b) The Friedmann equation for a radiation-dominated flat universe is

$$\frac{H^2}{H_0^2} = \frac{\Omega_{r,0}}{a^4}$$

Use this to find the Hubble parameter at the time of radiation-matter equality when $a_{\text{rm}} \approx 3 \times 10^{-4}$. Take the value of $\Omega_{r,0}$ from the Benchmark model.

Solution:

$$H_{\text{rm}} = \frac{H_0\sqrt{\Omega_{r,0}}}{a_{\text{rm}}^2} = \frac{(70 \text{ km s}^{-1} \text{ Mpc}^{-1})\sqrt{8.4 \times 10^{-5}}}{(3 \times 10^{-4})^2} = \boxed{\mathbf{71 \times 10^5 \text{ km s}^{-1} \text{ Mpc}^{-1}}}$$

(c) The photons remain coupled to the electrons as long as their mean free path λ_{mfp} is shorter than the Hubble distance c/H. Verify that they are coupled at the time of radiation-matter equality.

Solution:

Converting H_{rm} to SI units, we find $c/H_{\text{rm}} = 1.3 \times 10^{21}$ m.

Therefore, clearly

$$\lambda_{\text{rm}}\Big(= 1.8 \times 10^{18} \text{ m}\Big) < \frac{c}{H_{\text{rm}}}\Big(= 1.3 \times 10^{21} \text{ m}\Big)$$

and so the photons are coupled to the electrons at the time of radiation-matter equality.

3. Given that the Universe is described by the Benchmark model, and the redshift of the last scattering surface is $z_{\rm ls} = 1100$,

(a) Find the angular diameter distance to the surface of last scattering.

Solution:

From equation (6.42) in Lecture 11, we know that the horizon distance in the Benchmark model is given by

$$d_{\rm hor}(t_0) = 3.24\frac{c}{H_0} = 14,000 \text{ Mpc}$$

And, from equation (7.41), we know that as $z \to \infty$, the angular distance

$$d_A \simeq \frac{d_{\rm hor}(t_0)}{z}$$

Therefore, the angular distance to the surface of last scattering will be

$$d_A \simeq \frac{d_{\rm hor}(t_0)}{z_{\rm ls}} = \frac{14,000 \text{ Mpc}}{1100} = \boxed{\textbf{13 Mpc}}$$

(b) Find the luminosity distance to the surface of last scattering.

Solution:

From equation (7.37) in Lecture 12, we get

$$d_L = d_A\Big(1+z\Big)^2 = 13 \text{ Mpc}\Big(1+1100\Big)^2 = \boxed{\mathbf{1.6 \times 10^7} \textbf{ Mpc}}$$

(c) Find the proper distance $d_p(t_0)$ to the surface of last scattering.

Solution:

From equation (3.28) in Lecture 5, we get

$$d_p(t_0) = r$$

Meanwhile from equation (7.36) in Lecture 12, we get

$$d_A = \frac{S_\kappa(r)}{1+z} = \frac{r}{1+z}$$

because $S_\kappa(r) = r$ in a flat universe, which is the case here since we're assuming the Benchmark model.

Combining the two equations above, we get

$$d_p(t_0) = d_A\Big(1+z\Big) = 13 \text{ Mpc}\Big(1+1100\Big) \approx \boxed{\textbf{14,000 Mpc}}$$

4. Big Bang Nucleosynthesis of ^4He was a race against time to bind neutrons before they decayed to protons.

(a) First, prove that the maximum possible value of the primordial ^4He fraction is

$$\Big[Y_p\Big]_{\max} = \frac{2f}{1+f}$$

where $f = n_n/n_p \leq 1$ is the neutron-proton ratio at the time of nucleosynthesis.

Solution:

First, since $f = n_n/n_p$, we get $n_n = f n_p$.

Now, we get the maximum ^4He fraction if all available n_n neutrons bind to protons. Since there are 2 neutrons (and 2 protons) in one ^4He nucleus, this implies that we would get $n_n/2$ nuclei of ^4He.

Furthermore, since one ^4He nucleus has a mass of $4m_p$, where m_p is the mass of a proton, $n_n/2$ such nuclei will have a mass of $(n_n/2)\,4m_p$. We are assuming $m_p = m_n$.

And, since we have a total of (n_n+n_p) neutrons and protons, their mass will be $(n_n+n_p)\,m_p$.

From the above considerations, we get

$$\Big[Y_p\Big]_{\max} = \frac{(n_n/2)\,4m_p}{(n_n+n_p)m_p} = \frac{2n_n}{n_n+n_p} = \frac{2(fn_p)}{fn_p+n_p}$$

Therefore, canceling common terms and rearranging, we get

$$\Big[Y_p\Big]_{\max} = \frac{2f}{1+f}$$

(b) Assuming that the neutron-proton ratio remained constant at $f = n_n/n_p = 1/5$ after freeze-out, and that all available neutrons were incorporated into ^4He, find the value of $[Y_p]_{\max}$.

Solution: For $f = 1/5$, we get

$$\Big[Y_p\Big]_{\max} = \frac{2f}{1+f} = \frac{2(1/5)}{1+1/5} = \frac{2/5}{6/5} = \frac{2}{6} = \frac{1}{3}$$

Therefore, we get

$$\Big[Y_p\Big]_{\max} = \boxed{\mathbf{0.33}}$$

(c) In reality, $[Y_p]_{\max}$ was less than this value because some of the free neutrons decayed into protons, thereby decreasing the number of neutrons available to combine with protons to form ^4He.

Show that if nucleosynthesis starts after a time delay of $t_{\rm nuc}$, neutron decay makes the neutron-to-proton ratio decrease from its freeze-out value of $n_n/n_p = 1/5$ to

$$\frac{n_{nf}}{n_{pf}} = \frac{\exp\,(-t_{\rm nuc}/\tau_n)}{5 + \Big[1 - \exp\,(-t_{\rm nuc}/\tau_n)\Big]}$$

where τ_n is the decay time of the neutron.

Solution:

At freeze-out, we have $n_{ni}/n_{pi} = 1/5$, so $n_{pi} = 5n_{ni}$.

Given that if you start out with a population of free neutrons n_{ni}, the number of free neutrons remaining after time $t_{\rm nuc}$ will be

$$n_{nf} = n_{ni}\,\exp\left(-\,\frac{t_{\rm nuc}}{\tau_n}\right)$$

The remaining neutrons become protons, so the number of protons after time $t_{\rm nuc}$ is

$$n_{pf} = n_{pi} + \Big[n_{ni} - n_{nf}\Big]$$

Putting $n_{pi} = 5n_{ni}$ and the expression for n_{nf} into this expression, we get

$$n_{pf} = 5n_{ni} + \Big[n_{ni} - n_{ni}\,\exp(-t_{\rm nuc}/\tau_n)\Big]$$

and this may be written as

$$n_{pf} = n_{ni}\,\Big\{5 + \Big[1 - \exp(-t_{\rm nuc}/\tau_n)\Big]\Big\}$$

From the expressions above, we obtain

$$\frac{n_{nf}}{n_{pf}} = \frac{n_{ni}\exp(-t_{\rm nuc}/\tau_n)}{n_{ni}\Big\{5 + \Big[1 - \exp(-t_{\rm nuc}/\tau_n)\Big]\Big\}}$$

Therefore, we get the desired expression:

$$\frac{n_{nf}}{n_{pf}} = \frac{\exp\,(-t_{\rm nuc}/\tau_n)}{5 + \Big[1 - \exp\,(-t_{\rm nuc}/\tau_n)\Big]}$$

(d) As a result of the process described in part (c), find the neutron-proton ratio $f_{\text{new}} = n_{nf}/n_{pf}$, and the corresponding value of $[Y_p]_{\text{max}}$, if nucleosynthesis starts after a delay of $t_{\text{nuc}} = 200$ s, and the decay time of the neutron is $\tau_n = 890$ s. Assume that all available neutrons (that haven't decayed) are incorporated into ^4He nuclei.

Solution:

Given $t_{\text{nuc}} = 200$ s, and $\tau_n = 890$s, we get

$$f_{\text{new}} = \frac{n_{nf}}{n_{pf}} = \frac{\exp\left(-200/890\right)}{5 + \left[1 - \exp\left(-200/890\right)\right]} = \boxed{\mathbf{0.15}}$$

and

$$\text{New } \left[Y_p\right]_{\text{max}} = \frac{2f_{\text{new}}}{1 + f_{\text{new}}} = \frac{2(0.15)}{1 + 0.15} = \boxed{\mathbf{0.26}}$$

(e) Suppose, instead, that the neutron decay time were $\tau_n = 89$ s, with all other physical parameters unchanged (including that we still have $t_{\text{nuc}} = 200$ s). Calculate $[Y_p]_{\text{max}}$, again assuming that all available neutrons are incorporated into ^4He nuclei.

Solution:

With $t_{\text{nuc}} = 200$ s, and a smaller $\tau_n = 89$s, we get

$$f_{\text{new}} = \frac{n_{nf}}{n_{pf}} = \frac{\exp\left(-200/89\right)}{5 + \left[1 - \exp\left(-200/89\right)\right]} = \boxed{\mathbf{0.018}}$$

and

$$\text{New } \left[Y_p\right]_{\text{max}} = \frac{2f_{\text{new}}}{1 + f_{\text{new}}} = \frac{2(0.018)}{1 + 0.018} = \boxed{\mathbf{0.04}}$$

This makes sense — the neutrons decay faster, so there are fewer neutrons left to form ^4He, so the maximum ^4He fraction is smaller.

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5. Consider the inflaton field

$$V(\phi) = A\phi^4$$

Equation (11.53) can be used to find when the slow roll condition breaks down by setting

$$\left(\frac{E_P}{V}\frac{dV}{d\phi}\right)^2 \sim 1$$

where $E_P = (\hbar c^5/G)^{1/2}$ is the Planck energy.

(a) Use the limiting condition given above to find the value of ϕ at which the slow roll conditions break down; this marks the end of the period of inflation. *Leave your answer as an expression, don't substitute numerical values.*

Solution:

Setting

$$\frac{dV}{d\phi} = 4A\phi^3$$

in the expression

$$\frac{E_P^2}{V^2}\left(\frac{dV}{d\phi}\right)^2 \sim 1$$

we get

$$\frac{\hbar c^5/G}{(A\phi^4)^2}\left(4A\phi^3\right)^2 \sim 1$$

which works out to

$$\frac{\hbar c^5(16)\phi^6}{G\phi^8} \sim 1$$

or

$$\frac{16\hbar c^5}{G} \sim \frac{\phi^8}{\phi^6}$$

Therefore, we get

$$\phi \sim \left(\frac{16\hbar c^5}{G}\right)^{1/2}$$

We can either leave it in this form, or simplify by inserting $E_P = (\hbar c^5/G)^{1/2}$ to get

$$\phi \sim \left(\frac{\mathbf{16}\hbar \boldsymbol{c^5}}{\boldsymbol{G}}\right)^{\mathbf{1/2}} \sim \boxed{\mathbf{4}\boldsymbol{E_P}}$$

(b) Find the number of e-foldings N.

Solution:

Equation (11.48):

$$3H\dot{\phi} = -\hbar c^3 \frac{dV}{d\phi}$$

with $dV/d\phi = 4A\phi^3$ gives

$$-\frac{3H}{\hbar c^3}\frac{d\phi}{(4A\phi^3)} = dt$$

Then, we get

$$N = \ln\left[\frac{a(t_f)}{a(t_i)}\right] = \int_{t_i}^{t_f} H(t)\,dt$$

$$= \int_{\phi_i}^{\phi_f} H\left[-\frac{3H}{\hbar c^3}\frac{d\phi}{(4A\phi^3)}\right]$$

$$= \int_{\phi_i}^{\phi_f} \frac{-3H^2}{4A\hbar c^3\phi^3}\,d\phi$$

$$= \int_{\phi_i}^{\phi_f} \frac{-3}{4A\hbar c^3\phi^3}\left[\frac{8\pi GV}{3c^2}\right]d\phi$$

where we have put $H = (8\pi GV/3c^2)^{1/2}$.

Putting $V = A\phi^4$, and canceling terms we get

$$N = -\frac{2\pi G}{\hbar c^5}\int_{\phi_i}^{\phi_f}\phi\,d\phi = -\frac{2\pi G}{\hbar c^5}\left[\frac{\phi^2}{2}\right]_{\phi_i}^{\phi_f}$$

Therefore

$$N = -\frac{\pi G}{\hbar c^5}\left[\phi_f^2 - \phi_i^2\right]$$

We can leave it in this form, or use the fact that inflation ends of $\phi_f = 4E_P$, together with $E_P^2 = \hbar c^5/G$ and write this as

$$N = \frac{\pi}{E_P^2}\left[\phi_i^2 - \phi_f^2\right] = \boxed{\boldsymbol{\pi\left[\frac{\phi_i^2}{E_P^2} - 16\right]}}$$