PHY 475/375
Lecture 18
(May 30, 2012)

The Formation of Large Scale Structure

In this chapter, we will study the formation of large scale structure. Remember, we are not going to discuss the formation of stars and galaxies, formidable courses in their own right — yet, relative to the scale of the Universe, these would be considered "small scale" structure!

Instead, we will focus on the formation of structures larger than galaxies - clusters, superclusters, and voids.

The basic mechanism for growing large structures, such as voids and superclusters, is *gravitational instability*.

Suppose that at some time in the past, the density of the universe had slight inhomogeneities. We know, for instance, that such density fluctuations occurred at the time of last scattering, since they left their stamp on the CMB.

When the universe is matter-dominated, the over-dense regions expand less rapidly than the universe as a whole. If their density is sufficiently great, they will collapse and become gravitationally bound objects such as clusters. The dense clusters will, in addition, draw matter to themselves from the surrounding under-dense regions. We will now examine this process in greater detail.

Density Fluctuations in a Static, Pressureless Medium

To define density fluctuations, consider some component in the universe whose energy density $\varepsilon(\vec{r}, t)$ is a function of position as well as time.

At a given time t, the spatially averaged energy density is

$$\overline{\varepsilon}(t) = \frac{1}{V} \int_V \varepsilon(\vec{r}, t)\, d^3r \tag{12.1}$$

In order to ensure a true average, the volume V must be large compared to the size of the largest structure in the universe.

We define a dimensionless density fluctuation

$$\delta(\vec{r}, t) = \frac{\varepsilon(t) - \overline{\varepsilon}(t)}{\overline{\varepsilon}(t)} \tag{12.2}$$

Equation (12.2) means that δ will be negative in underdense regions, and positive in overdense regions. Its minimum possible value is $\delta = -1$ (corresponding to $\varepsilon = 0$), but it has no maximum value.

In order to study how large scale structure evolves with time, we need to know how small density fluctuations $\delta \ll 1$ grow in amplitude under gravity.

We consider a region in the universe which is approximately static and homogeneous.

In this region, add a small amount of mass within a sphere of radius R and write two relations:

- Write down a relation between the gravitational acceleration $\ddot{R}$ at the sphere's surface due to the excess mass and the density excess $\delta(t)$ within the sphere. See equations (12.3) and (12.4) in your text for details.

- Write down a relation between R and $\delta(t)$ starting from the conservation of mass within the sphere, that is, that the mass of the sphere remains constant during the collapse. See equations (12.5)-(12.10) in your text for details.

Combining these equations, you will find the equation that tells us how the (small) overdensity δ evolves as the sphere collapses to be:

$$\ddot{\delta} = 4\pi G\overline{\rho}\,\delta \tag{12.11}$$

The most general solution of equation of equation (12.11) has the form

$$\delta(t) = A_1\, e^{t/t_{\rm dyn}} + A_2\, e^{-t/t_{\rm dyn}} \tag{12.12}$$

where A_1 and A_2 depend on initial conditions in the sphere, and $t_{\rm dyn}$ gives the dynamical time for collapse:

$$t_{\rm dyn} = \frac{1}{(4\pi G\overline{\rho})^{1/2}} = \left(\frac{c^2}{4\pi G\overline{\varepsilon}}\right)^{1/2} \tag{12.13}$$

After a few dynamical times, however, only the exponentially growing term of equation (12.12) is significant.

Thus, gravity tends to make small density fluctuations in a static, *pressureless* medium grow exponentially with time.

Density Fluctuations in a Static Medium

In reality, of course, thermal pressure is always present, and acts to counter the effects of gravity.

In a star, the outward force provided by a pressure gradient exactly balances the inward force of gravity — this is known as *hydrostatic equilibrium.*

However, hydrostatic equilibrium cannot always be attained, especially in the cosmological situations we're discussing.

Instead, in cases where we don't have hydrostatic equilibrium, we can define a length scale called the *Jeans length*, given by

$$\lambda_J = c_s \left(\frac{\pi c^2}{G\overline{\varepsilon}}\right)^{1/2} = 2\pi c_s\, t_{\rm dyn} \qquad (12.14)$$

where c_s is the sound speed.

A method for setting the conditions to crudely define λ_J is given in equation (12.15)-(12.19) in your text.

Overdense regions larger than the Jeans length collapse under their own gravity, whereas overdense regions smaller than the Jeans length merely oscillate in density, and constitute stable sound waves.

For example, in the Earth's atmosphere, $c_s = 330$ km s^{-1}, and $t_{\rm dyn} \sim 9$ hr, so $\lambda_{\rm J} \sim 10^5$ km.

Since the scale height of the Earth's atmosphere is 9 km, much smaller than the Jeans length of 10^5 km, it will not collapse under gravity.

Note: To find scale height, set up a relation for how atmospheric pressure changes with height z. When you integrate, you will get a relation like $P(z) = P_0\, e^{-z/H}$ The scale height of the atmosphere is defined as H, that is, the altitude z at which the atmospheric pressure falls to 1/e of its value at the surface of the Earth.

Read pages 263-5 of your text to see how the baryonic Jeans mass decreased abruptly at the time of decoupling, allowing perturbations in baryon density to grow with amplitude.

Density Fluctuations in an Expanding Universe

Now, let us relax the static assumption and consider an expanding universe. There is now a competition between processes.

- The timescale for the growth of density perturbation by self-gravity is

$$t_{\rm dyn} \sim \left(\frac{c^2}{G\overline{\varepsilon}}\right)^{1/2}$$

  This causes overdense regions to become denser with time (in the absence of expansion).

- The timescale for expansion is

$$t_{\rm exp} \sim H^{-1} \sim \left(\frac{c^2}{G\overline{\varepsilon}}\right)^{1/2}$$

  This causes overdense regions to become less dense with time (in the absence of self-gravity)

To find an equation for $\delta(t)$ taking into account the competition between these two processes, it is easiest to consider a Newtonian analysis of perturbations. See equations (12.36)-(12.47) in your text for details.

The basic idea of a perturbation analysis is to study the effect of increasing a quantity by a small amount, so all of the subsequent discussion will be limited to the case $\delta \ll 1$.

While we won't write the Newtonian derivation and result here, we note that from a fully relativistic calculation for the growth of density perturbations, we get

$$\ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}\Omega_m H^2 \delta = 0 \tag{12.51}$$

Remember that δ in equation (12.51) represents the fluctuations in matter alone:

$$\delta = \frac{\varepsilon_m - \overline{\varepsilon}_m}{\overline{\varepsilon}_m} \tag{12.49}$$

When the universe is not matter-dominated, the density perturbations in matter (i.e., δ) don't grow rapidly in amplitude.

- For example, in the radiation-dominated phase ($\Omega_m \ll 1, H = 1/2t$), equation (12.51) becomes
$$\ddot{\delta} + \frac{1}{t}\dot{\delta} \approx 0 \tag{12.52}$$
and its solution is
$$\delta(t) \approx B_1 + B_2 \ln t \tag{12.53}$$
Therefore, in the radiation-dominated phase, density fluctuations in matter (including dark matter) grew only at a logarithmic rate.

- Meanwhile, in the phase dominated by a cosmological constant ($\Omega_m \ll 1, H = H_\Lambda$), equation (12.51) becomes
$$\ddot{\delta} + 2H_\Lambda \dot{\delta} \approx 0 \tag{12.54}$$
and its solution is
$$\delta(t) \approx C_1 + C_2 e^{-2H_\Lambda t} \tag{12.55}$$
Therefore, in the phase dominated by a cosmological constant, density fluctuations in matter (including dark matter) eventually reach a constant value.

Fluctuations can grow at a significant rate only when matter dominates the energy density. In such cases ($\Omega_m \approx 1, H = 2/3t$), equation (12.51) becomes

$$\ddot{\delta} + \left(\frac{4}{3t}\right)\dot{\delta} - \left(\frac{2}{3t^2}\right)\delta = 0 \tag{12.56}$$

If we guess a power law solution of the form $\delta \sim Dt^n$, and substitute in equation (12.56), we will get

$$\delta(t) \approx D_1\, t^{2/3} + D_2\, t^{-1} \tag{12.59}$$

where D_1 and D_2 can be found from the initial conditions for $\delta(t)$.

The decaying mode ($\sim t^{-1}$) in the solution above eventually becomes negligibly small compared to the growing mode ($\sim t^{2/3}$).

So, eventually, density perturbations in a matter-dominated universe grow as

$$\delta \propto t^{2/3} \propto a(t) \propto \frac{1}{1+z} \tag{12.60}$$

as long as $\delta \ll 1$.

Large Amplitude Density Fluctuations

When an overdense region attains $\delta \sim 1$, its evolution can no longer be treated with a simple linear perturbation approach.

In such cases, we must use numerical computer simulations.

The usual method is to model matter (including dark matter) as a distribution of a large number point masses interacting via Newtonian gravity (e.g., an N-body simulation). For example, the Millennium simulation that we viewed in class had 10^{10} particles.

- In the simulation, one can start with a small overdensity δ and watch it grow. When a region reaches overdensity $\delta \sim 1$ (in the simulation, as well as the real Universe), it breaks away from the Hubble flow and collapses.

- Soon, the overdense region becomes a gravitationally bound structure.

- If the baryonic matter within this structure is able to cool efficiently, e.g., via "bremsstrahlung" (free-free) radiation, it will radiate away energy and fall to the center.

- The centrally concentrated baryons then proceed to form stars, becoming the visible portions of galaxies that we see today. The less concentrated nonbaryonic dark matter forms the dark halo within which the stellar component is embedded.

- We looked at an example simulation in class, the so-called Millennium simulation. If you missed class, or wish to view the movie in more detail, it is available at the link: http://www.mpa-garching.mpg.de/galform/millennium/ (please cut and paste the address into your browser if the link doesn't work directly from this page).

The Power Spectrum

It is convenient to study the density fluctuations in Fourier space.

This is analogous to what we did with the CMB temperature anisotropies, except there we expanded $\delta T/T$ in spherical harmonics since we were dealing with a surface (the "last scattering" surface). Here, δ is defined in 3-dimensional space.

The Fourier transform pairs are

$$\delta(\vec{r}) = \frac{V}{(2\pi)^3} \int \delta_{\vec{k}} \exp\left(-i\vec{k}\cdot\vec{r}\right) d^3k \tag{12.63}$$

and

$$\delta_{\vec{k}} = \frac{1}{V} \int \delta(\vec{r}) \exp\left(i\vec{k}\cdot\vec{r}\right) d^3r \tag{12.64}$$

Be warned that different authors put the factors of π in different places.

Essentially, equation (12.63) breaks up $\delta(\vec{r})$ into an infinite number of sine waves, each with comoving wave number $\vec{k}$ and comoving wavelength $\lambda = 2\pi/k$.

Each Fourier component in equation (12.63) is a complex number (the Fourier amplitude):

$$\delta_{\vec{k}} = \left|\delta_{\vec{k}}\right| \exp\left(i\phi_{\vec{k}}\right) \tag{12.65}$$

If the phases $\phi_{\vec{k}}$ of the different Fourier components are *uncorrelated*, then $\delta(\vec{r})$ is called a *Gaussian field*, that is, the value of δ at a randomly selected point is drawn from the Gaussian probability distribution

$$p(\delta) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{\delta^2}{2\sigma^2}\right) \tag{12.68}$$

where the variance

$$\sigma \equiv \frac{V}{2\pi^2} \int_0^\infty P(k)\, k^2\, dk \tag{12.69}$$

The study of Gaussian density fields is of particular interest to cosmologists because most inflationary scenarios predict that the density fluctuations created by inflation will be an *isotropic and homogenous Gaussian field*.

The expected power spectrum for inflationary fluctuations has the form of a power law:

$$P(k) \propto k^n \tag{12.70}$$

The favored value of $n = 1$, sometimes called a Harrison-Zel'dovich spectrum.

The value $n = 1$ is favored because it is the only power law which prevents divergence of fluctuations on both large and small scales. This is explained in greater detail at the end of Section 12.4 in your text.

Cold Dark Matter or Hot Dark Matter?

We discussed above how the power spectrum at the end of inflation has the form $P(k) \propto k^n$, with $n = 1$ preferred.

The shape of $P(k)$ will be modified, though, between the end of inflation (t_f) and the time of radiation-matter equality ($t_{rm} \approx 47,000$ yr).

The shape of $P(k)$ at t_{rm} depends on the properties of dark matter, specifically whether it is predominantly cold dark matter (CDM) or hot dark matter (HDM).

- Recall from an earlier lecture that CDM is comprised of particles which were *non-relativistic* at the time they decoupled from the other components of the universe, e.g., WIMPS and primordial black holes (if any).
- On the other hand, HDM would be comprised of particles which were *relativistic* when they decoupled, e.g., photons.

So, how does HDM modify the spectrum of density perturbations?

- Particles which were relativistic at the time of decoupling cool as the Universe expands, until a time t_h at which their thermal velocities have dropped well below c.
- Prior to $t = t_h$, the HDM particles move freely in random directions with a speed close to c. This motion is called *free streaming*, and its net effect is to wipe out any density fluctuations in the HDM whose wavelength is smaller than $\sim ct_h$.
- If you calculate this (see equations 12.78-12.80), you will find that if dark matter is contributed by neutrinos with rest energy of a few eV, then free streaming will wipe out all density fluctuations on scales smaller than superclusters.

So, if most dark matter is HDM, then the first structures in the Universe would have been superclusters, so we expect the oldest structures in the Universe to be superclusters, and galaxies would be relatively young.

In fact, the opposite appears to be true in our Universe. Superclusters have started collapsing only recently, but galaxies have been around since at least $z \sim 10$.

So, most of the dark matter in the Universe must be CDM. For a more detailed discussion of how a CDM spectrum works, see Section 12.5 in your text. In brief, in a CDM universe, the first objects to form are the smallest — galaxies, then clusters, then superclusters. This "bottom up" scenario is consistent with the observed ages of galaxies and superclusters.

Finally, one may ask if the dark matter is exclusively CDM. Since there is strong evidence that neutrinos do have some mass, so the Universe must contain at least some HDM. The current constraint for $\Omega_{m,0} = 0.3$ in the Benchmark model is $\Omega_{\mathrm{HDM},\,0} \leq 0.04$ — any more and free streaming would make the Universe too smooth on small scales.

End Game and Wish List

In less than a 100 years, we have certainly come a long way! In the 1920's, the Shapley-Curtis debate centered around whether our Galaxy was the whole Universe or not. Today, we are aware that a significant part of the Universe may be far removed from our ability to detect it directly.

So, what are the major issues in cosmology that occupy theorists and observers today?

Perhaps the question that is most prominent in the minds of researchers is the nature of the dark energy that appears to be accelerating the expansion of the Universe. We have discussed how the simplest, and currently favored model, is a cosmological constant with negative pressure $P = -\varepsilon$, where ε is a positive energy density. However, no one has a satisfactory explanation as to why such a component should exist in the Universe. Furthermore, could dark energy have an equation of state with w different from -1?

Another aspect that continues to befuddle researchers is the nature of dark matter. It seems likely that it is primarily cold dark matter, as we have discussed above, but no one knows what constitutes dark matter.

What events took place in the very early Universe ($t < 10^{-10}$ s)? For example, why is there a matter-antimatter asymmetry? What set the ratio of baryons to photons?

Did inflation really take place? If so, how will we find a convincing direct signature of an early phase of vacuum-dominated expansion?

Can we formulate a theory of quantum gravity to study the earliest moments of the Universe?

There are a number of documents on the web that present pathways to resolving the above problems, if you're interested in further reading.