PHY 475/375
Lecture 17
(May 23, 2012)

Inflation in the Early Universe

The inflationary scenario was proposed by Alan Guth in 1981 to solve some of the problems inherent in the standard Big Bang scenario, including the horizon and flatness problems.

The key point of the inflationary scenario was a brief phase of accelerated expansion early in the history of our Universe.

Recall from the acceleration equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\Big(\varepsilon + 3P\Big) \tag{11.18}$$

that acceleration takes place ($\ddot{a} > 0$) when $P < -\varepsilon/3$.

Since the equation of phase is $P = w\varepsilon$, this means that inflation would have taken place if the universe was temporarily dominated by a component with equation-of-state parameter $w < -1/3$.

Now, recall from equation (4.62) that the Friedmann equation in the presence of a cosmological constant Λ is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\,\varepsilon - \frac{\kappa c^2}{R_0^2 a^2} + \frac{\Lambda}{3}$$

To get a brief phase of acceleration, the usual practice is to make the universe be temporarily dominated by a positive cosmological constant Λ_i (with $w = -1$), so that the Friedmann equation becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\Lambda_i}{3} \tag{11.20}$$

During the inflationary phase, therefore, the Hubble factor H_i was constant, with the value

$$H_i = \left(\frac{\Lambda_i}{3}\right)^{1/2}$$

and the scale factor grew exponentially with time:

$$a(t) \propto e^{H_i t} \tag{11.21}$$

We will now see how this can solve the flatness and horizon problems.

Suppose that the Universe had such a period of exponential expansion sometime during its radiation-dominated phase.

For simplicity, we will switch on the exponential growth instantaneously at a time t_i and switch it off instantaneously at some later time t_f, at which point the Universe returns to its state of radiation-dominated expansion. The scale factor for this case is given by

$$a(t) = \begin{cases} a_i \left(\dfrac{t}{t_i}\right)^{1/2} & t < t_i \\ a_i \, e^{H_i(t-t_i)} & t_i < t < t_f \\ a_i \, e^{H_i(t_f-t_i)} \left(\dfrac{t}{t_f}\right)^{1/2} & t > t_f \end{cases} \tag{11.22}$$

so that, between the time t_i when the exponential inflation began and the time t_f when it stopped, the scale factor increased by a factor

$$\frac{a(t_f)}{a(t_i)} = \frac{a_i \, e^{H_i(t_f-t_i)}}{a_i}$$

so that we can write

$$\frac{a(t_f)}{a(t_i)} = e^N \tag{11.23}$$

where N is the number of e-foldings of inflation, given by

$$N \equiv H_i \left(t_f - t_i\right) \tag{11.24}$$

If the duration of inflation $(t_f - t_i)$ was long compared to the Hubble time during inflation (H_i^{-1}), then N would have been large, and the growth in scale factor during inflation would have been enormous.

For a concrete example, let us take one model for inflation, in which exponential growth in the scale factor started around the GUT time. Here, GUT stands for Grand Unified Theory — the moment before which the strong nuclear force was believed to be coupled together with the electroweak (electromagnetic and weak nuclear) force. So, in our example, inflation would have started at

$$t_i \approx t_{\mathrm{GUT}} \approx 10^{-36} \text{ s}$$

and let us suppose it lasted for $N \sim 100$ Hubble times. This would mean a growth in the scale factor during the period of inflation of

$$\frac{a(t_f)}{a(t_i)} = e^N \sim e^{100} \sim 10^{43} \tag{11.25}$$

This means that, during inflation, the scale factor can grow by an enormous amount in a moderate number of Hubble times.

Solving the Flatness Problem

We will now see how inflation solves the flatness problem.

Equation (11.1) can be written in the following form for any universe which is not perfectly flat:

$$\left|1 - \Omega(t)\right| = \frac{\kappa c^2}{R_0^2\, a(t)^2 H(t)^2} \tag{11.27}$$

During the inflationary phase, the Universe is expanding exponentially with constant H_i, and $a \propto e^{H_i t}$, so we get

$$\left|1 - \Omega(t)\right| \propto \frac{1}{(e^{H_i t})^2}$$

or

$$\left|1 - \Omega(t)\right| \propto e^{-2H_i t} \tag{11.29}$$

Equation (11.29) tells us that the difference between Ω and 1 decreases exponentially with time.

Let us compare the density parameter at the beginning of the inflationary phase ($t = t_i$) with the density parameter at the end of the inflationary phase ($t = t_f$).

To do this, recall from equation (11.24) that $(t_f - t_i) = N/H_i$, so $t_f = t_i + N/H_i$.

So, equation (11.29) gives

$$\frac{\left|1 - \Omega(t_f)\right|}{\left|1 - \Omega(t_i)\right|} = \frac{e^{-2H_i t_f}}{e^{-2H_i t_i}} = \frac{e^{-2H_i (t_i + N/H_i)}}{e^{-2H_i t_i}} = \frac{e^{-2H_i\, t_i} e^{-2N}}{e^{-2H_i t_i}}$$

so that

$$\left|1 - \Omega(t_f)\right| = e^{-2N} \left|1 - \Omega(t_i)\right| \tag{11.30}$$

Now, suppose that prior to inflation, the Universe was fairly strongly curved, with

$$\left|1 - \Omega(t_i)\right| \sim 1 \tag{11.31}$$

Then, after a 100 e-foldings of inflation, the deviation of Ω from 1 would be

$$\left|1 - \Omega(t_f)\right| = e^{-2N} \underbrace{\left|1 - \Omega(t_i)\right|}_{\sim 1} \sim e^{-2N} \sim e^{-2(100)} \sim 10^{-87} \tag{11.32}$$

That is, even if the universe wasn't particularly close to being flat at the beginning of the inflationary epoch, a 100 e-foldings of inflation would have rendered it close to near-perfect flatness!

Solving the Horizon Problem

Recall from Lecture 9 that the horizon distance is the proper distance at the current time to the most distant object we can see, for which the light emitted at $t = 0$ is just now reaching us at $t = t_0$. Therefore, the horizon distance at the current epoch t is given by (5.55):

$$d_{\rm hor}(t_0) = c\int_0^t \frac{dt}{a(t)}$$

and since

$$d_{\rm hor}(t) = a(t)d_{\rm hor}(t_0)$$

we get

$$d_{\rm hor}(t) = a(t)\left[c\int_0^t \frac{dt}{a(t)}\right] \tag{11.33}$$

Be careful you don't get confused by the $a(t)$ inside and outside the integral. The one inside is to evaluate the integral from 0 to t; having found it, you multiply by the value of a at the instant t to get $d_{\rm hor}(t)$ at that instant.

Since the universe was radiation dominated during the inflationary epoch, we can write

$$a(t) = a_i\left(\frac{t}{t_i}\right)^{1/2}$$

from equation (11.22).

Then we get for the horizon time at the beginning of inflation ($t = t_i$):

$$\begin{aligned} d_{\rm hor}(t_i) &= a_i\left[c\int_0^{t_i} \frac{dt}{a_i(t/t_i)^{1/2}}\right] \\ &= a_i\frac{c\,t_i^{1/2}}{a_i}\left[\frac{t^{-1/2+1}}{-1/2+1}\right]_0^{t_i} \\ &= 2c\,t_i^{1/2}\left[t_i^{1/2}\right] \\ \Rightarrow \quad d_{\rm hor}(t_i) &= 2c\,t_i \end{aligned} \tag{11.34}$$

Meanwhile, the horizon size at the end of inflation ($t = t_f$) is given by

$$d_{\rm hor}(t_f) = a_f c\left[\int_0^{t_i} \frac{dt}{a_i(t/t_i)^{1/2}} + \int_{t_i}^{t_f} \frac{dt}{a_i \exp[H_i(t-t_i)]}\right] \tag{11.35}$$

You should be able to show on your own that equation (11.35) works out to

$$d_{\rm hor}(t_f) = e^N c\Big(2t_i + H_i^{-1}\Big) \tag{11.36}$$

Equations (11.34) and (11.36) show that the horizon size grows exponentially during inflation.

For example, if inflation starting at $t_i \approx t_{\rm GUT} \approx 10^{-36}$ s^{-1}, with a Hubble parameter $H_i \approx 10^{36}$ s^{-1}, and lasted for $n \approx 100$ e-foldings, then:

- The horizon size immediately before inflation was

$$d_{\rm hor}(t_i) = 2c\,t_i \approx 10^{-28}\ {\rm m} \tag{11.37}$$

- The horizon size immediately after inflation was

$$d_{\rm hor}(t_f) \approx e^N\,3c\,t_i \approx 2 \times 10^{16}\ {\rm m} \approx 0.8\ {\rm pc} \tag{11.38}$$

During the brief period of inflation ($\sim 10^{-34}$ s), the horizon size is boosted from submicroscopic scales to nearly a parsec, a factor of 10^{40}.

Thus, the net result of inflation is to increase the horizon length in the post-inflationary universe by a factor $\sim e^N$ over what it would have been without inflation.

Thus it makes sense that regions so far apart now are in thermal equilibrium — they were very close together before inflation, close enough to be in causal contact.

An important point to note is that the weaker the acceleration, the longer inflation must last to solve the horizon and flatness problems.

The Physics of Inflation

A question naturally arises — what triggers inflation at $t = t_i$ and what turns it off at $t = t_f$?

There is not a consensus among cosmologists about the exact mechanism driving inflation.

We will look at one plausible scenario.

The general strategy is to posit a scalar field $\phi(\vec{r}, t)$, the so-called *inflaton field*. Notice that, in general, its value can vary as a function of position and time. The scalar field is associated with potential $V(\phi)$.

If the scalar field ϕ has units of energy, and its potential has units of energy density, then the energy density and pressure of the inflaton field are, respectively

$$\varepsilon_\phi = \frac{1}{2\hbar c^3}\,\dot{\phi}^2 + V(\phi) \tag{11.42}$$

$$P_\phi = \frac{1}{2\hbar c^3}\,\dot{\phi}^2 - V(\phi) \tag{11.43}$$

If the inflaton field changes only very slowly with time, with the $(\ldots)\,\dot{\phi}$ term much smaller than $V(\phi)$, i.e.,

$$\dot{\phi}^2 \ll \hbar\, c^3\, V(\phi) \tag{11.44}$$

then we have $P_\phi \approx -\varepsilon_\phi$, and thus a component that can produce exponential expansion.

Therefore, inflation requires a phase in which $V(\phi)$ dominates the energy and pressure budget for a sufficiently long time.

The requirement of slow time-evolution of ϕ is known as the *slow roll* condition.

To write a dynamical equation for ϕ, start with the fluid equation for the energy density of the inflaton field:

$$\dot{\varepsilon}_\phi + 3H(t)\Big[\varepsilon_\phi + P_\phi\Big] = 0 \tag{11.46}$$

where we have written the usual $\dot{a}/a$ as $H(t)$.

From equation (11.42), we get

$$\dot{\varepsilon}_\phi = \frac{1}{2\hbar c^3}\Big[2\dot{\phi}\,\ddot{\phi}\Big] + \frac{dV}{d\phi}\,\dot{\phi}$$

and substituting this in equation (11.46), we get

$$\frac{1}{\hbar c^3}\Big[\dot{\phi}\,\ddot{\phi}\Big] + \frac{dV}{d\phi}\,\dot{\phi} + 3H(t)\left[\frac{1}{2\hbar c^3}\,\dot{\phi}^2 + \cancel{V(\phi)} + \frac{1}{2\hbar c^3}\,\dot{\phi}^2 - \cancel{V(\phi)}\right] = 0$$

Dividing by $\dot{\phi}$ and moving the term involving V to the right hand side, this becomes

$$\frac{1}{\hbar c^3}\Big[\ddot{\phi} + 3H(t)\,\dot{\phi}\Big] = -\frac{dV}{d\phi}$$

This gives us a dynamical equation for ϕ:

$$\ddot{\phi} + 3H(t)\,\dot{\phi} = -\hbar c^3\,\frac{dV}{d\phi} \tag{11.47}$$

Equation (11.47) resembles the equation of motion for a particle which is being accelerated by a force proportional to $-dV/d\phi$, and subject to a frictional force proportional to the speed of the particle.

Just as a skydiver reaches terminal velocity when the downward force of gravity is balanced by air resistance acting upward, so too the inflation field can reach "terminal velocity" (with $\ddot{\phi} = 0$) when

$$3H\dot{\phi} = -\hbar c^3\,\frac{dV}{d\phi} \tag{11.48}$$

This is called the slow roll evolution equation.

If the inflaton field has reached the terminal velocity given by equation (11.48), then the requirement (11.44) for the inflaton field to cause an exponential expansion translates into

$$\left(\frac{E_P}{V}\frac{dV}{d\phi}\right)^2 \ll 1 \tag{11.53}$$

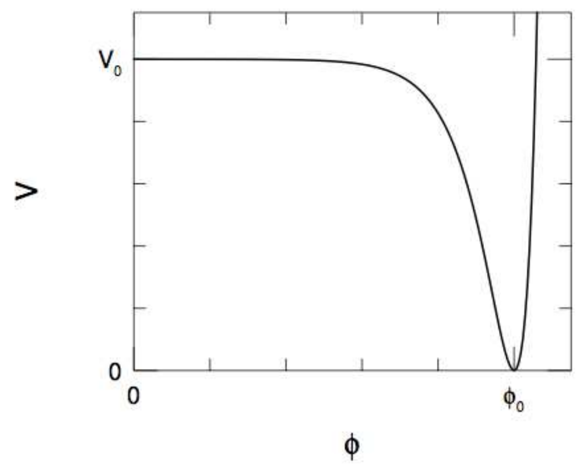
Inflation occurs during the slow roll phase, when ϕ and $V(\phi)$ are approximately constant. In other words, the slope of the inflaton field must be sufficiently shallow, and the amplitude of the potential sufficiently large, to dominate the energy density of the universe, for the inflaton field to be capable of giving rise to exponential expansion.

For a concrete example of a potential $V(\phi)$ that can give rise to inflation, consider the potential shown in the figure below.

The global minimum in this potential occurs when the value of the inflaton field is $\phi = \phi_0$, this is also called the "true vacuum" position.

Suppose, however, that the inflaton field starts at $\phi \approx 0$, where the potential is $V(\phi) \approx V_0$. If condition (11.53) is satisfied on the plateau where $V \approx V_0$, then while ϕ is slowly rolling toward ϕ_0, the inflaton field contributes an energy density $\varepsilon_\phi \approx V_0 \approx$ constant to the universe.

When the inflaton field is near $\phi \approx 0$, it is said to be in a "false vacuum" state.



Note that the word "vacuum" is used here in the sense of (the local) *lowest minimum potential*. The word "false" alludes to the fact that the "vacuum" is not stable in the sense that if the inflaton field is nudged from $\phi = 0$ to $\phi = d\phi$, it will continue to slowly roll toward the true vacuum state at $\phi = \phi_0$ and $V = 0$.

However, if the plateau is sufficiently broad as well as shallow, it can take many Hubble times for the inflaton field to roll down to the true vacuum state.

Whether the inflaton field is dynamically significant during its transition from the false vacuum to the true vacuum state depends on the value of V_0, specifically the condition in (11.53), which we can rewrite as

$$\left(\frac{dV}{d\phi}\right)^2 \ll \frac{V_0^2}{E_P^2} \tag{11.54}$$

As long as we have a sufficiently large value of V_0 to satisfy equation (11.54), exponential inflation driven by the energy density V_0 of the inflaton field will begin at a temperature that may be found from the αT^4 relation for the energy density that we studied in Lecture 3 (equation 2.24).

$$T_i \approx \left(\frac{V_0}{\alpha}\right)^{1/4} \tag{11.55}$$

Since $\alpha = \dfrac{\pi^2}{15}\dfrac{k^4}{\hbar^3\, c^3}$ from (2.27), we can write equation (11.55) as

$$kT_i \approx \left(\hbar^3\, c^3\, V_0\right)^{1/4} \tag{11.56}$$

This corresponds to a time

$$t_i \approx \left(\frac{c^2}{GV_0}\right)^{1/2} \tag{11.57}$$

While the inflaton field is slowly rolling toward the true vacuum state, it produces exponential expansion, with a constant Hubble parameter

$$H_i \approx \left(\frac{8\pi GV_0}{3c^2}\right)^{1/2} \approx t_i^{-1} \tag{11.58}$$

The exponential expansion ends as the inflaton field reaches the true vacuum at $\phi = \phi_0$.

The duration of inflation, therefore, depends on the exact shape of the potential $V(\phi)$. Large values of ϕ_0 and V_0 (i.e., a broad high plateau), and small values of $dV/d\phi$ (i.e., a shallow, sloped plateau) lead to more e-foldings of inflation.

After rolling off the plateau shown in the above figure, the inflaton field oscillates about the minimum at ϕ_0.

The amplitude of these oscillations is damped by the Hubble friction term proportional to $H\dot{\phi}$ in (11.47).

If the inflaton field is coupled to any other fields in the universe, however, the oscillations in ϕ are damped more rapidly, with the energy of the inflaton being carried away by photons or other relativistic particles.

This has an important consequence. Remember that if the scale factor increases by

$$\frac{a(t_f)}{a(t_i)} = e^N \tag{11.60}$$

then the temperature of the Universe will drop by a factor e^{-N}. So, after 100 e-foldings of inflation, the temperature would have dropped from $T(t_i) \sim T_{\rm GUT} \sim 10^{28}$ K, down to $T(t_f) \sim e^{-100}\, T_{\rm GUT} \sim 10^{-15}$ K.

The energy lost by the inflaton field after its phase transition from the false vacuum to the true vacuum can be thought of as the "latent heat" of that transition, just like the energy released when water freezes. The energy released during the transition from false to true vacuum goes to heat the Universe back to its pre-inflationary value of T_i.

Models for $V(\phi)$

The following discussion is based on a paper presented by Alan Guth (Proceedings of the National Academy of Sciences, 1993, vol. 90, pp. 4871.

Old Inflation

Guth's original (1981) model was based on the false vacuum corresponding to a local minimum in the potential, where the energy density ϕ could stay for a long time in a metastable state trapped by a potential barrier. The model contained a fatal flaw in that the potential energy function would decay by bubble nucleation (a process that resembles the way water boils). The bubbles needed to collide to reheat the Universe, but in reality they would never have done so because they were being carried apart by the expansion. Guth realized this flaw and pointed it out, but published in the hope that others would find more plausible models.

New Inflation

In this model, $V(\phi)$ has a nearly flat plateau but no barrier, like the example we graphed in a previous page. In such models, there is a slow roll down the plateau to the true vacuum, where oscillation and reheating can take place.

Chaotic Inflation

Andrei Linde showed that the severe flat plateau restriction in the new inflation was not necessary. Linde showed that inflation can work for a scalar field potential as simple as $V(\phi) = \lambda\phi^4$, provided that one makes some assumptions about the initial conditions. Linde proposed that the scalar field begins in a chaotic state, and that some regions must exceed a minimal size, usually estimated to be several times the Hubble parameter. Calculations indicate that there is an adequate amount of inflation when ϕ rolls down the hill of such a potential. The Hubble "constant" is not a constant in this case, but instead is slowly varying, so the expansion can be called "quasi-exponential."

More details are available in the paper, which is freely available.

Finally, inflation is such a finely tuned theory that it is disturbing to many. Yet it works, so it will be hard to dislodge (remember, the geocentric system endured for 1400 years until the time after Galileo)! Still, if you want to read about speculations on alternatives, a good place to start is Steinhardt's Scientific American article from last year.