PHY 475/375
Lecture 12
(May 7, 2012)

Measuring Cosmological Parameters

So far, we have looked at a number of model universes containing one or more components. In order to confirm any particular model, we need to know $a(t)$. For the model universes we studied, the contents were known with precision, so $a(t)$ could just be computed from the Friedmann equation. For the real universe, however, finding $a(t)$ turns out to be much more difficult. Since it is not directly observable, it must be inferred from imperfect or incomplete observations.

In general, determining $a(t)$ from observations involves measuring distances, which is why so much of observational cosmology is dedicated to making accurate and precise measurements of distances.

The Hubble Constant and Deceleration Parameter

Determining the exact functional form of $a(t)$ is difficult, so instead we do a Taylor series expansion of $a(t)$ around the present moment $t = t_0$:

$$a(t) = a(t_0) + \Big(t - t_0\Big)\ \frac{da}{dt}\bigg|_{t=t_0} + \frac{(t-t_0)^2}{2}\ \frac{d^2a}{dt^2}\bigg|_{t=t_0} + \ldots \tag{7.1}$$

Now, while equation (7.1) is an infinite series, the variation in $a(t)$ around a point can be captured well by just the first few terms in this series, as long as it doesn't fluctuate wildly — this seems like a reasonable assumption, since none of our model universes had wildly fluctuating a's.

So, if we keep the first 3 terms, and divide by the current scale factor $a(t_0)$, we get

$$\frac{a(t)}{a(t_0)} \approx 1 + \Big(t - t_0\Big)\ \frac{\dot{a}}{a}\bigg|_{t=t_0} + \frac{(t-t_0)^2}{2}\ \frac{\ddot{a}}{a}\bigg|_{t=t_0} \tag{7.3}$$

With the usual normalization $a(t_0) = 1$, we get

$$a(t) \approx 1 + \Big(t - t_0\Big)H_0 - \frac{1}{2}\,q_0\Big(t - t_0\Big)^2 H_0^2 \tag{7.4}$$

where H_0 is the Hubble constant:

$$H_0 \equiv \frac{\dot{a}}{a}\bigg|_{t=t_0} \tag{7.5}$$

and q_0 is a dimensionless number called the deceleration parameter, and defined as

$$q_0 \equiv -\left(\frac{\ddot{a}}{a\,H^2}\right)\bigg|_{t=t_0} \tag{7.6}$$

Note that a positive value for q_0 corresponds to $\ddot{a} < 0$, meaning that the expansion of the universe is decelerating. On the other hand, a negative value for q_0 corresponds to $\ddot{a} > 0$, meaning that the expansion of the universe is accelerating. The name and choice of sign are historical, from a period in the 1950's when the universe was believed to be matter-dominated and decelerating.

Note that the Taylor expansion in equation (7.4) is *model-independent*, that is, it does not depend on any particular cosmology, or in fact, any underlying physics. In other words, this is a kinematic determination of the scale factor $a(t)$, simply a mathematical description of how the universe expands at time $t \sim t_0$, and is even more fundamental than the Friedmann equation; recall that the Friedmann equation assumes that the universe is controlled by gravity, and that gravity is accurately described by Einstein's general relativity, which then requires a metric to be defined for space-time.

The importance of equation (7.4) was highlighted by the cosmologist Allan Sandage in 1970, when he described cosmology as the quest for two numbers, H_0 and q_0. While the scope of cosmology has widened considerably since then, efforts aimed at the precise and accurate measurement of H_0 and q_0 continue today.

Next, although H_0 and q_0 are kinematic and free of theoretical assumptions, one could use the Friedmann equation to predict what q_0 will be in a given model universe. Let us use the acceleration equation (derived by combining the Friedmann equation and the fluid equation) to do this.

Recall that in a model universe containing several components, each with a different value of the equation-of-state parameter w, the acceleration equation can be written as

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2} \sum_w \varepsilon_w \Big(1 + 3w\Big) \tag{7.7}$$

Dividing each side of equation (7.7) by $H(t)^2$ and changing sign, we get

$$-\frac{\ddot{a}}{aH^2} = \frac{4\pi G}{3c^2 H^2} \sum_w \varepsilon_w \Big(1 + 3w\Big)$$

or

$$-\frac{\ddot{a}}{aH^2} = \frac{1}{2} \left[\frac{8\pi G}{3c^2 H^2}\right] \sum_w \varepsilon_w \Big(1 + 3w\Big) \tag{7.8}$$

Recall that the quantity in square brackets is $1/\varepsilon_c$, where ε_c is the critical energy density. Since it does not depend on w, it can be taken inside the summation, so that

$$-\frac{\ddot{a}}{aH^2} = \frac{1}{2} \sum_w \frac{\varepsilon_w}{\varepsilon_c} \Big(1 + 3w\Big)$$

and since $\varepsilon_w/\varepsilon_c = \Omega_w$, we get finally

$$-\frac{\ddot{a}}{aH^2} = \frac{1}{2} \sum_w \Omega_w \Big(1 + 3w\Big) \tag{7.9}$$

If we choose to evaluate equation (7.9) at the present moment, then the left hand side is just the deceleration parameter q_0 from equation (7.6). Therefore, we get the relationship between the deceleration parameter q_0 and the density parameters of the different components of the universe:

$$q_0 = \frac{1}{2} \sum_w \Omega_{w,0} \Big(1 + 3w\Big) \tag{7.10}$$

So, for a universe containing radiation ($w = 1/3$), matter ($w = 0$), and a cosmological constant ($w = -1$), we get

$$q_0 = \frac{1}{2}\Big[\Omega_{r,0}\Big(1 + 3\{1/3\}\Big) + \Omega_{m,0}\Big(1 + 0\Big) + \Omega_{\Lambda,0}\Big(1 + 3\{-1\}\Big)\Big]$$

or

$$q_0 = \Omega_{r,0} + \frac{1}{2}\Omega_{m,0} - \Omega_{\Lambda,0} \tag{7.11}$$

If $\Omega_{\Lambda,0} > (\Omega_{r,0} + \Omega_{m,0}/2)$, then equation (7.11) gives $q_0 < 0$, and the universe will currently be accelerating outward. For example, the Benchmark model has $q_0 = -0.55$.

Next, let us look at H_0. For small redshifts, we know that

$$cz = H_0 d \tag{7.12}$$

so if we can measure the redshift z and distance d for a large sample of galaxies, and fit a straight line to a plot of cz vs. d, the slope of the plot gives the value of H_0.

Measuring the redshift of a galaxy is pretty simple, but the distance to a galaxy is not only difficult to measure, it is somewhat difficult to define in an expanding universe.

Perhaps the most straightforward definition of distance in an expanding universe is the proper distance, e.g., the proper distance at the time of observation:

$$d_p(t_0) = c\int_{t_e}^{t_0} \frac{dt}{a(t)} \tag{7.13}$$

If we Taylor expand $a(t)$ (see equations (7.14)-(7.18) in your text), we find that $d_p(t_0)$ depends on H_0 and q_0:

$$d_p(t_0) \approx \frac{cz}{H_0}\left[1 - \left(\frac{1 + q_0}{2}\right) z\right] \tag{7.19}$$

In practice, though, we need some way of computing a distance to an object based on the observed properties of that object.

Luminosity Distance

Since proper distance is not a measurable quantity, one must look to other alternatives. One property that we can measure for objects at cosmological distances is the flux of light from the object (f, in units of watts m^{-2}). While we would like to measure the flux integrated over all wavelengths (called the *bolometric flux*), in practice, we can only measure it over a limited range of wavelengths. If the object is an extended source rather than a point source of light, we can measure its angular diameter $\delta\theta$. And, of course, we can measure the redshift z.

One way of using measured properties to assign a distance is by using a so-called standard candle, which is an object whose luminosity is known. If, by some means, you could find the luminosity L of some class of astronomical object, then you can use the measured flux f of that object to define a function called the *luminosity distance*:

$$d_L \equiv \left(\frac{L}{4\pi f}\right)^{1/2} \tag{7.21}$$

The function d_L is called a "distance" not only because it has the dimensionality of a distance, but also because it is what the proper distance to the standard candle would be if the Universe were static and Euclidean, where the light would follow the inverse square law $f = L/(4\pi d^2)$.

However, in a universe described by a Robertson-Walker (*FLRW*) metric (written in equation 3.25):

$$ds^2 = -c^2\,dt^2 + a(t)^2\Big[dr^2 + S_\kappa(r)^2\,d\Omega^2\Big] \tag{7.22}$$

with

$$S_\kappa(r) = \begin{cases} R_0 \sin(r/R_0), & \kappa = +1 \\ r, & \kappa = 0 \\ R_0 \sinh(r/R_0), & \kappa = -1 \end{cases} \tag{7.23}$$

Suppose you are located at the origin and are observing, at the present moment $t = t_0$, light that was emitted by a standard candle at the comoving coordinate location (r, θ, ϕ) at a time t_e. The photons which were emitted at time t_e are spread at the present moment, over a sphere of proper radius $d_p(t_0) = r$ (e.g., see equation (5.33) in Lecture 9) and proper surface area $A_p(t_0)$, given by

$$A_p(t_0) = 4\pi S_\kappa(r)^2 \tag{7.24}$$

- If space is flat, then $A_p(t_0) = 4\pi r^2$, the familiar Euclidean relation.
- If space is positively curved, then $A_p(t_0) < 4\pi r^2$, and photons are spread over a *smaller* area than they would be in flat space.
- If space is negatively curved, then $A_p(t_0) > 4\pi r^2$, and photons are spread over a *larger* area than they would be in flat space.

Moreover, the expansion of the universe causes the observed flux of light from a standard candle at redshift z to be decreased by an additional factor $(1 + z)^{-2}$.

- First, the wavelength of a photon emitted with λ_e is changed to

$$\lambda_0 = \frac{a(t_0)}{a(t_e)}\,\lambda_e = \frac{1}{a(t_e)}\,\lambda_e = (1+z)\,\lambda_e \tag{7.25}$$

- Second, if two photons are emitted in the same direction separated by a time interval δt_e, the proper distance between them will be stretched to $c\,(\delta t_e)\,(1+z)$, and we will detect them separated by a time interval $\delta t_0 = \delta t_e\,(1+z)$.

Combining all of the observations on the previous page, we get that in an expanding, spatially curved universe, the relation between the observed flux f and the luminosity L of a distant light source is

$$f = \frac{L}{4\pi S_\kappa(r)^2\,(1+z)^2} \tag{7.27}$$

so that the luminosity distance is

$$d_L = S_\kappa(r)\,(1+z) \tag{7.28}$$

Angular Diameter Distance

Yet another distance measure based on observable properties of cosmological objects is the angular diameter distance.

To define such a measure, consider measuring a *standard yardstick*, instead of a standard candle. A standard yardstick is one whose proper length l is known.

Suppose such a yardstick of constant proper length l is aligned perpendicular to the line of sight. We measure an angular distance $\delta\theta$ between the ends of the yardstick, and a redshift z for the light emitted by the yardstick. If $\delta\theta \ll 1$, and we know the length l of the yardstick, we can compute a distance to the yardstick using the small-angle formula

$$d_A \equiv \frac{l}{\delta\theta} \tag{7.33}$$

Then dA is called the *angular-diameter distance*.

In a universe described by the Robertson-Walker ($FLRW$) metric in equation (7.22), the angular-diameter distance d_A to a standard yardstick is given by

$$d_A = \frac{S_\kappa(r)}{1+z} \tag{7.36}$$

Comparison with equation (7.28) shows that the relation between angular-diameter distance and luminosity distance is

$$d_A = \frac{d_L}{(1+z)^2} \tag{7.37}$$

It is of interest to note that at low redshifts $z \to 0$, we have

$$d_A \approx d_L \approx d_p(t_0) \approx \frac{cz}{H_0}$$

Since standard yardsticks can be hard to identify in practice, however, more attention has been focussed in recent years on standard candles to determine H_0.

Standard Candles and the Hubble Constant

The usual recipe for finding the Hubble constant H_0 using standard candles is:

- Identify a population of standard candles with luminosity L.
- Measure the redshift z and flux f for each standard candle.
- Compute $d_L = \sqrt{L/(4\pi f)}$ for each standard candle.
- Plot cz vs. d_L.
- Measure the slope of the cz vs. d_L relation when $z \ll 1$; this gives H_0.

Of course, a good standard candle is not easy to find!

We've already learned how one of the most frequently used standard candles are the Cepheid variable stars.

In the early 1910's, Henrietta Leavitt, working at Harvard College Observatory, discovered that there was a clear relation between the period P and mean flux f of Cepheids in the Small Magellanic Cloud (SMC), with stars having the longest period of variability also having the largest flux. Since the depth of the SMC along the line of sight is small compared to its distance from us, she was justified in assuming that the difference in mean flux of the Cepheids was due to their mean luminosity, and not due to differences in their distance from us. Therefore, Leavitt had discovered a period-luminosity relation for Cepheids, allowing them to be used as standard candles.

Note that, by measuring the ratio of fluxes to Cepheids in two galaxies, we can only know the relative distances to these galaxies. To find an absolute distance to either galaxy, we need to know the luminosity L for a Cepheid of a given period P. This normalization of the period-luminosity relation for Cepheids has constituted a significant effort by many astronomers ever since the discovery by Leavitt. Trigonometric parallax is the only method to get a good distance to a star, but nearby Cepheids are rare. Until future space-based missions can measure trigonometric parallaxes to a sizable number of Cepheids, we must rely on alternate methods of normalizing the period-luminosity relation for Cepheids. One of these involves finding the distance to the Large Magellanic Clouds (LMC) by secondary methods, then using this distance to compute the mean luminosity of the LMC Cepheids.

Successful measurements of the periods and fluxes of Cepheids have been accurately measured to luminosity distances of $d_L \sim 20$ Mpc with the Hubble Space Telescope (HST). In fact, measurement of H_0 was designated as a Key Project for the HST, with a final report in 2001 showing that Cepheid data are best fit with a Hubble constant of $H_0 = 72 \pm 8$ km s^{-1} Mpc^{-1} (Freedman, et al. 2001, *Astrophysical Journal*, 553, 47).

Unfortunately, Cepheid variables can only take us out to about 20 Mpc. On this scale, we know that the Universe is not homogenous and isotropic. In fact, we know that our Local Group is gravitationally attracted toward the Virgo cluster of galaxies, causing it to have a peculiar motion in that direction. Dynamical models must therefore be used to estimate what effect this has on the recession velocities we measure at these distances; for example, the recession velocity cz we measure for the Virgo cluster is found to be 250 km s^{-1} less than if the Universe were perfectly homogenous. Plots of cz vs. d_L must then correct for this "Virgocentric flow" in their recession velocities.

The Accelerated Expansion of the Universe

Since peculiar velocities affect nearby measurements of recession velocities, we need to determine the luminosity distance to standard candles with $d_L > 100$ Mpc ($z > 0.02$). It is difficult to find standard candles that are luminous enough at such large distances. Initial attempts in finding standard candles focussed on using entire galaxies, but didn't meet with much success due to lack of standardization among galaxies.

In the last two decades, the standard candle of choice among cosmologists has been Type Ia supernovae.

- We've already discussed how Type Ia supernovae occur in binary systems where one of the two stars is a white dwarf. The transfer of mass from the companion star to the white dwarf eventually takes it over the Chandrasekhar limit of 1.4 $M_\odot$, at which point the white dwarf undergoes a runaway nuclear fusion reaction; the resulting spectacle is called a Type Ia supernovae.

- We've also discussed how such supernovae can outshine all the stars in the entire galaxy combined. Since moderately bright galaxies can be seen at $z \sim 1$, this means Type Ia supernovae can also be seen at $z \sim 1$.

Therefore, even though Type Ia supernovae are not all that common in our Galaxy (they occur about once per century in our Galaxy, on average), their tremendous brightness means they can be detected out to great distances. Moreover, in rich clusters like Virgo, we see several Type Ia supernovae in a year. Therefore, they are reasonably good standard candles out to large distances.

The search for distant Type Ia supernovae has been led by two teams: the *Supernova Cosmology Project*, and the *High-z Supernova Search Team*. In addition to H_0, observations of Type Ia supernovae at high redshift have allowed them to measure q_0. Their data resulted in the remarkable discovery that the expansion of our Universe is accelerating, instead of slowing down. We have already incorporated their result in our model for the Universe. Rather than lecture about this phenomenal discovery in class, however, I'm going to have you read their papers, together with explanatory notes in your textbook, and write a report.

About half the class was spent on working problems from the Midterm exam.