# PHY 475
## Homework 3 solutions
*(Due by beginning of class on Wednesday, April 25, 2012)*

1. The cosmological constant has come under renewed scrutiny in recent years (with a different value from Einstein's, of course), because it may be a contributor to the dark energy that is responsible for the acceleration of the expansion of the Universe.

(a) Calculate the energy density of the cosmological constant in the current epoch, assuming $\Omega_\Lambda = 0.7$ and $H_0 = 70$ km s$^{-1}$ Mpc$^{-1}$.

**Solution:** Since $\Omega_\Lambda = \varepsilon_{\Lambda,0}/\varepsilon_{c,0}$, we get

$$\varepsilon_{\Lambda,0} = \Omega_\Lambda\,\varepsilon_{c,0} = 0.7\Big(8.3\times 10^{-10}\ \mathrm{J\ m^{-3}}\Big) = 5.8\times 10^{-10}\ \mathrm{J\ m^{-3}}$$

or if we want this in MeV m$^{-3}$

$$\varepsilon_{\Lambda,0} = \Omega_\Lambda\,\varepsilon_{c,0} = 0.7\Big(5200\ \mathrm{MeV\ m^{-3}}\Big) = 3640\ \mathrm{MeV\ m^{-3}}$$

Therefore the energy density of the cosmological constant in the current epoch is

$$\boldsymbol{\varepsilon_{\Lambda,0} = 5.8\times 10^{-10}\ \mathrm{J\ m^{-3}} = 3640\ \mathrm{MeV\ m^{-3}}}$$

(b) What is the total energy of the cosmological constant within a sphere 1 AU in radius?

**Solution:** The total energy with a sphere $r = 1$ AU in radius is then

$$E_{1\ \mathrm{AU}} = \varepsilon_{\Lambda,0}\left[\frac{4\pi}{3}r^3\right] = 5.8\times 10^{-10}\ \mathrm{J\ m^{-3}}\left[\frac{4\pi}{3}\Big(150\times 10^9\,\mathrm{m}\Big)^3\right]$$

Therefore, the total energy within a sphere 1 AU in radius is equal to

$$E_{1\ \mathrm{AU}} = \mathbf{8.2\times 10^{24}\ J} \equiv= \mathbf{5.1\times 10^{37}\ MeV}$$

(c) What is the rest energy of the Sun ($E_\odot = M_\odot\,c^2$)?

**Solution:** The rest energy of the Sun is equal to

$$E_\odot = M_\odot\,c^2 = \Big(1.99\times 10^{30}\ \mathrm{kg}\Big)\Big(3\times 10^8\ \mathrm{m\ s^{-1}}\Big)^2 = \mathbf{1.8\times 10^{47}\ J}$$

(d) Comparing your answers above, do you expect the cosmological constant to have a significant effect on the motion of planets within the Solar System?

**Solution:** Since the energy in the solar neighborhood is dominated by that of the Sun, whose rest energy is many orders of magnitude larger than the energy of the cosmological constant in a sphere of radius 1 AU, we don't expect the cosmological constant to have any significant effect on the motion of planets within the Solar System.

2. While Einstein introduced his cosmological constant to get a static universe, he was never satisfied with it. This wasn't merely due to the aesthetics of imposing such a solution. A significant problem with his cosmological constant was that while it made the universe static, it also made it unstable. To illustrate this, consider Einstein's static universe to be comprised only of non-relativistic matter with mass density $\rho$, and a cosmological constant, $\Lambda = 4\pi G\rho$. Suppose, now, that *some* of the non-relativistic matter is converted into radiation (e.g., by stars). Will the universe start to expand or contract? *You must show calculations justifying your answer; a mere statement like "expands" or "contracts" will be awarded zero points.*

   **Solution:** Start with the acceleration equation that includes Einstein's cosmological constant:
$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\Big(\varepsilon + 3P\Big) + \frac{\Lambda}{3}$$
   With a mass density of $\rho$ corresponding to an energy density of $\varepsilon = \rho\, c^2$ for non-relativistic matter, $\Lambda = 4\pi G\rho$, and pressure $P = 0$ in a static universe, the acceleration equation becomes
$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\Big(\rho c^2\Big) + \frac{4\pi G\rho}{3}$$
   This is clearly works out to zero, which was Einstein's original motivation for introducing the cosmological constant.

   Now, suppose some of the matter is changed into radiation. Let us write the energy density of matter as $\varepsilon_m$ and that of radiation as $\varepsilon_r$. Then, the acceleration equation becomes
$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\Big(\varepsilon_m + \varepsilon_r + 3P\Big) + \frac{4\pi G\rho}{3}$$
   Using the equation of state $P = w\varepsilon$, with $w = 0$ for non-relativistic matter, and $w = 1/3$ for radiation, this becomes
$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\left(\varepsilon_m + \varepsilon_r + 3\left\{\frac{1}{3}\varepsilon_r\right\}\right) + \frac{4\pi G\rho}{3}$$
   Since energy must be conserved when the matter is converted to radiation, we must have $\varepsilon_m + \varepsilon_r = \rho c^2$. Substituting this into the above equation, we finally get the acceleration equation in the form
$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\left(\rho c^2 + 3\left\{\frac{1}{3}\varepsilon_r\right\}\right) + \frac{4\pi G\rho}{3}$$
   so that
$$\frac{\ddot{a}}{a} = -\frac{4\pi G\rho}{3} - \frac{4\pi G\epsilon_r}{3c^2} + \frac{4\pi G\rho}{3}$$
   Therefore
$$\frac{\ddot{a}}{a} = -\frac{4\pi G\epsilon_r}{3c^2}$$
   Since the right hand side is negative, this means that $\ddot{a} < 0$. Since we are starting in an initial static state, $\ddot{a} < 0$ implies that Einstein's universe will **contract**.

   (**Note:** *Be careful when you apply this condition to other problems you may encounter. If the universe is expanding, $\ddot{a} < 0$ would mean the expansion would slow down.*)

3. In a *flat universe* with $H_0 = 70$ km s$^{-1}$ Mpc$^{-1}$, you observe a galaxy at a redshift $z = 7$. Carry out calculations to find the current proper distance to the galaxy, $d_p(t_0)$, in each of the following 3 cases. Also, carry out calculations to find the proper distance at the time the light was emitted, $d_p(t_e)$, again in each of the following 3 cases.

(a) Show your calculations for $d_p(t_0)$ and $d_p(t_e)$ if the universe contains *only radiation*?

**Solution:**

In a spatially flat universe, the proper distance at the time of observation and at the time of emission respectively is given by:

$$d_p(t_0) = \frac{c}{H_0}\frac{2}{1+3w}\left[1 - \frac{1}{(1+z)^{(1+3w)/2}}\right] \qquad \text{and} \qquad d_p(t_e) = \frac{d_p(t_0)}{1+z}$$

if $w \neq -1$.

Therefore, in a universe containing only radiation ($w = 1/3$), we get

$$d_p(t_0) = \frac{c}{H_0}\left[1 - \frac{1}{(1+z)}\right] = \frac{3\times 10^8 \text{ m/s}}{70,000 \text{ m/s Mpc}^{-1}/(3.1\times 10^{22}\text{ m/Mpc})}\left[1 - \frac{1}{(1+7)}\right]$$

so we get

$$d_p(t_0) = \frac{3\times 10^8\ (3.1\times 10^{22})}{70,000}\left(\frac{7}{8}\right) = \mathbf{1.2\times 10^{26}\ m \equiv 3750\ Mpc}$$

and

$$d_p(t_e) = \frac{d_p(t_0)}{1+z} = \frac{3750}{1+7} = \mathbf{470\ Mpc}$$

(b) Show your calculations if the universe contains *only matter*?

**Solution:**

Using the same expression as above, except with $w = 0$ for matter, we get

$$d_p(t_0) = \frac{2c}{H_0}\left[1 - \frac{1}{\sqrt{1+z}}\right] = \frac{(2)\ 3\times 10^8 \text{ m/s}}{70,000 \text{ m/s Mpc}^{-1}/(3.1\times 10^{22}\text{ m/Mpc})}\left[1 - \frac{1}{\sqrt{1+7}}\right]$$

so we get

$$d_p(t_0) = \frac{(2)\ 3\times 10^8\ (3.1\times 10^{22})}{70,000}\left(\frac{1.828}{2.828}\right) = \mathbf{1.7\times 10^{26}\ m \equiv 5500\ Mpc}$$

and

$$d_p(t_e) = \frac{d_p(t_0)}{1+z} = \frac{5500}{1+7} = \mathbf{690\ Mpc}$$

(c) Show your calculations if the universe contains *only a cosmological constant*?

**Solution:**

In a universe containing only a cosmological constant, the proper distance is given by equation (5.79) from Lecture 9:

$$d_p(t_0) = \frac{c}{H_0} z = \frac{3 \times 10^8 \text{ m/s}}{70,000 \text{ m/s Mpc}^{-1} / (3.1 \times 10^{22} \text{ m/Mpc})} \; \Big(7\Big)$$

so that

$$d_p(t_0) = \frac{3 \times 10^8 \; (3.1 \times 10^{22}) \; 7}{70,000} = \mathbf{9.3 \times 10^{26} \; m \equiv 30,000 \; Mpc}$$

and

$$d_p(t_e) = \frac{d_p(t_0)}{1+z} = \frac{30,000}{1+7} = \mathbf{3750 \; Mpc}$$

(d) Put all your answers for $d_p(t_0)$ and $d_p(t_e)$ for the three cases above in a table.

**Solution:**

| | Nature of universe | $d_p(t_0)$ (Mpc) | $d_p(t_e)$ (Mpc) |
|---|---|---|---|
| (a) | Radiation only | 3800 | 470 |
| (b) | Matter only | 5500 | 690 |
| (c) | Cosmological Constant only | 30,000 | 3800 |