

**PHY 475**  
**Homework 1 solutions**  
(Due by beginning of class on Monday, April 9, 2012)

1. The Hubble “constant”  $H_0$  can be used to obtain a rough estimate of the age of the Universe under a certain assumption.
- (a) We discussed the assumption in class. What is it?

**Solution:**

The assumption is that the Universe has been expanding at constant velocity.

- (b) In class, we wrote that under the current consensus value of  $H_0 = 70 \text{ km s}^{-1} \text{ Mpc}^{-1}$  and the assumption above, the approximate age of the Universe is  $14 \times 10^9 \text{ yr}$ , or 14 Gyr. Due to severe underestimates of his measured distances to galaxies, Hubble originally measured  $H_0 = 500 \text{ km s}^{-1} \text{ Mpc}^{-1}$ . What would this value of  $H_0$  give you for the approximate age of the Universe? *For full credit, you must show all your calculation steps clearly.*

**Solution:**

All we need to do is find  $H_0^{-1}$ , along with a straightforward conversion from Mpc to km.

Recall that  $1 \text{ Mpc} \equiv 3.1 \times 10^{19} \text{ km}$

$$\begin{aligned} H_0 &= 500 \text{ km s}^{-1} \text{ Mpc}^{-1} \\ &= 500 \frac{\text{km}}{\text{s}} \frac{1}{\text{Mpc}} \\ &= 500 \frac{\text{km}}{\text{s}} \frac{1}{\text{Mpc}} \frac{\text{Mpc}}{3.1 \times 10^{19} \text{ km}} \end{aligned}$$

so that

$$H_0^{-1} = \frac{3.1 \times 10^{19} \text{ km}}{500 \text{ km s}^{-1}} = 6.2 \times 10^{16} \text{ s}$$

Converting to yr, we get

$$H_0^{-1} = \frac{6.2 \times 10^{16} \text{ s}}{3600 \times 24 \times 365 \text{ s/yr}} = 1.966 \times 10^9 \text{ yr} \equiv \boxed{2 \text{ Gyr}}$$

2. Suppose that you are in an infinitely large, infinitely old universe in which standard Euclidean geometry holds true.
- (a) The density of stars in this universe is  $n_\star = 10^9 \text{ Mpc}^{-3}$  and the average radius of a star is equal to the Sun's radius:  $R_\star = R_\odot = 7 \times 10^8 \text{ m}$ . How far, on average, could you see in any direction before your line of sight struck a star?

### Solution:

The easiest way to solve this is by analogy with the mean free path of atoms/molecules in a gas. The treatment below is adapted from the chapter on Kinetic Theory of Gases by *Halliday & Resnick*.

The strategy in this derivation is to invert the situation. That is, instead of considering a ray of light (hence a point) moving along and striking a star surface of radius  $R_\star$ , think instead of the light as a circular surface of radius  $R_\star$  and the star as a point. In other words, the light moves along as a circular disk, thereby defining a long cylinder of radius  $R_\star$ . So, the average distance between collisions (and, in our case, we need only one collision) is given by the length of this cylinder divided by the number of collisions in the cylinder. Since we are talking about light, the length of the cylinder in time  $t$  will be  $ct$ , whereas the number of collisions in the cylinder in time  $t$  will be the volume of the cylinder ( $V_{\text{cylinder}}$ ) times the number of stars per unit volume ( $n_\star$ ) in the cylinder. So,

$$\text{Distance before striking star} = \frac{\text{length of cylinder}}{\text{number of collisions}} = \frac{ct}{V_{\text{cylinder}} n_\star} = \frac{ct}{\pi R_\star^2 (ct) n_\star} = \frac{1}{n_\star \pi R_\star^2}$$

Let's calculate the denominator first; recall  $1 \text{ Mpc} \equiv 3.1 \times 10^{22} \text{ m}$ :

$$\begin{aligned} n_\star \pi R_\star^2 &= \left[ 10^9 \text{ Mpc}^{-3} \right] \pi \left( 7 \times 10^8 \text{ m} \right)^2 \\ &= \left[ 10^9 \frac{1}{\text{Mpc}^3} \left\{ \frac{1 \text{ Mpc}^3}{(3.1 \times 10^{22} \text{ m})^3} \right\} \right] \pi \left( 7 \times 10^8 \text{ m} \right)^2 \\ &= 5.16726 \times 10^{-41} \text{ m}^{-1} \end{aligned}$$

where more digits than significant have been retained temporarily to avoid rounding errors.

Therefore

$$\text{Distance before striking star} = \frac{1}{n_\star \pi R_\star^2} = \frac{1}{5.16726 \times 10^{-41} \text{ m}^{-1}} = 1.9 \times 10^{40} \text{ m}$$

On average, therefore, you could see out to  $1.9 \times 10^{40} \text{ m}$  before your line of sight struck a star.

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## 2. (Continued from previous page)

- (b) If the stars are clumped into galaxies with a density of  $n_{\text{gal}} = 1 \text{ Mpc}^{-3}$ , and average radius  $R_{\text{gal}} = 2000 \text{ pc}$ , how far, on average, could you see in any direction before your line of sight struck a galaxy?

**Solution:**

As in part (a), let us calculate the denominator first, but this time with  $n_{\text{gal}} = 1 \text{ Mpc}^{-3}$  and  $R_{\text{gal}} = 2000 \text{ pc}$ ; recall,  $1 \text{ pc} = 3.1 \times 10^{16} \text{ m}$ , and  $1 \text{ Mpc} \equiv 3.1 \times 10^{22} \text{ m}$ :

$$\begin{aligned} n_{\text{gal}} \pi R_{\text{gal}}^2 &= \left[ 1 \text{ Mpc}^{-3} \right] \pi \left( 2000 \text{ pc} \right)^2 \\ &= \left[ 1 \frac{1}{\text{Mpc}^3} \left\{ \frac{1 \text{ Mpc}^3}{(3.1 \times 10^{22} \text{ m})^3} \right\} \right] \pi \left( 2000 \text{ pc} \left\{ 3.1 \times 10^{16} \frac{\text{m}}{\text{pc}} \right\} \right)^2 \\ &= 4.05367 \times 10^{-28} \text{ m}^{-1} \end{aligned}$$

where more digits than significant have been retained temporarily to avoid rounding errors.

$$\text{So, distance before striking star} = \frac{1}{n_{\text{gal}} \pi R_{\text{gal}}^2} = \frac{1}{4.05367 \times 10^{-28} \text{ m}^{-1}} = 2.5 \times 10^{27} \text{ m}$$

On average, therefore, you could see out to  $\boxed{2.5 \times 10^{27} \text{ m}}$  before your line of sight struck a galaxy.

- (c) To make sense of your results, convert your answers in parts (a) and (b) to Mpc, and compare them to the approximate size of the Universe  $c/H_0$ , then comment on how this helps you with resolving Olbers' paradox.

**Solution:**

The distance that you could see, on average, before your line of sight struck a star was found in part (a); it is

$$1.9 \times 10^{40} \text{ m} = 1.9 \times 10^{40} \text{ m} \left[ \frac{1 \text{ Mpc}}{3.1 \times 10^{22} \text{ m}} \right] = \boxed{6.1 \times 10^{17} \text{ Mpc}}$$

The distance that you could see, on average, before your line of sight struck a galaxy was found in part (b); it is

$$2.5 \times 10^{27} \text{ m} = 2.5 \times 10^{27} \text{ m} \left[ \frac{1 \text{ Mpc}}{3.1 \times 10^{22} \text{ m}} \right] = \boxed{8.1 \times 10^4 \text{ Mpc}}$$

Since  $c/H_0 = 3 \times 10^5 \text{ km/s} / 70 \text{ km/s Mpc}^{-1} = 4300 \text{ Mpc}$ , these distances are orders of magnitude larger than the approximate size of the Universe. Therefore, on average, it is rare that your line of sight will end in a star or galaxy, which resolves Olber's paradox — the night sky is dark because only a few lines of sight end on a star or galaxy.

3. Since you're made mostly of water, you're very efficient at absorbing microwave photons.
- (a) The number density of CMB photons is  $n_\gamma = 4.11 \times 10^8 \text{ m}^{-3}$ . If you were in intergalactic space, approximately how many CMB photons would you absorb per second? *If you like, you could assume you are spherical. Alternatively, you could be lazy like I was, and assume your surface area is  $1 \text{ m}^2$ .*

**Solution:**

Assuming my surface area is  $A = 1 \text{ m}^2$ , and the photons are traveling at  $c = 3 \times 10^8 \text{ m s}^{-1}$ , I would absorb

$$n_\gamma c A = 4.11 \times 10^8 \text{ m}^3 \left( 3 \times 10^8 \text{ m s}^{-1} \right) 1 \text{ m}^2 = \boxed{1.23 \times 10^{17} \text{ s}^{-1}}$$

- (b) What is the approximate rate, in watts, at which you would absorb radiative energy from the CMB?

**Solution:**

The mean energy per CMB photon is given by the energy density of the CMB ( $\varepsilon_\gamma$ ) divided by the number density of CMB photons ( $n_\gamma$ ), and with the temperature of the CMB equal to  $T = 2.725 \text{ K}$ , is given by

$$E_{\text{mean}} = \frac{\varepsilon_\gamma}{n_\gamma} = \frac{\alpha T^4}{\beta T^3} = \frac{\alpha}{\beta} T = \frac{7.56 \times 10^{-16}}{2.03 \times 10^7} 2.725 \text{ K} = 1.015 \times 10^{-22} \text{ J}$$

so I would absorb radiative energy from the CMB at the rate

$$1.015 \times 10^{-22} \text{ J} \left( 1.23 \times 10^{17} \text{ s}^{-1} \right) = \boxed{1.25 \times 10^{-5} \text{ watts} \equiv 12.5 \text{ } \mu\text{W}}$$

- (c) Ignoring any other energy inputs and outputs, how long would it take the CMB to raise your temperature by 1 nano Kelvin (i.e.,  $10^{-9} \text{ K}$ )? *Since your body is mostly water, assume your specific heat capacity is the same as water ( $= 4200 \text{ J kg}^{-1} \text{ K}^{-1}$ ).*

**Solution:**

This answer will depend on the mass you assume for your body, so let us keep it as  $M \text{ kg}$  for now. Using  $\text{heat} = mc(\Delta T)$  that you learned in introductory physics classes, we can solve for the time  $t$  it would take the CMB to raise your temperature by 1 nK:

$$\left( 1.25 \times 10^{-5} \text{ W} \right) t = M \left( 4200 \text{ J kg}^{-1} \text{ K}^{-1} \right) 10^{-9} \text{ K} = \boxed{0.336 \left( \text{M kg} \right) \text{ s}}$$

For example, a 100 kg person would have their temperature raised by the CMB by 1 nK in

$$0.336 \left( 100 \text{ kg} \right) = \boxed{33.6 \text{ s}}$$