PHY 475/375

Lecture 7

(April 16, 2012)

The Fluid and Acceleration Equations

While the Friedmann equation is an important part of the effort to study the Universe, it is not enough. Even with accurate boundary conditions, e.g., precise values for H_0 and ε_0, the Friedmann equation still contains two unknowns in $a(t)$ and $\varepsilon(t)$. Or, if we like, we can say it contains two unknowns $H(t)$ and $\Omega(t)$ — the key point is that there are still two unknowns. Therefore, we need another equation connecting $a(t)$ and $\varepsilon(t)$.

To find another equation, let us look at the first law of thermodynamics

$$dQ = dE + PdV \tag{4.32}$$

where dQ is the amount of heat flowing into or out of a region, dE is the change in internal energy, P is the pressure, and dV is the change in volume of the region. Recall that we had applied this in an earlier lecture to a photon gas to study aspects of the Cosmic Microwave Background. Now, we will apply it to a comoving volume of the Universe.

If the universe is perfectly homogenous, then there should not be a bulk flow of heat into or out of any volume. Applying this to a comoving volume (recall that this means a volume that is expanding with the universe), we set $dQ = 0$ in equation (4.32), and write its time derivative as:

$$\dot{E} + P\dot{V} = 0 \tag{4.33}$$

where we are using the usual notation that $dx/dt = \dot{x}$.

Consider now a sphere of comoving radius r_s expanding along with the Universe, so that its proper radius is

$$R_s(t) = a(t)\, r_s$$

The volume of this sphere is

$$V(t) = \frac{4\pi}{3}\, R_s(t)^3 = \frac{4\pi}{3}\, r_s^3\, a(t)^3 \tag{4.34}$$

Its time derivative is

$$\begin{aligned} \dot{V} &= \frac{4\pi}{3}\, r_s^3 \Big(3a^2\dot{a}\Big) \\ &= \frac{4\pi}{3}\, r_s^3 a^3 \left(\frac{3\dot{a}}{a}\right) \qquad (4.34.a) \end{aligned}$$

where we have suppressed writing (t) after the quantities, and will continue to do so in most of the equations below, for better visibility.

Substituting equation (4.34) into the above equation (4.34.a), we get

$$\dot{V} = V\left(\frac{3\dot{a}}{a}\right) \tag{4.35}$$

Meanwhile, since $\varepsilon(t)$ is the energy density, the internal energy of the sphere of volume $V(t)$ can be written as

$$E(t) = V(t)\,\varepsilon(t) \tag{4.36}$$

so that its time derivative is

$$\begin{aligned}\dot{E} &= V\,\dot{\varepsilon} + \dot{V}\,\varepsilon \\ &= V\,\dot{\varepsilon} + \underbrace{V\left(\frac{3\dot{a}}{a}\right)}_{\text{from eq. (4.35)}}\,\varepsilon\end{aligned}$$

$$\Rightarrow \quad \dot{E} = V\left(\dot{\varepsilon} + \frac{3\dot{a}}{a}\,\varepsilon\right) \tag{4.37}$$

Substituting from equation (4.33) for $\dot{E}$, this becomes

$$-P\,\dot{V} = V\left(\dot{\varepsilon} + \frac{3\dot{a}}{a}\,\varepsilon\right)$$

Inserting $\dot{V}$ from equation (4.35), and rearranging all terms to one side, we obtain

$$V\left(\dot{\varepsilon} + \frac{3\dot{a}}{a}\,\varepsilon + \frac{3\dot{a}}{a}\,P\right) = 0 \tag{4.38}$$

Since the volume V has to be finite, we can equate the quantity within parentheses to zero, to get

$$\dot{\varepsilon} + \frac{3\dot{a}}{a}\left(\varepsilon + P\right) = 0 \tag{4.39}$$

Equation (4.39) is called the *fluid equation*, and is the second of the key equations describing the expansion of the universe.

Both the Friedmann equation and fluid equation are statements about energy conservation. By combining the two, we can derive an acceleration equation which tells us how the expansion of the universe speeds up or slows down over time. Recall that we wrote the Friedmann equation in (4.13) as:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\,\varepsilon - \frac{\kappa c^2}{R_0^2}\frac{1}{a^2}$$

Multiplying by a^2, this becomes

$$\dot{a}^2 = \frac{8\pi G}{3c^2}\,\varepsilon a^2 - \frac{\kappa c^2}{R_0^2} \tag{4.40}$$

Since the second term on the right hand side of equation (4.40) only contains constants, its time derivative is zero. So, the time derivative of equation (4.40) is

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3c^2}\Big(\dot{\varepsilon}a^2 + 2\varepsilon a\dot{a}\Big) - 0 \tag{4.41}$$

Dividing by $2\dot{a}a$, this becomes

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3c^2}\left(\dot{\varepsilon}\frac{a}{\dot{a}} + 2\varepsilon\right) \tag{4.42}$$

Meanwhile, $\dot{\varepsilon} = -\frac{3\dot{a}}{a}\Big(\varepsilon + P\Big)$ from equation (4.39), and multiplying by $a/\dot{a}$, we get

$$\dot{\varepsilon}\frac{a}{\dot{a}} = -3\Big(\varepsilon + P\Big) = -3\varepsilon - 3P$$

Substituting the above equation in (4.42), we get

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3c^2}\left(-3\varepsilon - 3P + 2\varepsilon\right)$$

which gives us the usual form of the *acceleration equation*:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\left(\varepsilon + 3P\right) \tag{4.44}$$

Note that while we tend to think of pressure as force per unit area, its dimensions are consistent with the energy density, so that the above equation has consistent units; that is, the unit of P is N m^{-2} ≡ (N.m) m^{-3} ≡ J m^{-3}, which is also the unit of the energy density ε.

Equation (4.44) leads us to some interesting conclusions:

- The pressure P is associated with the material filling the universe. A gas made of ordinary baryonic matter has a positive pressure P resulting from the random thermal motions of its constituent molecules, atoms, or ions. A gas of photons also has a positive pressure, as does a gas of neutrinos. So if the energy density ε is also positive, we have a net negative sign on the right hand side of equation (4.44). This means that we have a negative acceleration; that is, a decrease in the value of $\dot{a}$ with time, and hence a reduction in the relative velocity of any two points in the universe. In other words, a positive energy density together with a positive pressure due to baryonic matter and photons causes the expansion of the universe to slow down.

- If, however, the universe had a component with a pressure

$$P < -\frac{\varepsilon}{3} \tag{4.45}$$

then we would have a net positive sign on the right hand side of equation (4.44). This would mean a positive acceleration, an increase in the value of $\dot{a}$ with time. In other words, a component of pressure given by equation (4.45) would cause the expansion of the universe to speed up rather than slow down!!! In the mid-1990's, it was found that the expansion of the Universe was accelerating, instead of slowing down. Therefore, there must be a component in the Universe with negative pressure; it has been given the name "dark energy" (indicating that we know nothing about it). One possible explanation for the dark energy is Einstein's notorious "cosmological constant" which he had inserted into his general relativity solutions to prevent the Universe from expanding (this was done prior to Hubble's discovery that the Universe is indeed expanding). In a later section, we will discuss how the cosmological constant has $P = -\varepsilon$, which would cause a positive acceleration for the expansion of the Universe.

The Equations of State

We now have three equations which describe the dynamics of the universe: the Friedmann equation (4.13), the fluid equation (4.39), and the acceleration equation (4.44). Only two of these are independent, however, since we've just demonstrated how equation (4.44) can be derived from equations (4.13) and (4.39).

So, we now have a system of two independent equations and three unknowns: the scale factor $a(t)$, the energy density $\varepsilon(t)$, and the pressure $P(t)$. To solved for these three quantities as a function of cosmic time t, we need another equation — an equation of state, that is, a mathematical relation between the pressure and energy density.

In other words, we need a relation of the form $P = P(\varepsilon)$.

In general, it is not an easy matter to write such an equation of state. Fortunately, cosmology usually deals with dilute gases, so the equation of state can be written in a simple linear form:

$$P = w\varepsilon \tag{4.50}$$

where w is a dimensionless number.

Let us look at some concrete examples to see what values w might take.

Consider first a low density gas of non-relativistic massive particles, where the random thermal motions of the gas particles have velocities that are tiny compared to the speed of light. Such a gas obeys the ideal gas law

$$P = \frac{\rho}{\mu} kT \tag{4.51}$$

where μ is the mean mass per gas particle. The energy density of such a gas is almost entirely contributed by the mass of the gas particles, so that $\varepsilon \approx \rho c^2$, and we can rewrite equation (4.51) by multiplying and dividing by c^2 as

$$P \approx \frac{\rho c^2}{\mu c^2} kT$$

which gives

$$P \approx \frac{kT}{\mu c^2} \varepsilon \tag{4.52}$$

One way to characterize the speed of such non-relativistic molecules is through the rms (root mean square) speed, which is given by

$$v_{\rm rms} = \sqrt{\frac{3kT}{\mu}} \tag{4.53}$$

so equation (4.52) can be written with kT replaced by $\mu v_{\rm rms}^2/3$ as

$$P \approx \frac{\mu v_{\rm rms}^2}{3\mu c^2}\,\varepsilon$$

so that the equation of state for a non-relativistic gas is

$$P_{\text{non-rel gas}} = w\,\varepsilon_{\text{non-rel gas}} \tag{4.55}$$

with

$$w \approx \frac{v_{\rm rms}^2}{3c^2} \ll 1 \tag{4.56}$$

since, e.g., for nitrogen molecules in air at room temperature, $v_{\rm rms} \sim 500$ m s^{-1}, so that $w \sim 10^{-12}$; and even in astronomical contexts, in a gas of ionized hydrogen for example, the electrons are non-relativistic as long as $T \ll 6 \times 10^9$ K, and protons are non-relativistic as long as $T \ll 10^{13}$ K.

Note that w cannot take on arbitrary values. Small perturbations in a substance with pressure P will travel at the speed of sound. For adiabatic perturbations in a gas with pressure P and energy density ε, the sound speed (c_s) is given by

$$c_s^2 = c^2\left(\frac{dP}{d\varepsilon}\right) \tag{4.57}$$

But from the equation of state, $dP/d\varepsilon = w$, so in a substance with $w > 0$, the sound speed is $c_s = c\sqrt{w}$. Therefore, sound waves cannot travel faster than the speed of light, so $w \leq 1$.

It is worth tabulating some values of w that are of particular interest.

- For non-relativistic matter, as we have shown above, $w \approx 0$.
- For photons and other relativistic particles, $w = 1/3$.
- The case $w < -1/3$ is of interest because such a component will provide a positive acceleration ($\ddot{a} > 0$ in equation 4.44); such a component with $w < -1/3$ is generically referred to as *dark energy*.
- A component of the universe that has $w = -1$ (and hence has $P = -\varepsilon$) is called the *cosmological constant* (Λ). It may be one form of the dark energy.

The Cosmological Constant

When Einstein published his first paper on general relativity in 1915, the expansion of the Universe had not yet been discovered. So, Einstein (and all of his contemporaries) believed the Universe was static.

Upon writing his field equations, Einstein realized the following, which we will describe in a Newtonian context. If the mass density of the universe is ρ, then the gravitational potential Φ is given by the so-called Poisson's equation:

$$\nabla^2 \Phi = 4\pi G \rho \tag{4.58}$$

The force per unit mass, or gravitational acceleration is then given by

$$\vec{a} = -\vec{\nabla}\Phi \tag{4.59}$$

In a static universe, $\vec{a}$ must vanish everywhere in space. Then equation (4.58) leads to

$$\rho = \frac{1}{4\pi G} \nabla^2 \Phi = 0 \tag{4.60}$$

But $\rho = 0$ means there is zero mass density in the universe, that there is nothing in the universe. So, *the only possible static universe is one with nothing in it, an empty universe*!!!

This was an embarrassing situation for Einstein. His field equations had turned out to be a description of a universe that cannot exist (or even if it did, there would be no one to measure it, since it would contain nothing).

The material below and on the following pages was done in class on W (4/18) but is included here for continuity.

So, how did Einstein fix this problem? He probably looked at Maxwell's equations in electromagnetism for an idea. We know that the magnetic field can be written as the curl of a vector potential $\vec{A}$, that is, $B = \vec{\nabla} \times \vec{A}$. But if we add the gradient of an arbitrary scalar field F to $\vec{A}$, then

$$\begin{aligned}\vec{B} &= \vec{\nabla} \times \left(\vec{A} + \vec{\nabla} F\right) \\ &= \vec{\nabla} \times \vec{A} + \vec{\nabla} \times \vec{\nabla} F \\ &= \vec{\nabla} \times \vec{A}\end{aligned}$$

because the curl of the gradient of any scalar function is zero.

In other words, the addition of the gradient of a scalar field to the vector potential has no effect on the magnetic field $\vec{B}$.

Likewise, Einstein found he could add a constant to his field equations that would keep a matter-filled universe static. This new term, Λ, came to be known as the *cosmological constant*.

With this new term added to Einstein's field equations, the Friedmann equation becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\,\varepsilon - \frac{\kappa c^2}{R_0^2 a^2} + \frac{\Lambda}{3} \tag{4.62}$$

while the acceleration equation becomes

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\Big(\varepsilon + 3P\Big) + \frac{\Lambda}{3} \tag{4.64}$$

Meanwhile, the fluid equation is not affected by the presence of a Λ term, so

$$\dot{\varepsilon} + 3\,\frac{\dot{a}}{a}\Big(\varepsilon + P\Big) = 0 \tag{4.63}$$

Rewriting equation (4.62) as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\left[\varepsilon + \frac{c^2}{8\pi G}\,\Lambda\right] - \frac{\kappa c^2}{R_0^2 a^2}$$

we see that adding the Λ term is equivalent to adding a new component to the universe with energy density

$$\varepsilon_\Lambda \equiv \frac{c^2}{8\pi G}\,\Lambda \tag{4.65}$$

Now, if Λ remains constant with time, then so does ε_Λ. So, from the fluid equation (4.63), we get

$$P_\Lambda = -\varepsilon_\Lambda = -\frac{c^2}{8\pi G}\,\Lambda \tag{4.66}$$

This means that we can think of the cosmological constant as a component of the universe which has a *constant density* ε_Λ and a *constant pressure* $P_\Lambda = -\varepsilon_\Lambda$.

Next, in a static universe, both $\dot{a}$ and $\ddot{a}$ must be equal to zero. Moreover, $P = 0$, since a static universe is also a pressure-less universe, and $\varepsilon \propto \rho c^2$ as we know already from our discussion of equation (4.51), that the energy density of a non-relativistic gas is contributed entirely by the mass of the particles. So the acceleration equation (4.64) reduces to

$$\frac{\ddot{a}}{a} = 0 = -\frac{4\pi G}{3c^2}\Big(\rho c^2 + \underbrace{3P}_{=0}\Big) + \frac{\Lambda}{3}$$

or

$$0 = -\frac{4\pi G}{3c^2}\,\rho + \frac{\Lambda}{3} \tag{4.67}$$

So, Einstein had to set $\Lambda = 4\pi G\rho$ in order to get a static universe.

With $\Lambda = 4\pi G\rho$, $\varepsilon = \rho c^2$, and $\dot{a} = 0$, the Friedmann equation (4.62) becomes

$$0 = \frac{8\pi G}{3c^2}\rho c^2 - \frac{\kappa c^2}{R_0^2 a^2} + \frac{4\pi G\rho}{3}$$

from which we get

$$0 = \frac{(8+4)\pi G\rho}{3} - \frac{\kappa c^2}{R_0^2 a^2}$$

Rearranging terms, we get

$$4\pi G\rho = \frac{\kappa c^2}{R_0^2 a^2}$$

so that we can write

$$R_0 = \sqrt{\frac{\kappa c^2}{4\pi G\rho\, a^2}} = \sqrt{\frac{\kappa c^2}{4\pi G\rho}}$$

where we have written $a = 1$; since $\dot{a} = 0$, a must be constant, and so we might as well take the constant to be the value of a in the current epoch (recall that $a(t_0) = 1$).

The only possible value of κ is then $+1$, since $\kappa = -1$ would make R_0 imaginary, and $\kappa = 0$ would make $R_0 = 0$, which makes no sense. So we get finally that

$$R_0 = \frac{c}{\Lambda^{1/2}} \tag{4.69}$$

as the radius of curvature of Einstein's static model.

Einstein was never satisfied with having to insert a term by hand into his elegant theory, and promptly dropped the cosmological constant when it was found that the Universe was expanding. However, in recent years, the cosmological constant has made a comeback as one of the possible explanations for the dark energy.