PHY 475/375
Lecture 4
(April 4, 2012)

The first 45 minutes of this lecture were spent on discussing aspects about the Cosmic Microwave Background; the notes have been written into the posted notes for April 2, to preserve continuity with the material for that day.

The Geometry of the Universe

Consider the four fundamental forces.

- The strong (nuclear) force is responsible for holding the nuclei of atoms together against the enormous repulsion of the protons, so as its name implies, it is very strong! However, it is a very short range force and acts only over distances $\sim 10^{-15}$ m (about the size of a nucleus). So while it plays a role in holding together the constituents of the Universe on the smallest scales, it plays no role on the large scales that are the domain of cosmology.

- The electromagnetic force is responsible for electric and magnetic effects, such as the force between charged particles and the interactions of magnets. On small scales, the forces of attraction and repulsion between electric charges are responsible for holding atoms and molecules together, and dominate over all the other forces. The range of the electromagnetic force is infinite, but the Universe is electrically neutral on large scales, so there are no electrostatic forces on cosmological scales. Moreover, intergalactic magnetic fields are sufficiently small that magnetic forces are also negligibly tiny on cosmological scales.

- The weak (nuclear) force is responsible for radioactive decays and neutrino interactions. Without it, one of the critical steps in the proton-proton cycle by which the Sun produces energy would not be able to take place. As its name implies, it is about 10^{-6} times weaker than the strong (nuclear) force. It is also a short range force like the strong force, but has an even smaller range than the strong force of about 10^{-18} m.

- The force of gravity is the weakest force, about 10^{-39} weaker than the strong (nuclear) force, but it can act over very large distances. Moreover, it is always attractive and acts between any two objects in the Universe that have mass. Therefore, on the cosmological scales of 100 Mpc, or above, it is gravity that plays the dominant role in determining the evolution of the Universe.

We can think of gravity in either the Newtonian or Einsteinian pictures, and implicit to talking about gravity as a force is that we are adopting the Newtonian picture. The Newtonian view characterizes gravity as a force which causes objects possessing mass to be accelerated. On the other hand, Einstein postulated gravity as a manifestation of the curvature of space-time. Before delving into these details, the idea of space-time itself deserves some investigation.

In the first of many seminal works that would transform the discipline of Physics, Einstein in 1905 postulated the Special Theory of Relativity. In it, he introduced a fundamental change for viewing physical space and time, which was now unified as 4-dimensional space-time. He postulated that the speed of light is the same for all observers, regardless of their motion relative to the source of the light. A second postulate that all observers moving at constant speed should observe the same physical laws established the equivalence of all unaccelerated frames of reference (inertial observers). From these postulates, Einstein showed that the length of any object in a moving frame will appear to be foreshortened (contracted) in the direction of motion, and a clock in a moving frame will seem to be moving slow or "dilated" so that the time will always be shortest as measured in its rest frame ("proper time"). Another equally revolutionary insight was that matter and energy are related, even equivalent, as summed in the now famous equation: $E = mc^2$.

Einstein's General Theory of Relativity introduced an even more fundamental change in viewing space-time and matter/energy. In it, he postulated the equivalence of ***all*** frames of reference (including accelerated ones).

- Via the equivalence principle, there is no way for an observer to distinguish locally between gravity and acceleration.
- The presence of mass/energy determines the geometry of space-time, whereas the geometry of space-time determines the motion of mass/energy. Therefore, gravity is a purely geometric consequence of the properties of space-time.

Your text provides an excellent discussion of the above ideas, so the discussion won't be repeated here; please read pages 26-30 of your text for details.

Describing Curvature

In order to develop a mathematical theory of general relativity, Einstein needed a way of mathematically describing curvature. In other words, on large scales, he needed to find a space-time manifold — a shape in 3 spatial dimensions and 1 time dimension. In order to impose isotropy and homogeneity, this manifold must look the same in all directions and in all places at any given time.

In order to assist with the visualization of such a manifold (decidedly difficult in 4-D), let us begin by considering ways to describe curvature in 2-dimensions, and then extrapolate to higher dimensions.

Let us reminds ourselves of the 2-dimensional surfaces that satisfy isotropy and homogeneity.

- An infinite plane looks the same everywhere at any point, looks the same in every direction
- A sphere, since it doesn't have any special points anywhere
- Another such 2-D surface that may not be so obvious is a saddle (or "hyperbolic paraboloid" in technical terms)

Let us now look at each of these in more detail.

Infinite Plane: An infinite plane is the simplest of 2-D surfaces, and Euclidean geometry holds on it. The geodesic on a plane is a straight line.

> We will be using the word "geodesic" a lot, so it is important to know what it means. A geodesic is simply the shortest distance between two points. On a flat surface, it is easy to pick out a geodesic — it will be a straight line. In other geometries, picking out a geodesic might not be so easy.

If we construct a triangle on a plane by connecting 3 points with geodesics (i.e., straight lines, in this case), the angles at the vertices of the triangle (α, β, and γ) obey the relation

$$\alpha + \beta + \gamma = \pi \tag{3.5}$$

where the angles are measured in radians.

We can set up a cartesian coordinate system on the plane, and assign to every point on the plane a coordinate (x, y).

The distance ds between two points (x, y) and $(x + dx, y + dy)$ on the plane can then be obtained using the Pythagorean theorem as

$$\left(ds\right)^2 = \left(dx\right)^2 + \left(dy\right)^2$$

Since it is inconvenient to keep writing the parenthesis, one usually tends to write this as

$$ds^2 = dx^2 + dy^2 \tag{3.6}$$

with the understanding that $ds^2 = \left(ds\right)^2$, and not $d(s^2)$. ***We will use this convention throughout the quarter.***

Alternatively, stating that equation (3.6) holds true everywhere in a 2-D space *is equivalent to saying that the space is a plane.*

Instead of cartesian coordinates, if we use a polar coordinate system, a point on the plane will be specified by the coordinates (r, θ), and the distance ds between the points (r, θ) and $(r+dr, \theta+d\theta)$ will be

$$ds^2 = dr^2 + r^2 d\theta^2 \tag{3.7}$$

Even though equations (3.6) and (3.7) look different, they both represent the same flat geometry. One can be transformed into the other by substituting $x = r\cos\theta, y = r\sin\theta$.

Sphere: Next, consider the surface of a sphere. On the surface of a sphere, the geodesic (i.e., minimum distance between two points) is a portion of a great circle, that is, a circle whose center corresponds to the center of the sphere; if you've ever looked at flight paths of airlines, especially for international travel, you should know this already.

If we construct a triangle on the surface of the sphere by connecting three points with geodesics (i.e., portions of great circles, in this case), the angles at the vertices of the triangle (α, β, and γ) obey the relation

$$\alpha + \beta + \gamma = \pi + \frac{A}{R^2} \tag{3.8}$$

where A is the area of the triangle, and R is the radius of the sphere.

All spaces in which $\alpha + \beta + \gamma > \pi$ are called *positively curved* spaces. The surface of a sphere is a positively curved 2-D space.

Moreover, the surface of a sphere is a space where the curvature is homogenous and isotropic; no matter where we draw a triangle on the surface of a sphere, or how we orient it, it will always satisfy equation (3.8).

Again, instead of cartesian coordinates, we can use a polar coordinate system. Setting one up on a sphere involves a little more work, however. We need to pick a pair of antipodal points on the sphere, which we will designate as the *north pole* and the *south pole*. Then, we must pick a geodesic from the designated north pole to the south pole to be the *prime meridian*. If r is the distance from the north pole, and θ is the azimuthal angle measured relative to the prime meridian, then the distance ds between the points (r, θ) and $(r + dr, \theta + d\theta)$ will be given by the relation

$$ds^2 = dr^2 + R^2 \sin^2\left(\frac{r}{R}\right) d\theta^2 \tag{3.9}$$

where, again, R is the radius of the sphere.

Note that, unlike an infinite plane whose surface has infinite area, the surface of a sphere has a finite area equal to $4\pi R^2$. On a sphere, there is also a maximum possible distance between points; the distance between antipodal points, at the maximum possible separation, is πR. In contrast, there is no upper limit on the distance between two points on an infinite plane.

Saddle or hyperbolic paraboloid: So far, we have looked at flat spaces and positively curved spaces. In addition, there exist negatively curved spaces. An example of a negatively curved two-dimensional space is the *saddle-shape or hyperbolic paraboloid*. Such a saddle-shape has constant curvature only in the central region, near the seat of the saddle.

Now, while it is difficult to visualize a 2-D surface of constant negative curvature throughout, its properties can easily be written down mathematically. We might as well start getting used to this situation, since in 4-dimensional space-time, we will only be able to operate mathematically, and visualization won't be possible in most circumstances. Consider, therefore, a 2-D surface of constant negative curvature, with radius of curvature R. If a triangle is constructed on this surface by connecting three points with geodesics, the angles at the vertices of the triangle (α, β, and γ) will obey the relation

$$\alpha + \beta + \gamma = \pi - \frac{A}{R^2} \tag{3.10}$$

where A is the area of the triangle.

On a surface of constant negative curvature, we can set up a polar coordinate system by choosing some point as the pole, and some geodesic leading away from the pole as the prime meridian. If r is the distance from this pole, and θ is the azimuthal angle measured relative to the prime meridian, then the distance ds between the points (r, θ) and $(r + dr, \theta + d\theta)$ will be given by the relation

$$ds^2 = dr^2 + R^2 \sinh^2\left(\frac{r}{R}\right) d\theta^2 \tag{3.11}$$

where, again, R is the radius of the sphere.

Like an infinite plane, a surface of constant negative curvature has infinite area, and has no upper limit on the possible distance between points.

Relations such as (3.7), (3.9), and (3.11), which give the distance ds between two nearby points in space, are known as *metrics*.

In general, curvature is a local property. For example, a bagel (or other toroidal object) is positively curved on part of its surface, and negatively curved in other parts. However, if we want a 2-dimensional space to be homogenous and isotropic, there are only three possibilities:

- The space can be uniformly flat.
- The space can have uniform positive curvature.
- The space can have uniform negative curvature.

Thus, if a 2-dimensional space is homogenous and isotropic, its geometry can be specified by two quantities, κ and R. The number κ is called the *curvature constant*, and we have $\kappa = 0$ for a flat space, $\kappa = +1$ for a positively curved space, and $\kappa = -1$ for a negatively curved space. Meanwhile, if the space is curved, then R, which has the dimensions of length, is the radius of curvature.