PHY 475/375

Lecture 9

(April 23, 2012)

Single Component Universes

An Empty Universe: Curvature only

Consider first a very simple, albeit non-physical, universe — one which is empty and contains no radiation, no matter, no cosmological constant, and therefore, no contribution to ε whatsoever. For such a universe, the Friedmann equation takes a very simple form:

$$\dot{a}^2 = -\frac{\kappa c^2}{R_0^2} \tag{5.28}$$

What kind of a universe does equation (5.28) allow? In other words, what are the solutions to this equation?

- Straight away, we can rule out positively curved ($\kappa = +1$) empty universes, since they would lead to an imaginary value for $\dot{a}$ in equation (5.28).
- A permissible solution to equation (5.28) is with $\kappa = 0$, which gives $\dot{a} = 0$. Since $\kappa = 0$ corresponds to a spatially flat universe, and $\dot{a} = 0$ means a static universe, this means that we get an empty, static, spatially flat universe. This is the universe whose geometry is described by the Minkowski metric that we wrote in an earlier lecture (equation 3.20).
- Yet another solution to equation (5.28) is with $\kappa = -1$, which corresponds to a negatively curved universe. So, it is possible to have a negatively curved empty universe.

Let us look at an empty, negatively curved universe in more detail. Equation (5.28) tells us such a universe should be expanding or contracting with

$$\dot{a}^2 = -\frac{(-1)c^2}{R_0^2} = +\frac{c^2}{R_0^2}$$

corresponding to

$$\dot{a} = \pm\frac{c}{R_0} \tag{5.29}$$

Integrating, we get

$$a(t) = \frac{c}{R_0}\, t$$

where we've chosen the positive sign for an expanding universe. Meanwhile, since c/R_0 has the dimensions of s^{-1}, if we set $t_0 = R_0/c$, we can write the above equation as

$$a(t) = \frac{t}{t_0} \tag{5.30}$$

Since the universe is empty, there is no gravitational force at work (in Newtonian terms), therefore the relative velocity of any two points is constant. This is why the scale factor a increases linearly with time in an empty universe.

Meanwhile, with nothing to speed or slow the expansion in an empty universe, we get $t_0 = H_0^{-1}$, that is, the age of the universe is exactly equal to the Hubble time in an empty universe.

While an empty expanding universe might seem like nothing more than a mathematical curiosity, a universe which has an energy density that is much smaller than the critical density (i.e., $\varepsilon \ll \varepsilon_c$, equivalent to $\Omega \ll 1$) would be one in which the linear scale factor of equation (5.30) could be used as a good approximation of the true scale factor.

Now, suppose we observe a distant galaxy with redshift z. The light which we observe now at $t = t_0$ was emitted at an earlier time $t = t_e$. We know from an earlier lecture that

$$1 + z = \frac{a(t_0)}{a(t_e)}$$

Since we usually set $a(t_0) = 1$, the above equation gives

$$1 + z = \frac{1}{a(t_e)}$$

From equation (5.30), the scale factor $a(t_e) = t_e/t_0$ in an empty expanding universe, so the above equation becomes

$$1 + z = \frac{1}{a(t_e)} = \frac{t_0}{t_e} \tag{5.31}$$

Note that, as we have shown already, an empty expanding universe is negatively curved (since an empty flat universe would be static).

Equation (5.31) makes it easy to compute the time when the observed light was emitted from the source:

$$t_e = \frac{t_0}{1+z} = \frac{H_0^{-1}}{1+z} \tag{5.32}$$

since $t_0 = H_0^{-1}$ in an empty universe.

In an earlier lecture, we discussed how in any universe described by a Robertson-Walker metric, the proper distance $d_p(t_0)$ from an observer at the origin to a galaxy located at space coordinates (r, θ, ϕ) is given by:

$$\begin{aligned} d_p(t_0) &= a(t_0) \int_0^r dr \\ &= a(t_0)\, r \\ \Rightarrow d_p(t_0) &= r \end{aligned} \tag{5.33}$$

since $a(t_0) = 1$.

Equation (5.33) is identical to equation (3.28) in Lecture 5, except that we wrote it there for a general t. Recall that proper distance is the distance between two points that you could measure with a ruler if you could freeze time; here, we have chosen to freeze the time at the moment of observation.

In that same lecture, we also discussed that when light was emitted by a galaxy at time t_e and observed by us at time t_0, we would end up with equation (3.39):

$$c\int_{t_e}^{t_0}\frac{dt}{a(t)} = \int_0^r dr = r \tag{5.34}$$

From equation (5.33) and (5.34), we obtain

$$d_p(t_0) = c\int_{t_e}^{t_0}\frac{dt}{a(t)} \tag{5.35}$$

Equation (5.35) holds true in any universe whose geometry is described by a Robertson-Walker metric.

In the specific case of an empty expanding universe where equation (5.30) tells us that $a(t) = t/t_0$, we get from equation (5.35) that

$$\begin{aligned}
d_p(t_0) &= c\int_{t_e}^{t_0}\frac{dt}{a(t)} \\
&= c\int_{t_e}^{t_0}\frac{dt}{t/t_0} \\
&= c\,t_0\int_{t_e}^{t_0}\frac{dt}{t} \\
&= c\,t_0\Big[\ln t\Big]_{t_e}^{t_0}
\end{aligned}$$

$$\Rightarrow \quad d_p(t_0) = c\,t_0\,\ln\left(\frac{t_0}{t_e}\right) \tag{5.36}$$

Moreover, we can use $t_0 = 1/H_0$ and equation (5.31) to express this in terms of the redshift z of the observed galaxy:

$$d_p(t_0) = \frac{c}{H_0}\ln\Big(1+z\Big) \tag{5.37}$$

- In the limit $z \ll 1$, there is a linear relation between d_p and z, as seen in Hubble's law.
- In the limit $z \gg 1$, we get $d_p \propto \ln z$ in an empty expanding universe.

Equation (5.37) also tells us that in an empty expanding universe, we can see objects which are currently at an arbitrarily large distance!

- This may seem counterintuitive, that we can see objects at a proper distance much greater than c/H_0, when age of the universe is only $1/H_0$.

- However, it is important to remember that $d_p(t_0)$ is the proper distance to the object at the *time of observation*.

- At the *time of emission*, the proper distance $d_p(t_e)$ was

$$d_p(t_e) = a(t_e) \int_0^r dr$$

$$\Rightarrow \quad d_p(t_e) = a(t_e)\, r \tag{5.38.a}$$

Now, recall that

$$\frac{a(t_e)}{a(t_0)} = \frac{1}{1+z}$$

This means that equation (5.38.a) can be written as

$$d_p(t_e) = \left[\frac{a(t_0)}{1+z}\right] r$$

$$\Rightarrow \quad d_p(t_e) = \frac{d_p(t_0)}{1+z}$$

where we have used equation (5.33) to write $a(t_0)\, r$ as $d_p(t_0)$.

Using the expression for $d_p(t_0)$ from equation (5.37), the above equation can finally be written as

$$d_p(t_e) = \frac{c}{H_0} \frac{\ln(1+z)}{1+z} \tag{5.38}$$

In other words, the proper distance at the time of emission was smaller by a factor $1/(1+z)$, so objects at extremely high redshifts today are seen as they were very early in the history of the universe, when their proper distance from the observer was very small.

Spatially Flat Universe

Consider again the Friedmann equation (5.1):

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\,\varepsilon - \frac{\kappa c^2}{R_0^2}\frac{1}{a^2}$$

We just finished discussing an empty universe, obtained by setting $\varepsilon = 0$.

Yet another way of simplifying the Friedmann equation is to set $\kappa = 0$ (therefore, stipulating a flat universe), and demand that the universe contain only a single component, with a single value of w.

In such a spatially flat ($\kappa = 0$), single-component universe, the Friedmann equation reduces to the simple form:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\,\varepsilon \qquad (5.39.\text{a})$$

and since, from equation (5.9), we have: $\varepsilon = \varepsilon_0\, a^{-3(1+w)} = \varepsilon_0\, a^{-3-3w}$, multiplication of equation (5.39.a) by a^2 on both sides further simplifies the Friedmann equation to

$$\dot{a}^2 = \frac{8\pi G}{3c^2}\,\varepsilon_0\, a^{-3-3w}\, a^2$$

from which we finally obtain

$$\dot{a}^2 = \frac{8\pi G\varepsilon_0}{3c^2}\, a^{-(1+3w)} \qquad (5.39)$$

We can solve equation (5.39) by guessing a power law solution of the form $a \propto t^q$. Substitution of this form for a in equation (5.39) then gives

$$\left[\frac{d}{dt}\Big(t^q\Big)\right]^2 = \frac{8\pi G\varepsilon_0}{3c^2}\, t^{-(1+3w)q}$$

so that

$$\Big[q\, t^{q-1}\Big]^2 = \frac{8\pi G\varepsilon_0}{3c^2}\, t^{-(1+3w)q}$$

Equating exponents, we get

$$(q-1)2 = -(1+3w)q$$

or

$$2q + q + 3wq = 2$$

so that

$$q = \frac{2}{3+3w} \qquad (5.40)$$

with the restriction $w \neq -1$ (which means that we will have to go back to the Friedmann equation to treat the case for a universe with only a cosmological constant, as done later in this lecture).

Based on equation (5.40), we can write the solution to the Friedmann equation (5.39) for a spatially flat universe as

$$a(t) = \left(\frac{t}{t_0}\right)^{2/(3+3w)} \tag{5.41}$$

It remains to determine how the normalization factor t_0 is connected to the quantities in equation (5.39).

To find the connection to the terms in equation (5.39) for t_0, let us differentiate equation (5.41) with respect to t:

$$\dot{a} = \frac{1}{t_0^{2/(3+3w)}}\left[\frac{2}{3+3w}\ t^{\frac{2}{3+3w}-1}\right]$$

$$= \frac{1}{t_0^{2/(3+3w)}}\left[\frac{2}{3(1+w)}\ t^{\frac{2-3-3w}{3+3w}}\right]$$

$$= \frac{1}{t_0^{2/(3+3w)}}\left[\frac{2}{3(1+w)}\ t^{-(1+3w)/(3+3w)}\right] \tag{5.42.a}$$

$$\Rightarrow \quad \dot{a}^2 = \frac{1}{t_0^{4/(3+3w)}}\left[\frac{4}{9(1+w)^2}\ t^{-2(1+3w)/(3+3w)}\right] \tag{5.42.b}$$

Now, to get the exponent of t_0 in the same form as the exponent of t, we see that

$$\frac{4}{3+3w} - 2 = \frac{4-2(3+3w)}{3+3w} = \frac{4-6-6w}{3+3w} = \frac{-2+6w}{3+3w} = \frac{-2(1+3w)}{3+3w}$$

which means we must multiply and divide equation (5.42.b) by t_0^{-2}, that is

$$\dot{a}^2 = \frac{t_0^{-2}}{t_0^{4/(3+3w)}t_0^{-2}}\left[\frac{4}{9(1+w)^2}\ t^{-2(1+3w)/(3+3w)}\right]$$

$$= \frac{4\,t_0^{-2}}{9(1+w)^2}\ \left(\frac{t}{t_0}\right)^{-2(1+3w)/(3+3w)}$$

$$= \frac{4\,t_0^{-2}}{9(1+w)^2}\ \left[\left(\frac{t}{t_0}\right)^{2/(3+3w)}\right]^{-(1+3w)}$$

$$\Rightarrow \quad \dot{a}^2 = \frac{4\,t_0^{-2}}{9(1+w)^2}\ a^{-(1+3w)} \tag{5.42.c}$$

Comparing equations (5.39) and (5.42.c), we can set

$$\frac{4\,t_0^{-2}}{9(1+w)^2} = \frac{8\pi G\varepsilon_0}{3c^2}$$

so that

$$\frac{t_0^{-2}}{3(1+w)^2} = \frac{2\pi G\varepsilon_0}{c^2}$$

Therefore, we finally obtain the normalization factor in equation (5.41), which gives a relation between the age t_0 and the energy density in the present epoch ε_0 to be

$$t_0 = \frac{1}{1+w}\left(\frac{c^2}{6\pi G\varepsilon_0}\right)^{1/2} \tag{5.42}$$

Meanwhile, when we multiply and divide equation (5.42.a) by t_0^{-1}, we get

$$\begin{aligned}
\dot{a} &= \frac{t_0^{-1}}{t_0^{2/(3+3w)}t_0^{-1}}\left[\frac{2}{3(1+w)}\;t^{-(1+3w)/(3+3w)}\right] \\
&= \frac{2\,t_0^{-1}}{3(1+w)}\;\left(\frac{t}{t_0}\right)^{-(1+3w)/(3+3w)} \\
\Rightarrow \qquad \frac{\dot{a}}{a} &= \frac{2\,t_0^{-1}}{3(1+w)}\;\left(\frac{t}{t_0}\right)^{-(1+3w)/(3+3w)}\;\frac{1}{a} \\
&= \frac{2\,t_0^{-1}}{3(1+w)}\;\left(\frac{t}{t_0}\right)^{-(1+3w)/(3+3w)}\;\left(\frac{t}{t_0}\right)^{-2/(3+3w)} \\
\Rightarrow \qquad \frac{\dot{a}}{a} &= \frac{2\,t_0^{-1}}{3(1+w)}\;\left(\frac{t}{t_0}\right)^{-1}
\end{aligned}$$

Therefore, the Hubble constant in such a universe is given by

$$H_0 \equiv \left(\frac{\dot{a}}{a}\right)_{t=t_0} = \frac{2\,t_0^{-1}}{3(1+w)} \tag{5.43}$$

so that the age of the universe, in terms of the Hubble time, is given by

$$t_0 = \frac{2\,H_0^{-1}}{3(1+w)} \tag{5.44}$$

Equation (5.44) tells us that in a spatially flat universe:

- If $w > -1/3$, the universe is younger than the Hubble time.
- If $w < -1/3$, the universe is older than the Hubble time.

Next, recall that the energy density of a component with equation-of-state parameter w is given by

$$\varepsilon(a) = \varepsilon_0 \, a^{-3(1+w)} \tag{5.45}$$

Substituting equation (5.41) for a, this can be written as

$$\varepsilon(t) = \varepsilon_0 \left[\left(\frac{t}{t_0} \right)^{2/3(1+w)} \right]^{-3(1+w)}$$

from which we obtain

$$\varepsilon(t) = \varepsilon_0 \left(\frac{t}{t_0} \right)^{-2} \tag{5.46}$$

no matter what the value of w.

Now, recall that we defined the critical energy density in a spatially flat universe in equation (4.25) in an earlier lecture as

$$\varepsilon_0 = \varepsilon_{c,0} = \frac{3c^2}{8\pi G} \, H_0^2 \tag{5.47}$$

Substituting equations (5.47) and (5.44) in equation (5.46), we get

$$\begin{aligned}
\varepsilon(t) &= \left(\frac{3c^2}{8\pi G} \, H_0^2 \right) \frac{t^{-2}}{t_0^{-2}} \\
&= \left(\frac{3c^2}{8\pi G} \, H_0^2 \right) t^{-2} \left[\frac{3(1+w)}{2H_0^{-1}} \right]^{-2} \\
&= \left(\frac{3c^2}{8\pi G} \, H_0^2 \right) t^{-2} \left[\frac{4H_0^{-2}}{9(1+w)^2} \right]
\end{aligned}$$

$$\Rightarrow \quad \varepsilon(t) = \frac{1}{6\pi \, (1+w)^2} \, \frac{c^2}{G} \, t^{-2} \tag{5.49}$$

Equation (5.49) gives the energy density as a function of time in a spatially flat universe containing only a single component with equation-of-state parameter w.

If we observe a distant source with redshift z, then equation (5.41) gives

$$1 + z = \frac{a(t_0)}{a(t_e)} = \left(\frac{t_0}{t_e} \right)^{2/(3+3w)} \tag{5.51}$$

Equation (5.51) implies that

$$\left(1+z\right)^{(3+3w)/2} = \frac{t_0}{t_e}$$

so that t_e is given by

$$t_e = \frac{t_0}{(1+z)^{3(1+w)/2}}$$

Substituting t_0 from equation (5.44), we finally obtain the time t_e at which the light from the distant galaxy was emitted:

$$t_e = \frac{2}{3\,(1+w)\,H_0}\,\frac{1}{(1+z)^{3(1+w)/2}} \tag{5.52}$$

The proper distance to the galaxy *at the current time* is then given by

$$\begin{aligned}
d_p(t_0) &= c\int_{t_e}^{t_0}\frac{dt}{a(t)} \\
&= c\int_{t_e}^{t_0}\frac{dt}{(t/t_0)^{2/(3+3w)}} \qquad \text{from equation (5.41)} \\
&= c\,t_0^{2/(3+3w)}\int_{t_e}^{t_0} t^{-2/(3+3w)}\,dt \\
&= c\,t_0^{2/(3+3w)}\left[\frac{t^{\{-2/(3+3w)\}+1}}{\{-2/(3+3w)\}+1}\right]_{t_e}^{t_0} \\
&= c\,t_0^{2/(3+3w)}\left[\frac{t^{(-2+3+3w)/(3+3w)}}{(-2+3+3w)/(3+3w)}\right]_{t_e}^{t_0} \\
&= c\,t_0^{2/(3+3w)}\left[\frac{t_0^{(1+3w)/(3+3w)} - t_e^{(1+3w)/(3+3w)}}{(1+3w)/(3+3w)}\right]_{t_e}^{t_0} \\
&= c\,\frac{(3+3w)}{(1+3w)}\;t_0^{2/(3+3w)}\;t_0^{(1+3w)/(3+3w)}\left[1-\left(\frac{t_e}{t_0}\right)^{(1+3w)/(3+3w)}\right] \\
&= c\,\frac{3(1+w)}{(1+3w)}\;t_0^{\cancel{(2+1+3w)/(3+3w)}}\left[1-\left(\frac{t_e}{t_0}\right)^{(1+3w)/(3+3w)}\right]
\end{aligned}$$

$$\Rightarrow \quad d_p(t_0) = c\,t_0\,\frac{3(1+w)}{1+3w}\left[1-\left(\frac{t_e}{t_0}\right)^{(1+3w)/(3+3w)}\right] \tag{5.53}$$

Equation (5.53) can be modified to be in terms of H_0 and z by using equation (5.44) for t_0 and (5.52) for t_e/t_0:

$$d_p(t_0) = c\left[\left\{\frac{2}{3(1+w)}\right\} H_0^{-1}\right] \frac{3(1+w)}{1+3w}\left[1-\left\{\frac{1}{(1+z)^{(3+3w)/2}}\right\}^{(1+3w)/(3+3w)}\right]$$

so that the proper distance in the current epoch in terms of H_0 and z is given by

$$d_p(t_0) = \frac{c}{H_0}\frac{2}{1+3w}\left[1-(1+z)^{-(1+3w)/2}\right] \tag{5.54}$$

The most distant object we can see (in theory) is one for which the light emitted at $t = 0$ is just now reaching us at $t = t_0$. The proper distance (at the time of observation, usually also in the current epoch) to such an object is called the *horizon distance*.

> Ryden gives the analogy that when we stand at a location on Earth, the horizon is a circle centered on us beyond which we cannot see due to the Earth's curvature. Similarly, in the universe the horizon is a spherical surface centered on us, beyond which we cannot see because light from more distant objects has not had enough time to reach us.

In a universe described by a Robertson-Walker metric, we can use the definition of proper distance with limits of integration from 0 to the current time t_0 to define the current horizon distance:

$$d_{\text{hor}}(t_0) = c\int_0^{t_0}\frac{dt}{a(t)} \tag{5.55}$$

Equation (5.55) tells us that we can compute $d_{\text{hor}}(t_0)$ from equation (5.53) by setting $t_e \to 0$, or equivalently by setting $z \to \infty$ in equation (5.54), to get

$$d_{\text{hor}}(t_0) = ct_0\frac{3(1+w)}{1+3w} = \frac{c}{H_0}\frac{2}{1+3w} \tag{5.56}$$

- Equation (5.56) implies that in a spatially flat universe, the horizon distance has a finite value if $w > -1/3$. Therefore, in a flat universe dominated by matter ($w = 0$) or by radiation ($w = 1/3$), an observer can see only a finite portion of the infinite volume of the universe. The portion of the universe lying within the horizon for a particular observer is usually referred to as the *visible universe* for that observer. The visible universe consists of all points in space which have had sufficient time to send information, in the form of photons or other relativistic particles, to the observer. In other words, the visible universe consists of all points which are causally connected to the observer.

- Meanwhile, in a flat universe with $w < -1/3$, the horizon distance is infinite, and all of space is causally connected to the observer. In such a universe, we would see every point, assuming the universe was transparent. For very distant points, we would see extremely redshifted versions of the way they looked early in the history of the universe.

Single Component Universes: Matter Only

We will now look at examples of spatially flat universes, starting with an universe containing only non-relativistic matter ($w = 0$), sometimes called an Einstein-de Sitter universe. The expressions for the age t_0, horizon distance $d_{\rm hor}(t_0)$, scale factor $a_m(t)$, proper distance to a galaxy with redshift z at the time of observation, $d_p(t_0)$, and at the time of emission, $d_p(t_e)$, for such a universe are shown in the table below. The equation numbers and expressions for the general case of spatially flat universes (from the discussion above) are listed first, followed by the expressions for spatially flat universes with matter only ($w = 0$).

General Expressions for Spatially Flat Universes		Spatially Flat Universes with Matter only ($w = 0$)	
(5.44) : $t_0 = \dfrac{2\,H_0^{-1}}{3(1+w)}$	$\Rightarrow$	$t_0 = \dfrac{2}{3H_0}$	(5.57)
(5.56) : $d_{\rm hor}(t_0) = ct_0\,\dfrac{3(1+w)}{1+3w} = \dfrac{c}{H_0}\,\dfrac{2}{1+3w}$	$\Rightarrow$	$d_{\rm hor}(t_0) = 3ct_0 = \dfrac{2c}{H_0}$	(5.58)
(5.41) : $a(t) = \left(\dfrac{t}{t_0}\right)^{2/(3+3w)}$	$\Rightarrow$	$a_m(t) = \left(\dfrac{t}{t_0}\right)^{2/3}$	(5.59)
(5.54) : $d_p(t_0) = \dfrac{c}{H_0}\,\dfrac{2}{1+3w}\left[1 - \dfrac{1}{(1+z)^{(1+3w)/2}}\right]$	$\Rightarrow$	$d_p(t_0) = \dfrac{2c}{H_0}\left[1 - \dfrac{1}{\sqrt{1+z}}\right]$	(5.60)
$d_p(t_e) = \dfrac{d_p(t_0)}{1+z}$	$\Rightarrow$	$d_p(t_e) = \dfrac{2c}{H_0(1+z)}\left[1 - \dfrac{1}{\sqrt{1+z}}\right]$	(5.61)

Equation (5.59) for the scale factor tells us that such a flat universe containing only matter expands forever. Such a fate is sometimes called the “Big Chill,” since the temperature of the universe decreases with time as the universe expands.

It is also interesting to plot $d_p(t_0)$ and $d_p(t_e)$ vs. the redshift z. The plots are shown by the dotted line in Figure 5.3 in your text. Notice that $d_p(t_e)$ reaches a maximum, and then decreases for higher-z galaxies, implying that objects at very high redshift are seen as they were very early in the history of the universe. One can easily determine this maximum by setting the time derivative $\dot{d}_p(t_e) = 0$, following which we find that $d_p(t_e)$ has a maximum for galaxies with a redshift $z = 5/4$, for which $d_p(t_e) = (8/27)\,c/H_0$.

Single Component Universes: Radiation Only

The next example of a spatially flat universe that we will consider is that of a universe containing only radiation. Such a case is of particular interest for us since, as discussed in an earlier lecture, the radiation term was dominant early in the history of our own universe. The expressions for the age t_0, horizon distance $d_{\rm hor}(t_0)$, scale factor $a_r(t)$, proper distance to a galaxy with redshift z at the time of observation, $d_p(t_0)$, and at the time of emission, $d_p(t_e)$, for such a universe are shown in the table below. Again, the equation numbers and expressions for the general case of spatially flat universes are listed first, followed by the expressions for spatially flat universes with radiation only ($w = 1/3$).

General Expressions for Spatially Flat Universes		Spatially Flat Universes with Radiation only ($w = 1/3$)	
$(5.44) : t_0 = \dfrac{2\,H_0^{-1}}{3(1+w)}$	$\Rightarrow$	$t_0 = \dfrac{1}{2H_0}$	(5.62)
$(5.56) : d_{\rm hor}(t_0) = ct_0\,\dfrac{3(1+w)}{1+3w} = \dfrac{c}{H_0}\,\dfrac{2}{1+3w}$	$\Rightarrow$	$d_{\rm hor}(t_0) = 2ct_0 = \dfrac{c}{H_0}$	(5.63)
$(5.41) : a(t) = \left(\dfrac{t}{t_0}\right)^{2/(3+3w)}$	$\Rightarrow$	$a_r(t) = \left(\dfrac{t}{t_0}\right)^{1/2}$	(5.64)
$(5.54) : d_p(t_0) = \dfrac{c}{H_0}\,\dfrac{2}{1+3w}\left[1 - \dfrac{1}{(1+z)^{(1+3w)/2}}\right]$	$\Rightarrow$	$d_p(t_0) = \dfrac{c}{H_0}\left[1 - \dfrac{1}{1+z}\right]$	(5.65)
$d_p(t_e) = \dfrac{d_p(t_0)}{1+z}$	$\Rightarrow$	$d_p(t_e) = \dfrac{c}{H_0}\left[\dfrac{z}{(1+z)^2}\right]$	(5.66)

Equation (5.63) shows that for a flat, radiation-only universe, the horizon distance is exactly equal to the Hubble distance (which is not generally the case).

Once again, the plots of $d_p(t_0)$ and $d_p(t_e)$ vs. the redshift z are shown in Figure 5.3 in your text, by the solid lines for this case (radiation-only). As usual, $d_p(t_e)$ reaches a maximum, and then decreases for higher-z galaxies, implying that objects at very high redshift are seen as they were very early in the history of the universe. Again, one can determine this maximum by setting the time derivative $\dot{d}_p(t_e) = 0$, following which we find that $d_p(t_e)$ has a maximum for galaxies with a redshift $z = 1$, for which $d_p(t_e) = (1/4)\,c/H_0$ for spatially flat universes containing only radiation.

Single Component Universes: Lambda Only

The solution $a(t) = (t/t_0)^{2/(3+3w)}$ that we have written for spatially flat universes containing only matter, or containing only radiation, does not work for $w = -1$. Therefore, if the energy density is contributed by a cosmological constant Λ only, we must go back to the Friedmann equation. Recall that we wrote the Friedmann equation in (5.1):

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\,\varepsilon - \frac{\kappa c^2}{R_0^2\,a^2}$$

For a flat universe ($\kappa = 0$) dominated by a cosmological constant Λ, the Friedmann equation takes the form

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\,\varepsilon_\Lambda \tag{5.75.a}$$

where ε_Λ is constant with time.

Now, $\dfrac{\dot{a}}{a} = H(t)$, so that $\left(\dfrac{\dot{a}}{a}\right)_{t=t_0} = H_0$.

So, if ε_Λ is constant with time, we can just choose to set H_0 equal to (the square root of) the right hand side of equation (5.75.a), to get

$$H_0 = \left(\frac{8\pi G\,\varepsilon_\Lambda}{3c^2}\right)^{1/2} \tag{5.77}$$

Moreover, for this case (of a spatial universe containing only the cosmological constant Λ), we have

$$\frac{\dot{a}}{a} = H_0 \tag{5.76}$$

Writing this as

$$\frac{da}{a} = H_0\,dt$$

and integrating, we obtain

$$\ln a = H_0\,t + C_I \tag{5.78.a}$$

where C_I is a constant of integration.

Now, at $t = t_0$, we have $a = a_0 = 1$, so

$$\ln a_0 = 0 = H_0\,t_0 + C_I \qquad \Rightarrow \qquad C_I = -H_0\,t_0$$

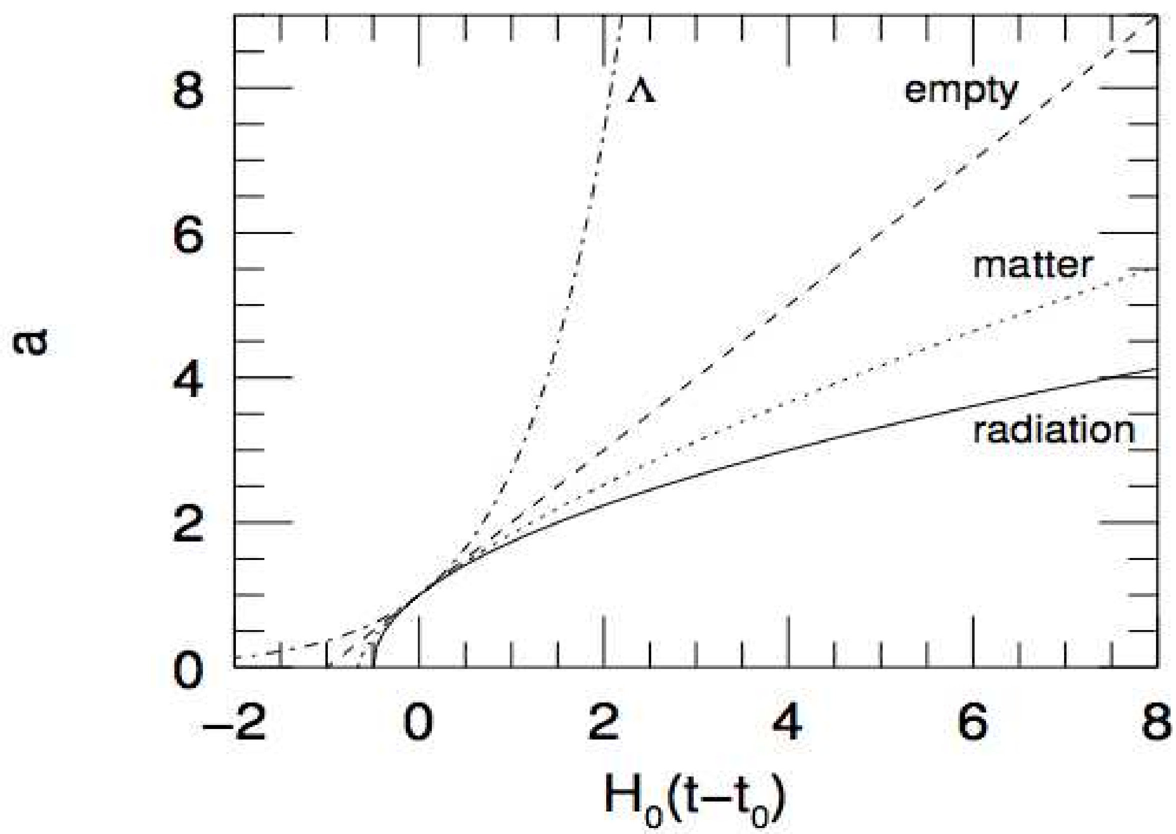
Substituting for C_I in (5.78.a), we get

$$\ln a = H_0\left(t - t_0\right)$$

so that the solution to equation (5.76) is

$$a(t) = e^{H_0\,(t-t_0)} \tag{5.78}$$

The scale factor in equation (5.78) as a function of time for a spatially flat universe containing only the cosmological constant Λ is shown in Figure 5.2 of your text, which is reproduced below.



Note that the plot above has been normalized to have the same scale factor and expansion rate in the current epoch for all the four cases shown. In other words, for $t = t_0$, we get $a(t_0) = 1$ for $t = t_0$ in the current epoch; the horizontal (x) axis is actually labeled in terms of the exponent in equation (5.78), hence in terms of $H_0\,(t - t_0)$. That is why the current epoch ($t = t_0$) corresponds to the "0" value on the horizontal axis.

It is clear from equation (5.78) and the graph above that a spatially flat universe containing only a cosmological constant expands exponentially. Also of interest in the above plot is that the curve for Λ asymptotically approaches the horizontal axis, implying that a spatially flat universe containing nothing but a cosmological constant is infinitely old, and has an infinite horizon distance d_{hor}.

It is also worth looking at the other three cases represented on the graph that we have already studied.

- From equation (5.30), the scale factor in an empty universe with curvature only is given by $a(t) = t/t_0$, and this linear dependence is shown in the above plot.
- In a spatially flat universe containing only matter, the scale factor is given by equation (5.59): $a_m(t) = (t/t_0)^{2/3}$, and is shown in the above plot to be less steep than the linear case.
- In a spatially flat universe containing only radiation, the scale factor is given by equation (5.64): $a_r(t) = (t/t_0)^{1/2}$, and is shown in the above plot to be the least steep of all the cases.

If we observe an object with a redshift z in a spatially flat universe containing only a cosmological constant Λ, then the proper distance to the object at the time of observation would be

$$d_p(t_0) = c \int_{t_e}^{t_0} \frac{dt}{a(t)} = c \int_{t_e}^{t_0} \frac{dt}{e^{H_0\,(t-t_0)}} = c \int_{t_e}^{t_0} e^{H_0\,(t_0-t)}\,dt$$

We can then pull $e^{H_0\,t_0}$ outside the integral and integrate the remaining term:

$$\begin{aligned}
d_p(t_0) = c\,e^{H_0\,t_0} \int_{t_e}^{t_0} e^{-H_0\,t}\,dt = c\,e^{H_0\,t_0} \left[\frac{e^{-H_0\,t}}{-H_0}\right]_{t_e}^{t_0} &= c\,e^{H_0\,t_0} \left[\frac{e^{-H_0\,t_0} - e^{-H_0\,t_e}}{-H_0}\right] \\
&= c\,e^{H_0\,t_0} \left[\frac{e^{-H_0\,t_e} - e^{-H_0\,t_0}}{H_0}\right] \\
&= \frac{c}{H_0}\left[e^{H_0\,(t_0-t_e)} - 1\right] \\
&= \frac{c}{H_0}\left[\frac{1}{a(t_e)} - 1\right] \qquad \text{from (5.78)} \\
&= \frac{c}{H_0}\left[(1+z) - 1\right] \\
\Rightarrow \quad d_p(t_0) &= \frac{c}{H_0}\,z
\end{aligned} \tag{5.79}$$

Meanwhile, the proper distance at the time of emission is

$$d_p(t_e) = \frac{d_p(t_0)}{1+z} = \frac{c}{H_0}\,\frac{z}{1+z} \tag{5.80}$$

These are plotted in Figure 5.3 in your text by the dot-dashed lines.

- Note that an exponentially growing universe, such as the spatially flat Λ-dominated model, is the only universe for which $d_p(t_0)$ is linearly proportional to z, for all values of z. In other universes, the relation $d_p(t_0) \propto z$ only holds true in the limit $z \ll 1$.
- Also note that in the limit $z \to \infty$, we have $d_p(t_0) \to \infty$, but $d_p(t_e) \to c/H_0$ from equations (5.79) and (5.80) respectively. This means that in a spatially flat, Λ-dominated universe, highly redshifted objects ($z \gg 1$) are at very large distances ($d_p(t_0) \gg c/H_0$) at the time of observation; the observer sees them as they were just before they reached a proper distance c/H_0.

In the next lecture, we will consider more complicated behavior by putting multiple terms in the right hand side of the Friedmann equation.