PHY 475
Final Examination
(Due by noon on Thursday, June 7, 2012)

Method of submission: You may submit *either* a hard copy or an electronic version.

- **Hard copies** should be submitted to Dr. Sarma in his office or left in his mailbox.
- **Electronic versions** should be uploaded in D2L in the dropbox set up for this purpose. *Do not send* these over email, they will not be accepted.

- **Answer all questions.**
- **Start each new question on a different page.** Do not start a new question on a page below another question. This does not apply to sub-parts like (a) and (b), but only to numbered questions.
- **Write down numbers of questions, including sub-parts (a), (b), etc., clearly.** Answers that are not numbered properly will not be graded.
- **Read the question carefully before you start.** Make sure you understand the setup, and check to see that you have addressed all the questions asked.
- **Show all steps.** No points will be awarded if only an answer is shown, even though the answer may be correct.
- **Attach this page to the top with your name and signature clearly visible in the box below.** This applies regardless of whether you are submitting a hard copy or an electronic version.

Name:	______________________________ (Please print)
	Allowed Materials: You are allowed the use of the Ryden text, the course website, and a table of integrals. Other than the course website, you may use the internet only to access a table of integrals. Do not look at any other materials on the internet. Do not look at any other textbooks. **Discussion Restrictions:** You are *not allowed* to discuss this test with any other student in this class, or any other person. You may discuss this test *only with* Dr. Sarma.
Pledge:	By signing below, I acknowledge that I have followed all of the above. **Signature:**

1. Write brief answers or show mathematical calculations, as appropriate, for the following.
   (a) What is the significance of the *linearity* of Hubble's law, $v = H_0 d$?

   (b) The deceleration parameter q_0 is defined as

   $$q_0 = -\left(\frac{\ddot{a}}{aH^2}\right)\Bigg|_{t=t_0}$$

   Show that in a matter-dominated universe, $q_0 = \Omega_m/2$, and in a radiation-dominated universe, $q_0 = \Omega_r$.

   (c) How can the observed temperature anisotropy of the Cosmic Microwave Background (CMB) be used to deduce that the Universe has a flat geometry?

   (d) What would be the effect on the CMB spectrum (plot of Δ_T^2 vs. multipole moment l, shown in Lecture 15) of increasing the dark matter content of the Universe? Explain why.

   (e) The expected power spectrum for inflationary fluctuations has the form of a power law: $P(k) \propto k^n$. If we pick out spheres of comoving radius L in such a universe, then the root mean square mass fluctuation within such spheres is given by

   $$\frac{\delta M}{M} \equiv \left\langle \left(\frac{M - \langle M \rangle}{\langle M \rangle}\right)^2 \right\rangle^{1/2} \propto \left[k^3\, P(k)\right]^{1/2}$$

   where $k = 2\pi/L$ is the comoving wavenumber associated with the sphere.
   Show that this can be expressed in the form

   $$\frac{\delta M}{M} \propto M^{-(3+n)/6}$$

2. Consider a flat universe containing only matter and a negative dark energy given by a cosmological constant $\Omega_\Lambda < 0$.
   (a) Show that in such a universe, the expansion will come to a stop at a maximum scale factor

   $$a_{\max} = \left(-\frac{\Omega_{m,0}}{\Omega_\Lambda}\right)^{1/3}$$

   (b) Show that the time from the beginning of such a universe to the Big Crunch (i.e., from the initial $a(0) = 0$ to the final $a(t_{\mathrm{BC}}) = 0$) is given by

   $$t_{\mathrm{BC}} = \frac{2\pi}{3H_0}\left(-\,\Omega_\Lambda\right)^{-1/2}$$

   Hint: You will need the integral

   $$\int \frac{dx}{\sqrt{b^2 - x^2}} = \sin^{-1}\left(\frac{x}{b}\right)$$

3. The Friedmann equation is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\,\varepsilon - \frac{\kappa c^2}{R_0^2}\frac{1}{a^2}$$

The density parameters Ω_Λ, Ω_m, and Ω_r have their standard definitions. Also define

$$\Omega_\kappa = -\frac{\kappa c^2}{R_0^2 a^2 H^2}$$

(a) Use the Friedmann equation to show that

$$\Omega_\Lambda + \Omega_\kappa + \Omega_m + \Omega_r = 1$$

(b) Show that the Friedmann equation can be written in the form

$$H(z) = H_0\Big[\Omega_{\Lambda,0} + \Omega_{\kappa,0}\,(1+z)^2 + \Omega_{m,0}\,(1+z)^3 + \Omega_{r,0}\,(1+z)^4\Big]^{1/2}$$

where, as usual, the subscripts 0 refer to the values of parameters at the present time.

(c) Show that the lookback time to an object at redshift z_t is

$$t = \frac{1}{H_0}\int_0^{z_t} \frac{dz}{(1+z)\Big[\Omega_{\Lambda,0} + \Omega_{\kappa,0}\,(1+z)^2 + \Omega_{m,0}\,(1+z)^3 + \Omega_{r,0}\,(1+z)^4\Big]^{1/2}}$$

(d) Show that

$$\Omega_m(z) = \frac{\Omega_{m,0}\,(1+z)^3}{\Omega_{\Lambda,0} + \Omega_{\kappa,0}\,(1+z)^2 + \Omega_{m,0}\,(1+z)^3 + \Omega_{r,0}\,(1+z)^4}$$

(e) Assuming that $\Omega_{r,0}$ and $\Omega_{\Lambda,0}$ are negligibly small, and can be set to zero in the equation in part (d) above, what are the asymptotic values of Ω_m and Ω_κ for $z \gg \Omega_{m,0}^{-1}$?

4. The binding energy of a deuterium nucleus is $B_D = 2.22$ MeV.

(a) Naively, one might expect deuterium to start forming when the temperature of the Universe drops to a value such that $kT \sim 2.22$ MeV. Yet, we know this doesn't happen. Why not?

(b) A better, but still crude, approximation to the temperature at which deuterium is synthesized can be obtained by setting $e^{-B_D/kT_{\rm nuc}} \approx \eta$, where $\eta = 5.5 \times 10^{-10}$ is the baryon-to-photon ratio. With $B_D = 2.22$ MeV, find the temperature $T_{\rm nuc}$ at which deuterium synthesis begins. *Be careful you don't copy the answers from your text — they won't match the correct answers here.*

(c) Find the age of the universe $t_{\rm nuc}$ when its temperature drops to the value $T_{\rm nuc}$ you determined in part (b). State clearly the assumptions you made in setting up this calculation.

4. ***(...continued from previous page)***

   (d) The neutron-to-proton ratio of 1/5 at freeze-out will change because some of the free neutrons decay to protons. After time t_{nuc}, the number of free neutrons remaining will be
   $$n_{nf} = n_{ni} \exp\left(-\frac{t_{\text{nuc}}}{\tau_n}\right)$$
   where $\tau_n = 890$ s is the decay time of the neutrons. Find the new neutron-to-proton ratio after time t_{nuc} when deuterium synthesis begins.

   (e) To a first approximation, all the neutrons present at time t_{nuc} are processed into primordial helium (^4He). Based on this, calculate the maximum primordial helium fraction $[Y_p]_{\text{max}}$.

   (f) For $\Omega_{\text{bary}} = 0.02$, numerical calculations predict a deuterium-to-hydrogen ratio of $D/H \approx 10^{-5}$. If, instead, $\Omega_{\text{bary}} = 0.20$, would the predicted deuterium abundance be higher or lower? *For full credit, explain your answer.*

   (g) Would raising Ω_{bary} to 0.20, as described in part (f) above, raise or lower the predicted helium abundance. *For full credit, explain your answer.*

5. The energy density and pressure of a scalar inflaton field can be written as
   $$\varepsilon_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \qquad \text{and} \qquad P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$
   This is the same as what we wrote in class, except that we are using high energy units in this problem like many cosmologists do, that is, we have set $\hbar = 1, c = 1$. This makes the dynamical equation for ϕ look like
   $$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0$$

   (a) In the slow roll approximation, the terms $\ddot{\phi}$ and $\dot{\phi}^2$ can be ignored (but not $\dot{\phi}$). Use this to derive the slow roll equations
   $$3H\dot{\phi} \approx -\frac{dV}{d\phi} \qquad \text{and} \qquad H^2 \approx \frac{8\pi G}{3}V$$

   (b) Show that the number of e-foldings of inflation is given by
   $$N = -8\pi G \int_{\phi_i}^{\phi_f} V(\phi) \left[\frac{dV}{d\phi}\right]^{-1} d\phi$$

   (c) Now consider the potential
   $$V(\phi) = \frac{1}{2}m^2\phi^2$$
   At what value of $\phi = \phi_f$ will the inflation end? *Leave your answer as an expression; don't substitute numerical values.*

   (d) For the potential given in part (c) above, what condition must be obeyed by ϕ_i so that an expansion of at least 10^{30} takes place?