PHY 475
Homework 2 solutions
(Due by beginning of class on Wednesday, April 18, 2012)

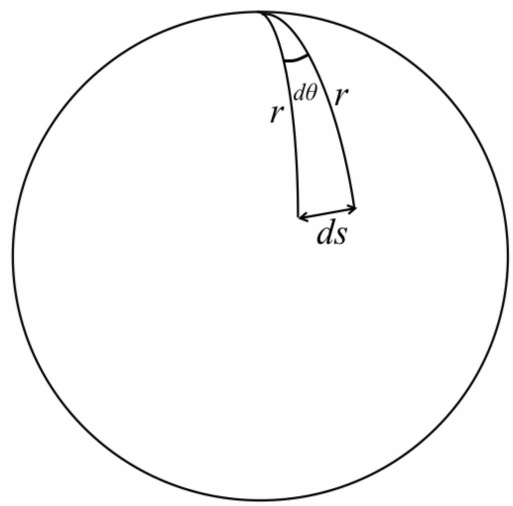
1. Suppose you are a two-dimensional being, living on the surface of a sphere with radius R. An object of width $ds \ll R$ is at a distance r from you (remember, all distances are measured on the surface of the sphere).

(a) What angular width $d\theta$ will you measure for the object?

Solution:

As discussed in lecture for 2-D surfaces, set up your location at the origin, which for convenience should be taken to be the north pole of the sphere.

This setup for the coordinate system is shown in the figure on the right. Recall that in setting up the metric for a 2-D surface, r is measured from the pole, and θ is the angle measured with respect to a great circle, so $d\theta$ as shown in the figure on the right matches this stipulation.



Now, since $ds \ll R$, whatever r may be, the distance from you to ds can be taken to be the same along the two sides of the triangle; another way of saying this is that you can take $dr = 0$.

The metric $ds^2 = dr^2 + R^2 \sin^2(r/R) d\theta^2$, with $dr = 0$, is

$$ds^2 = R^2 \sin^2 \left(\frac{r}{R}\right) d\theta^2$$

Taking the square root, the angular width you will measure is

$$\boldsymbol{d\theta = \frac{ds}{R \sin(r/R)}}$$

(b) Examine and explain the behavior of $d\theta$ as $r \to \pi R$.

Solution: Notice that, at $r = \pi R/2$, we get $d\theta = ds/R$, the smallest value for $d\theta$.

So, for fixed ds, we find that $d\theta$ decreases with increasing r up to $r = \pi R/2$, whereas for $r > \pi R/2$, $d\theta$ increases with increasing r. This makes perfect sense if you look at the figure: if you keep ds fixed and bring it near the poles it will subtend a much larger angle $d\theta$ at the poles, whereas the same ds will subtend a smaller angle at the poles when it is taken near the equatorial region of the sphere.

2. The Robertson-Walker (*FLRW*) metric may be written in the form

$$ds^2 = -c^2dt^2 + a(t)^2\left[dr^2 + S_\kappa(r)^2 d\Omega^2\right]$$

where $S_\kappa(r) =$

$$R_0 \sin\left(\frac{r}{R_0}\right), \text{ for } \kappa = +1; \qquad r, \text{ for } \kappa = 0; \qquad R_0 \sinh\left(\frac{r}{R_0}\right), \text{ for } \kappa = -1$$

(a) By inspection of the above equation, write down all the 16 elements (in matrix form) of $g_{\mu\nu}$ in the coordinate space (ct, r, θ, ϕ).

Solution: Recall that $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$. By inspection, therefore

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a(t)^2 & 0 & 0 \\ 0 & 0 & a(t)^2\, S_\kappa(r)^2 & 0 \\ 0 & 0 & 0 & a(t)^2\, S_\kappa(r)^2 \sin^2\theta \end{pmatrix}$$

(b) By putting $x \equiv S_\kappa(r)$, show explicitly (i.e., for all 3 cases) that the Robertson-Walker metric takes the form

$$ds^2 = -c^2dt^2 + a(t)^2\left[\frac{dx^2}{1 - \kappa\, x^2/R_0^2} + x^2\, d\Omega^2\right]$$

Solution: Putting $x \equiv S_\kappa(r)$, the form for the 2nd term in the spatial part is obvious, so let us work with the first term in the spatial part below.

$\kappa = +1$	$\kappa = 0$	$\kappa = -1$
$x = S_\kappa(r) = R_0 \sin\left(\frac{r}{R_0}\right)$	$x = r$	$x = S_\kappa(r) = R_0 \sinh\left(\frac{r}{R_0}\right)$
$\Rightarrow \quad dx = R_0\left[\cos\left(\frac{r}{R_0}\right)\right]\frac{1}{R_0}\, dr$	$\Rightarrow dx^2 = dr^2$	$\Rightarrow \quad dx = R_0\left[\cosh\left(\frac{r}{R_0}\right)\right]\frac{1}{R_0}\, dr$
$\Rightarrow \quad dx^2 = \cos^2\left(\frac{r}{R_0}\right) dr^2$		$\Rightarrow \quad dx^2 = \cosh^2\left(\frac{r}{R_0}\right) dr^2$
$\Rightarrow \quad dx^2 = \left[1 - \sin^2\left(\frac{r}{R_0}\right)\right] dr^2$		$\Rightarrow \quad dx^2 = \left[1 + \sinh^2\left(\frac{r}{R_0}\right)\right] dr^2$
$\Rightarrow \quad dx^2 = \left[1 - \left(\frac{x^2}{R_0^2}\right)\right] dr^2$		$\Rightarrow \quad dx^2 = \left[1 + \left(\frac{x^2}{R_0^2}\right)\right] dr^2$

Therefore, in all 3 cases, we can write $dr^2 = \dfrac{dx^2}{1 - \kappa\, x^2/R_0^2}$, which gives us the desired form.

3. The principle of wave-particle duality tells us that a particle with momentum p has an associated de Broglie wavelength of $\lambda = h/p$; this wavelength increases as $\lambda \propto a$, as the universe expands. The total energy density of a gas of particles can be written as $\varepsilon = nE_p$, where n is the number density of particles, and E_p is the energy per particle. For simplicity, let us assume that all the gas particles have the same mass m and momentum p. The energy per particle is then simply

$$E_p = \sqrt{(m^2c^4 + p^2c^2)}$$

(a) Compute the equation-of-state parameter w for this gas as a function of the scale factor a.

Solution:

Since $\lambda \propto a$, we get $p \propto 1/a$, so we can write $\dfrac{p}{p_0} = \dfrac{a}{a_0}$, and since $a_0 = 1$, we get $p = \dfrac{p_o}{a}$.

Likewise, since $n \propto a^{-3}$, we can write $\dfrac{n}{n_0} = \dfrac{a^{-3}}{a_0^{-3}}$, and again since $a_0 = 1$, we get $n = \dfrac{n_o}{a^3}$.

Using the equation of state $P = w\varepsilon$, we get

$$w = \frac{P}{\varepsilon} = \frac{P}{nE_p}$$

Since E_p is easily cast in terms of a, we need to find P in terms of a, and the easiest way to do it is by using the fluid equation $\dot{\varepsilon} + 3(\dot{a}/a)(\varepsilon + P) = 0$, so that

$$3\frac{\dot{a}}{a}(\varepsilon + P) = -\dot{\varepsilon} \qquad \Rightarrow \qquad \varepsilon + P = -\frac{a}{3\dot{a}}\,\dot{\varepsilon}$$

So

$$\begin{aligned}
P &= -\varepsilon - \frac{a}{3\dot{a}}\,\dot{\varepsilon} \\
&= -nE_p - \frac{a}{3\dot{a}}\left[\frac{d}{dt}\Big(nE_p\Big)\right] \\
&= -nE_p - \frac{a}{3\dot{a}}\left[\frac{d}{da}\Big(nE_p\Big)\frac{da}{dt}\right] \\
&= -nE_p - \frac{a}{3}\left[\frac{d}{da}\Big(nE_p\Big)\right] \\
&= -nE_p - \frac{a}{3}\left[\left(\frac{dn}{da}\right)E_p + n\left(\frac{dE_p}{da}\right)\right] \\
&= -nE_p - \frac{a}{3}\left[\left(-\frac{3n_0}{a^4}\right)E_p + n\,\frac{1}{2}\Big(m^2c^4 + p^2c^2\Big)^{1/2-1}\left(0 + 2p\frac{dp}{da}\,c^2\right)\right]
\end{aligned}$$

3. (a) . . . *Continued from previous page*

For convenience, let us write the last line from the previous page before continuing

$$P = -nE_p - \frac{a}{3}\left[\left(-\frac{3n_0}{a^4}\right)E_p + n\,\frac{1}{2}\Big(m^2c^4 + p^2c^2\Big)^{1/2-1}\left(0 + 2p\frac{dp}{da}\,c^2\right)\right]$$

$$= -nE_p - \frac{a}{3}\left[\left(-\frac{3}{a}\frac{n_0}{a^3}\right)E_p + n\,\frac{1}{2}\Big(m^2c^4 + p^2c^2\Big)^{-1/2}\left(2p\left\{-\frac{p_0}{a^2}\right\}c^2\right)\right]$$

$$= -nE_p + \frac{a}{3}\left[\left(\frac{3}{a}\,n\right)E_p + \frac{n}{2E_p}\left(2p\left\{\frac{p}{a}\right\}c^2\right)\right]$$

$$= -nE_p + nE_p + \frac{n}{E_p}\,\frac{p^2c^2}{3}$$

So

$$w = \frac{P}{nE_p} = \frac{1}{nE_p}\left[\frac{n}{E_p}\,\frac{p^2c^2}{3}\right]$$

Therefore, either one of the solutions below expresses w as a function of a.

$$w = \frac{p^2c^2}{3E_p^2} \qquad \text{or} \qquad w = \frac{p_0^2c^2}{3\Big(p_0^2c^2 + m^2c^4a^2\Big)}$$

(b) Use your result in part (a) to show that $w = 1/3$ in the highly relativistic limit ($p \to \infty$).

Solution: When $p \to \infty$, we can write

$$w = \frac{p^2c^2}{3E_p^2} = \frac{\cancel{p^2}\,c^2}{3\cancel{p^2}\left[c^2 + m^2c^4/p^2\right]} = \frac{c^2}{3c^2}, \quad \text{for } p \to \infty$$

Therefore, for $p \to \infty$, we get $w = 1/3$.

(c) Use your result in part (a) to show that $w = 0$ in the highly non-relativistic limit ($p \to 0$).

Solution: When $p \to 0$, we can write

$$w = \frac{p^2c^2}{3E_p^2} = \frac{0}{3\Big(0 + m^2c^4/p^2\Big)} = 0$$

Therefore, for $p \to 0$, we get $w = 0$.