PHY 475/375
Lecture 10
(April 25, 2012)

Multiple-Component Universes

So far, we have studied model universes that contain only one component — matter only, radiation only, etc. In this chapter, we will look at more sophisticated models containing two or more components.

We begin, as always, with the Friedmann equation; recall that it can be written in the form

$$H(t)^2 = \frac{8\pi G}{3c^2}\,\varepsilon(t) - \frac{\kappa c^2}{R_0^2\,a(t)^2} \tag{6.1}$$

where $H \equiv \dot{a}/a$ and $\varepsilon(t)$ is the energy density contributed by all the components of the universe.

Also, recall from an earlier lecture that equation (4.31) gave us a relation between κ, R_0, H_0, and Ω_0:

$$\frac{\kappa}{R_0^2} = \frac{H_0^2}{c^2}\Big(\Omega_0 - 1\Big) \tag{6.2}$$

Let us rewrite the Friedmann equation (6.1) without explicitly including the curvature by using equation (6.2):

$$H(t)^2 = \frac{8\pi G}{3c^2}\,\varepsilon(t) - \frac{H_0^2}{a(t)^2}\Big(\Omega_0 - 1\Big) \tag{6.3}$$

Dividing by H_0^2, this becomes

$$\frac{H(t)^2}{H_0^2} = \frac{\varepsilon(t)}{\varepsilon_{c,0}} + \frac{1-\Omega_0}{a(t)^2} \tag{6.4}$$

where $\varepsilon_{c,0}$ is the critical density in the current epoch, given by

$$\varepsilon_{c,0} \equiv \frac{3c^2 H_0^2}{8\pi G} \tag{6.5}$$

Now, we know that our Universe contains matter, for which the energy density ε_m has the dependence $\varepsilon_m = \varepsilon_{m,0}/a^3$, and radiation, for which the energy density ε_r has the dependence $\varepsilon_r = \varepsilon_{r,0}/a^4$. It may also have a cosmological constant, with energy density $\varepsilon_\Lambda = \varepsilon_{\Lambda,0} =$ constant. While it is certainly possible that the universe contains other components as well, we will consider only the contributions of matter ($w = 0$), radiation ($w = 1/3$), and the cosmological constant Λ ($w = -1$).

Recall from an earlier lecture that the energy density ε is additive, as written in equation (5.4):

$$\varepsilon = \sum_w \varepsilon_w$$

Using the additive property of the energy densities of different components, together with the various dependencies on the scale factor written above, equation (6.4) may be written as

$$\frac{H^2}{H_0^2} = \frac{\varepsilon_r + \varepsilon_m + \varepsilon_\Lambda}{\varepsilon_{c,0}} + \frac{1-\Omega_0}{a^2}$$

$$= \frac{\varepsilon_{r,0}/a^4 + \varepsilon_{m,0}/a^3 + \varepsilon_{\Lambda,0}}{\varepsilon_{c,0}} + \frac{1-\Omega_0}{a^2}$$

$$= \frac{\varepsilon_{r,0}/\varepsilon_{c,0}}{a^4} + \frac{\varepsilon_{m,0}/\varepsilon_{c,0}}{a^3} + \frac{\varepsilon_{\Lambda,0}}{\varepsilon_{c,0}} + \frac{1-\Omega_0}{a^2}$$

$$\Rightarrow \quad \frac{H^2}{H_0^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} + \frac{1-\Omega_0}{a^2} \tag{6.6}$$

where we have put $\Omega_{r,0} = \varepsilon_{r,0}/\varepsilon_{c,0}$, $\Omega_{m,0} = \varepsilon_{m,0}/\varepsilon_{c,0}$, $\Omega_{\Lambda,0} = \varepsilon_{\Lambda,0}/\varepsilon_{c,0}$, and $\Omega_0 = \Omega_{r,0} + \Omega_{m,0} + \Omega_{\Lambda,0}$.

Note also that the Benchmark model introduced in an earlier lecture has $\Omega_0 = 1$ (i.e., spatially flat), but while this is consistent with the observational data, it is not demanded by the data. Therefore, we will retain the curvature term $(1 - \Omega_0)/a^2$ in equation (6.6).

To proceed, we write $H = \dot{a}/a$ in equation (6.6), multiply both sides by a^2, and take the square root; this gives

$$H_0^{-1}\,\dot{a} = \left[\frac{\Omega_{r,0}}{a^2} + \frac{\Omega_{m,0}}{a} + a^2\,\Omega_{\Lambda,0} + (1-\Omega_0)\right]^{1/2} \tag{6.7}$$

Integrating, we get the cosmic time t as a function of a:

$$\int_0^a \frac{da}{[\Omega_{r,0}/a^2 + \Omega_{m,0}/a + a^2\,\Omega_{\Lambda,0} + (1-\Omega_0)]^{1/2}} = H_0\,t \tag{6.8}$$

In general, this integral does not have a simple analytic solution. There are instances, however, when simple analytic approximations may be found. For example, we discussed in an earlier lecture how in a multi-component universe containing radiation, matter, and Λ, the radiation dominates in the early stages. With radiation dominating, and $\Omega_0 = 1$ for a flat universe, equation (6.8) simplifies to

$$H_0\,t \approx \int_0^a \frac{da}{\sqrt{\Omega_{r,0}/a^2}} = \frac{1}{\sqrt{\Omega_{r,0}}}\int_0^a a\,da = \frac{1}{\sqrt{\Omega_{r,0}}}\left[\frac{a^2}{2}\right]_0^a$$

so that we get eventually

$$H_0\,t \approx \frac{a^2}{2\sqrt{\Omega_{r,0}}} \tag{6.9}$$

From equation (6.9), we get

$$a(t) \approx \left(2\sqrt{\Omega_{r,0}}H_0\,t\right)^{1/2} \tag{6.10}$$

In the limit $\Omega_{r,0} = 1$ (i.e., a spatially flat universe containing only radiation), this becomes

$$a(t) \approx \left(2\,H_0\,t\right)^{1/2}$$

Using $t_0 = 1/2H_0$ from equation (5.62) that we wrote in the last lecture for a flat universe containing only radiation, this becomes

$$a(t) \approx \left(\frac{t}{t_0}\right)^{1/2}$$

which matches equation (5.64) that we obtained for the scale factor in a spatially flat universe containing only radiation.

Likewise, there will be epochs where matter dominates, or Λ dominates, as we discussed in an earlier lecture.

During some epochs, however, two of the components are of comparable density, and provide terms of roughly equal size in the Friedmann equation. During these epochs, a single-component model is a poor description of the universe, and we need to use a two-component model. We will now look at some of these instances in more detail.

First, we will examine a universe which is of historical interest to cosmology: a universe containing both matter and curvature (either negative or positive). In the years following Einstein's revoking of Λ, cosmologists studied in detail the possibility of a spatially curved universe dominated by non-relativistic matter.

Matter + Curvature

In a curved universe containing only matter, we can put $\Omega_{r,0} = 0, \Omega_{\Lambda,0} = 0$.

Also, since $\Omega_0 = \Omega_{r,0} + \Omega_{m,0} + \Omega_{\Lambda,0}$, having made the above choices ($\Omega_{r,0} = 0, \Omega_{\Lambda,0} = 0$), we now have $\Omega_{m,0} = \Omega_0$.

With the above choices, the Friedmann equation (6.6) in a curved, matter-dominated universe can be written in the form

$$\frac{H(t)^2}{H_0^2} = \frac{\Omega_0}{a^3} + \frac{1-\Omega_0}{a^2} \tag{6.12}$$

Note that although $\Omega_{m,0} = \Omega_0$ is the only component, we retain the term $(1-\Omega_0)$ for the curvature which means that $\Omega_0 \neq 1$; in other words, $\varepsilon_{m,0} \neq \varepsilon_{c,0}$.

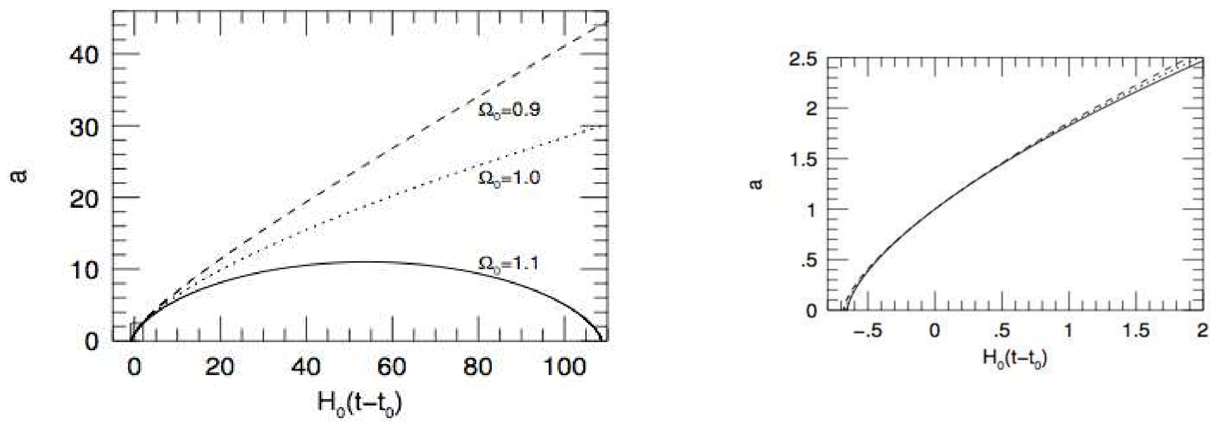
Let us now look at the characteristics of such a curved, matter-dominated universe. Suppose it is currently expanding ($H_0 > 0$).

- For it to stop expanding, we must have $H_0 = 0$. Since Ω_0/a^3 is always positive, $H(t) = 0$ requires the second term on the right hand side of equation (6.12) to be negative. This means that a matter-dominated universe will cease to expand only if $\Omega_0 > 1$, and hence from equation (6.2), $\kappa = +1$ (positively curved).
- We can find the scale factor $a_{\max}$ at maximum expansion by setting $H(t) = 0$ in equation (6.12):
$$0 = \frac{\Omega_0}{a_{\max}^3} + \frac{1-\Omega_0}{a_{\max}^2} \tag{6.13}$$
so that
$$\frac{a_{\max}^3}{\Omega_0} = \frac{a_{\max}^2}{\Omega_0 - 1}$$
from which we get
$$a_{\max} = \frac{\Omega_0}{\Omega_0 - 1} \tag{6.14}$$
Recall that Ω_0 is the density parameter measured at a scale factor $a(t_0) = 1$.
- Also, note that in equation (6.12), the Hubble parameter enters only as H^2, so there is a time symmetry, and the contraction phase is just the time reversal of the expansion phase (although not at small scales, that is, small-scale processes will not be reversed during the contraction phase; e.g., stars will not absorb the photons they previously emitted).
- The eventual collapse of a $\Omega_0 > 1$ universe is sometimes called the "Big Crunch" — the universe returns to the hot, dense state in which it had originally started. In its contracting stage, an observer will see galaxies with a blueshift proportional to their distance, and wavelengths in the cosmic microwave background will compress progressively to shorter ones to eventually end in a cosmic γ-ray background.

What happens, however, in a matter-dominated universe with $\Omega_0 < 1$, corresponding to $\kappa = -1$ (negatively curved)?

- Both terms on the right hand side of equation (6.12) are then positive, so if such a universe is expanding at $t = t_0$, it will continue to expand forever ("Big Chill").
- At early times, when the scale factor is small ($a \ll \Omega_0/\{1-\Omega_0\}$), the matter term in the Friedmann equation (6.12) will dominate, so we can write from equation (6.8) that
$$H_0\, t = \int_0^a \frac{da}{[\Omega_0/a + (1-\Omega_0)]^{1/2}} = \int_0^a \frac{da}{[1/a + (1-\Omega_0)/\Omega_0]^{1/2}} \approx \int_0^a \frac{da}{[1/a]^{1/2}} = \left[\frac{a^{3/2}}{3/2}\right]_0^a$$
where we have used the fact that if $a \ll \Omega_0/\{1-\Omega_0\}$, then $1/a \gg \Omega_0/\{1-\Omega_0\}$. From this expression, we see that the scale factor will grow at the rate $a \propto t^{2/3}$ at early times.
- Ultimately, the density of matter will be diluted far below the critical density, and the universe will expand like the negatively curved empty universe we discussed in the last lecture, with $a \propto t$.

A plot of the scale factor $a(t)$ vs. time, in units of $H(t - t_0)$ is shown in Figure 6.1 of your text, and reproduced below. The panel on the right is an enlarged view of the region enclosed by the small rectangle near "0" in the panel on the left.



- The solid line shows the fate of a matter-dominated universe with $\Omega_0 > 1$ (corresponding to $\kappa = +1$, positively curved). Such a universe expands until a maximum value of scale factor a_{max} given by equation (6.14) and then collapses back down to $a = 0$ in a "Big Crunch."

- The dashed line shows the perpetually expanding fate of a matter-dominated universe with $\Omega_0 < 1$ (corresponding to $\kappa = -1$, negatively curved). At early times, the scale factor grows at the rate $a \propto t^{2/3}$, but ultimately when the density of matter falls far below the critical density, the scale factor grows as $a \propto t$.

- For perspective, the fate of a spatially flat, matter-dominated universe ($\Omega_0 = 1, \kappa = 0$) that we discussed in the last lecture is shown by the dotted line. Such a universe grows as $a \propto t^{2/3}$ for all time.

Although the fate of the universe differs depending on whether Ω_0 is greater than, less than, or equal to one, it is clear from the panel on the right above that it is very difficult at $t \sim t_0$ to tell apart a universe with Ω_0 slight less than one from one with Ω_0 slightly greater than one; the right panel shows that the scale factors start to diverge significantly only after a Hubble time or more.

Matter + Lambda

Next, let us consider a universe which is spatially flat, but which contains both matter and a cosmological constant Λ. Such a universe may be a close approximation to our Universe in the present epoch.

Spatial flatness implies that

$$\Omega_{m,0} + \Omega_{\Lambda,0} = \Omega_0 = 1$$

so that

$$\Omega_{\Lambda,0} = 1 - \Omega_{m,0} \tag{6.22}$$

The Friedmann equation (6.6) for a flat universe containing matter and Λ then becomes

$$\frac{H^2}{H_0^2} = \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0}$$

and using equation (6.22) can be written as

$$\frac{H^2}{H_0^2} = \frac{\Omega_{m,0}}{a^3} + \left(1 - \Omega_{m,0}\right) \tag{6.23}$$

- The first term on the right hand side of equation (6.23) represents the contribution of matter and is always positive.
- The second term on the right hand side of equation (6.23) represents the contribution of a cosmological constant. It is positive if $\Omega_{m,0} < 1$, implying $\Omega_{\Lambda,0} > 0$, in which case the universe will continue to expand forever if it is expanding at $t = t_0$, another example of a "Big Chill" universe.
- On the other hand, the second term on the right hand side of equation (6.23) is negative if $\Omega_{m,0} > 1$, implying $\Omega_{\Lambda,0} < 0$; such a negative cosmological constant provides an attractive force. A flat universe with $\Omega_{\Lambda,0} < 0$ will cease to expand at a maximum scale factor $a_{\max}$ which can be found by setting $H^2 = 0$ in equation (6.23):

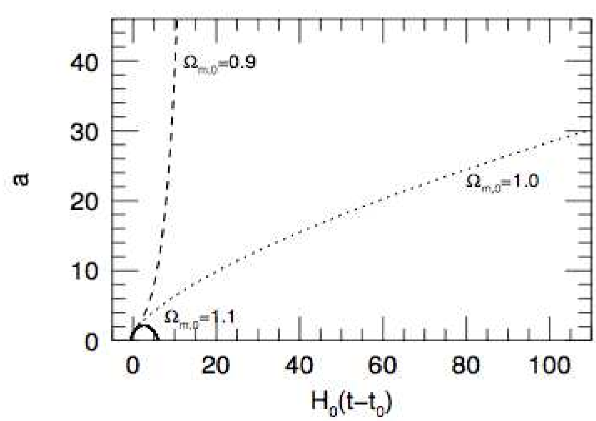
  $$0 = \frac{\Omega_{m,0}}{a^3} + \left(1 - \Omega_{m,0}\right)$$

  from which we obtain

  $$a_{\max} = \left(\frac{\Omega_{m,0}}{\Omega_{m,0} - 1}\right)^{1/3} \tag{6.24}$$

  Meanwhile, integration of the Friedmann equation (6.23) gives the solution

  $$H_0\, t = \frac{2}{3\sqrt{\Omega_{m,0} - 1}} \sin^{-1}\left[\left(\frac{a}{a_{\max}}\right)^{3/2}\right] \tag{6.26}$$



A plot of scale factor vs. time for the above cases is shown in Figure 6.2 in your text, reproduced on the right. Just as for a positively curved, matter-only universe that we discussed previously, a flat universe with negative cosmological constant ($\Omega_{\Lambda,0} < 0$) also ends in a "Big Crunch" (solid line), except that with the negative cosmological constant providing an attractive force, the lifetime of such a universe is extremely short. Also shown in the figure is the "Big Chill" expansion of a $\Omega_{m,0} < 1, \Omega_{\Lambda,0} > 0$ universe (dashed line), and for perspective, the $a \propto t^{2/3}$ behavior (dotted line) of a spatially flat, matter-dominated universe ($\Omega_{m,0} = 1, \Omega_{\Lambda,0} = 0$).

It is worth doing a comparison of "Big Crunch" lifetimes between a positively curved matter-only universe and a flat universe with a negative cosmological constant. While the graph in Figure 6.1 of your text (reproduced in this lecture on page 5) shows a "Big Crunch" after $\approx 110 H_0^{-1}$ in a positively curved universe containing only matter ($\Omega_{m,0} = \Omega_0 = 1.1$), a flat universe with a negative cosmological constant ($\Omega_{m,0} = 1.1, \Omega_{\Lambda,0} = -0.1$) has a lifetime of only $\approx 7H_0^{-1}$, as seen in Figure 6.2 of your text (reproduced on the previous page).

Of greater interest is a universe containing matter with a non-negative cosmological constant, which seems to resemble our Universe. In a flat universe with $\Omega_{m,0} < 1$ and $\Omega_{\Lambda,0} > 0$, we discussed in an earlier lecture how the density contributions of matter and the cosmological constant are equal at the scale factor

$$a_{m\Lambda} = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3} = \left(\frac{0.3}{0.7}\right)^{1/3} = 0.75 \tag{6.27}$$

With a flat, $\Omega_{\Lambda,0} > 0$ universe, the Friedmann equation (6.23) can be integrated to obtain

$$H_0\,t = \frac{2}{3\sqrt{1-\Omega_{m,0}}}\ \ln\left[\left(\frac{a}{a_{m\Lambda}}\right)^{3/2} + \sqrt{1+\left(\frac{a}{a_{m\Lambda}}\right)^{3}}\,\right] \tag{6.28}$$

The plot of a vs. t for such a "Big Chill" universe has already been discussed on the previous page.

At early times, when $a \ll a_{m\Lambda}$, equation (6.28) reduces to

$$H_0\,t = \frac{2}{3\sqrt{1-\Omega_{m,0}}}\ \ln\left[\left(\frac{a}{a_{m\Lambda}}\right)^{3/2} + 1\right]$$

which allows us to use the Taylor series expansion: $\ln\,(1+x) = x - x^2/2 + \ldots$, and retaining only the first term, we get

$$H_0\,t \approx \frac{2}{3\sqrt{1-\Omega_{m,0}}}\ \left(\frac{a}{a_{m\Lambda}}\right)^{3/2}$$

so that

$$a(t)^{3/2} \approx \frac{3}{2}\,H_0\,t\,\sqrt{1-\Omega_{m,0}}\,\Big[a_{m\Lambda}\Big]^{3/2} = \frac{3}{2}\,H_0\,t\,\sqrt{1-\Omega_{m,0}}\,\left[\left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3}\right]^{3/2}$$

and putting $\Omega_{\Lambda,0} = 1 - \Omega_{m,0}$, we get finally that at early times ($a \ll a_{m\Lambda}$),

$$a(t) \approx \left(\frac{3}{2}\,\sqrt{\Omega_{m,0}}\,H_0\,t\right)^{2/3} \tag{6.29}$$

with the $a \propto t^{2/3}$ dependence characteristic of a flat, matter-dominated universe.

On the other hand, at late times when $a \gg a_{m\Lambda}$, equation (6.28) reduces to

$$a(t) \approx a_{m\Lambda}\,\exp\left(\sqrt{1-\Omega_{m,0}}\,H_0\,t\right) \tag{6.30}$$

which gives the $a \propto e^{(\ldots)t}$ dependence characteristic of a flat, Λ-dominated universe.

If we measure H_0 and $\Omega_{m,0}$ in a flat universe containing only matter and Λ, then the age of the universe can be found from equation (6.28) by putting $a = a_0 = 1$:

$$t_0 = \frac{2H_0^{-1}}{3\sqrt{1-\Omega_{m,0}}} \ln \left[\left(\frac{1}{a_{m\Lambda}}\right)^{3/2} + \sqrt{1 + \left(\frac{1}{a_{m\Lambda}}\right)^3} \right]$$

$$= \frac{2H_0^{-1}}{3\sqrt{1-\Omega_{m,0}}} \ln \left[\left(\left\{ \frac{1-\Omega_{m,0}}{\Omega_{m,0}} \right\}^{1/3} \right)^{3/2} + \sqrt{1 + \left(\left\{ \frac{1-\Omega_{m,0}}{\Omega_{m,0}} \right\}^{1/3} \right)^3} \right]$$

$$= \frac{2H_0^{-1}}{3\sqrt{1-\Omega_{m,0}}} \ln \left[\sqrt{\frac{1-\Omega_{m,0}}{\Omega_{m,0}}} + \sqrt{\frac{1}{\Omega_{m,0}}} \right]$$

$$\Rightarrow \quad t_0 = \frac{2H_0^{-1}}{3\sqrt{1-\Omega_{m,0}}} \ln \left[\frac{\sqrt{1-\Omega_{m,0}} + 1}{\sqrt{\Omega_{m,0}}} \right] \qquad (6.31)$$

If we approximate our own Universe as having $\Omega_{m,0} = 0.3$ and $\Omega_{\Lambda,0} = 0.7$ (ignoring the contribution of radiation which, as we will see later has no significant effect on our estimate of t_0), we get its current age to be

$$t_0 = 0.964\, H_0^{-1} = (13.5 \pm 1.3) \text{ Gyr} \qquad (6.32)$$

assuming $H_0 = 70 \pm 7$ km s^{-1} Mpc^{-1}.

The epoch at which $\Omega_m = \Omega_\Lambda$ is obtained by putting $a = a_{m\Lambda}$ in equation (6.28):

$$t_0 = \frac{2H_0^{-1}}{3\sqrt{1-\Omega_{m,0}}} \ln \left[1 + \sqrt{2} \right] = 0.702\, H_0^{-1} = 9.8 \pm 1.0 \text{ Gyr} \qquad (6.33)$$

Equation (6.33) implies that, if our Universe is well fit by the Benchmark Model with $\Omega_{m,0} = 0.3$ and $\Omega_{\Lambda,0} \approx 0.7$, then the cosmological constant has been the dominant component of our Universe for the last 4 billion years or so.

Matter + Curvature + Lambda

In the above discussion, we showed that a flat universe with $\Omega_{m,0} > 1$ and $\Omega_{\Lambda,0} < 0$ is infinite in spatial extent, but has a finite duration in time.

On the other hand, we showed that a flat universe with $\Omega_{m,0} \leq 1$ and $\Omega_{\Lambda,0} \geq 0$ extends to infinity both in space and in time.

So then, if a universe containing both matter and Λ is curved, a wide range of behaviors is possible for the function $a(t)$. We can examine such behaviors by choosing different values of $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$ without constraining the universe to be flat.

Let us begin by writing the Friedmann equation for a curved universe containing both matter and a cosmological constant Λ:

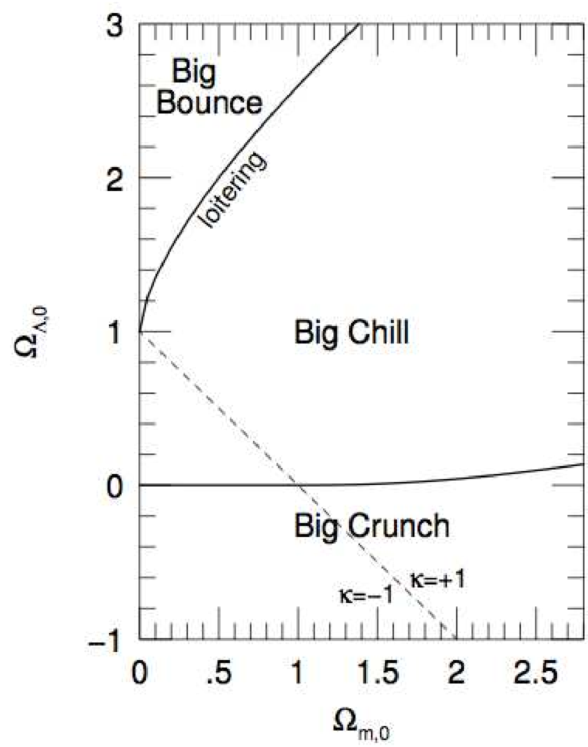
$$\frac{H^2}{H_0^2} = \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} + \frac{1 - \Omega_{m,0} - \Omega_{\Lambda,0}}{a^2} \tag{6.34}$$

Now, consider the following:

- If $\Omega_{m,0} > 0$ and $\Omega_{\Lambda,0} > 0$, then both the first and second terms on the right hand side of equation (6.34) are positive.
- If, however, $\Omega_{m,0} + \Omega_{\Lambda,0} > 1$, so that the universe is positively curved (based on equation 6.2), then the third term on the right hand side is negative.

As a result, for some choices of $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$, the value of H^2 will be positive for small values of a (where matter dominates) and for large values of a (where Λ dominates), but will be negative for intermediate values of a (where the curvature term dominates). Since negative values of H^2 are unphysical, this means that those universes have a forbidden range of scale factors.

The possibilities stemming from the above discussion are best studied via the plot in Figure 6.3 in your text, which shows the general behavior of the scale factor $a(t)$ as a function of $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$. The plot is reproduced below.



The dashed line in the plot is for $\Omega_{m,0} + \Omega_{\Lambda,0} = 1$, and hence indicates flat universes with $\kappa = 0$. Models lying above this line have positive curvature ($\kappa = +1$), whereas models lying below this line have negative curvature ($\kappa = -1$).

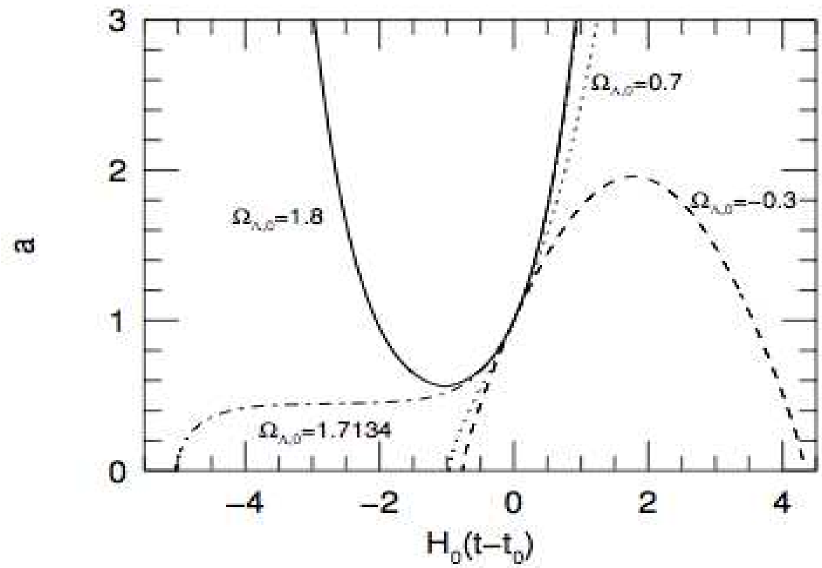
- In the region labeled "Big Crunch," the universe starts with $a = 0$ at $t = 0$, reaches a maximum scale factor a_{max}, then recollapses to $a = 0$ at a finite time $t = t_{\text{crunch}}$. Note that "Big Crunch" universes can be positively curved, negatively curved, or flat.
- In the region labeled "Big Chill," the universe starts with $a = 0$ at $t = 0$, then expands outward forever, with $a \to \infty$ as $t \to \infty$. Like "Big Crunch" universes, "Big Chill" universes can have any sign for their curvature.
- In the region labeled "Big Bounce," the universe starts out with $a \gg 1$ and $H_0 < 0$; that is, it is contracting from a low-density, Λ-dominated state. As it contracts, the negative curvature term in equation (6.34) becomes dominant, causing the contraction to stop at a minimum scale factor $a = a_{\text{min}} > 0$ at some time t_{bounce}. The universe then expands outward forever, with $a \to \infty$ as $t \to \infty$. That is, it is possible to have a universe which expands outward at late times, but which never had an initial Big Bang (with $a = 0$ at $t = 0$).

- Universes which fall just below the dividing line between "Big Bounce" and "Big Chill" universes are "loitering" universes, sometimes also called "Lemaitre" universes. Such a universe starts in a matter-dominated state, expanding outward with $a \propto t^{2/3}$. Then, it enters the "loitering" stage, in which a is very nearly constant for a long period of time (almost like a static universe). The closer such a universe lies to the "Big Bounce" – "Big Chill" dividing line, the longer its loitering stage lasts. Eventually, however, the cosmological constant takes over, and the universe starts to expand exponentially.

Yet another way to study the different types of expansion and contraction possible is by looking at a plot like Figure 6.4 in your text, which is reproduced below.

The figure shows $a(t)$ vs. time for four model universes. Each of these universes has the same current matter density parameter: $\Omega_{m,0} = 0.3$, measured at $t = t_0$ and $a = 1$.

The four model universes depicted in this plot cannot be distinguished from each other by measuring their current matter density and Hubble constant. Yet, due to having different values for the cosmological constant, they have very different pasts and very different futures.



- The dashed line in the figure above shows the scale factor for a universe with $\Omega_{\Lambda,0} = -0.3$. Since $\Omega_{m,0} + \Omega_{\Lambda,0} = 0.3 + (-0.3) = 0 < 1$, this universe has negative curvature. As shown in the plot, it is destined to end in a "Big Crunch."

- The dotted line shows $a(t)$ for a universe with $\Omega_{\Lambda,0} = 0.7$. Since $\Omega_{m,0} + \Omega_{\Lambda,0} = 0.3 + 0.7 = 1$, this universe is spatially flat. As shown in the plot, it is destined to end in an exponentially expanding "Big Chill."

- The solid line shows $a(t)$ for a universe with $\Omega_{\Lambda,0} = 1.8$. Since $\Omega_{m,0} + \Omega_{\Lambda,0} = 0.3 + 1.8 = 2.1 > 1$, this universe is positively curved. It is a "Big Bounce" universe which contracted until $a = a_{\rm bounce} \approx 0.56$, then started expanding outward, which it will continue to do forever.

- The dot-dash line shows $a(t)$ for a universe with $\Omega_{\Lambda,0} = 1.7134$. Since $\Omega_{m,0} + \Omega_{\Lambda,0} = 0.3 + 1.7 = 2.0 > 1$, this universe is positively curved. It is a "loitering" universe, which starts out by expanding outward, but then spend a long period of time near $a = a_{\rm loiter} \approx 0.44$. Eventually, it expands exponentially (*the exponential part is barely visible in the scanned figure, since it skims along the outer edge of the solid line; you should look at the figure in your text for better visibility*).

Since we are very close to accounting for all the components in our Universe, it is worth pausing to reflect whether any of the possibilities discussed above can be ruled out for our Universe. Strong observational evidence exists, in fact, to suggest that we do not live in a "loitering" or "Big Bounce" universe.

- If we lived in a "loitering" universe, then we would see nearly the same redshift for galaxies with a very large range of distances. For instance, with $a_{\rm loiter} \approx 0.44$ (the appropriate "loitering" scale factor for a universe with $\Omega_{m,0} = 0.3$, such as in ours), there should be a large excess of galaxies at redshift

$$z_{\rm loiter} = 1/a_{\rm loiter} - 1 \approx 1/0.44 - 1$$

  that is, at $z_{\rm loiter} \approx 1.3$. No such excess of galaxies is seen at this redshift, or indeed at any redshift, in our Universe.

- If we lived in a "Big Bounce" universe, then as we looked out into space, we would see redshifts increasing until a maximum $z_{\rm max} = 1/a_{\rm bounce} - 1$, after which redshifts would decrease until they became actually became blueshifts. In our Universe, we do not see very distant blueshifted galaxies.

The highest likelihood, at the moment, seems to be that our Universe is a "Big Chill" universe, fated to eternal expansion.

Radiation + Matter

In a previous lecture, we calculated that radiation-matter equality took place at a scale factor

$$a_{rm} \equiv \frac{\Omega_{r,0}}{\Omega_{m,0}} \approx 2.8 \times 10^{-4}$$

Near this scale factor, the universe is best described by a flat model containing both radiation and matter.

The Friedmann equation around the time of radiation-matter equality is then

$$\frac{H^2}{H_0^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} \tag{6.35}$$

After taking the square root, this may be rearranged as

$$\frac{1}{H_0}\left(\frac{\dot{a}}{a}\right) = \sqrt{\frac{\Omega_{r,0}}{a^4}}\left[1 + \frac{a\,\Omega_{m,0}}{\Omega_{r,0}}\right]^{1/2} = \frac{\Omega_{r,0}^{1/2}}{a^2}\left[1 + \frac{a}{a_{\rm rm}}\right]^{1/2}$$

and put in the form

$$H_0\,dt = \frac{a\,da}{\sqrt{\Omega_{r,0}}}\left[1 + \frac{a}{a_{\rm rm}}\right]^{-1/2} \tag{6.36}$$

Integrating equation (6.36), we get

$$H_0\, t = \frac{4a_{\rm rm}^2}{3\sqrt{\Omega_{r,0}}} \left[1 - \left(1 - \frac{a}{2a_{\rm rm}}\right)\left(1 + \frac{a}{a_{\rm rm}}\right)^{1/2}\right] \tag{6.37}$$

In the limit $a \ll a_{\rm rm}$ (radiation-dominated phase), we get

$$a \approx \left(2\sqrt{\Omega_{r,0}}\, H_0\, t\right)^{1/2} \tag{6.38}$$

which matches the result for the radiation-dominated phase obtained in equation (6.10).

The time of radiation-matter equation ($t_{\rm rm}$) can be found by setting $a = a_{\rm rm}$ in equation (6.37):

$$\begin{aligned} H_0\, t_{\rm rm} &= \frac{4a_{\rm rm}^2}{3\sqrt{\Omega_{r,0}}} \left[1 - \left(1 - \frac{1}{2}\right)\left(1 + 1\right)^{1/2}\right] \\ &= \frac{4a_{\rm rm}^2}{3\sqrt{\Omega_{r,0}}} \left[1 - \left(\frac{1}{2}\right)\sqrt{2}\right] \\ &= \frac{4a_{\rm rm}^2}{3\sqrt{\Omega_{r,0}}} \left[1 - \frac{1}{\sqrt{2}}\right] \end{aligned}$$

so that, putting back $a_{\rm rm} = \Omega_{r,0}/\Omega_{m,0}$, we get

$$t_{\rm rm} \approx 0.391\, \frac{\Omega_{r,0}^{3/2}}{\Omega_{m,0}^2}\; H_0^{-1} \tag{6.40}$$

For the Benchmark model, with $\Omega_{r,0} = 8.4 \times 10^{-5}, \Omega_{m,0} = 0.3$, we get

$$t_{\rm rm} = 0.391\, \frac{(8.4 \times 10^{-5})^{3/2}}{(0.3)^2}\; H_0^{-1} = 3.34 \times 10^{-6}\, H_0^{-1}$$

and with $H_0^{-1} = 14$ Gyr, the time of radiation-matter equality was

$$t_{\rm rm} = 3.34 \times 10^{-6} \left(14 \times 10^9 \text{ yr}\right) = 47,000 \text{ yr} \tag{6.41}$$

So the epoch when the Universe was radiation-dominated was very brief. This explains why our estimate of the age of the Universe as 13.5 Gyr in one of the previous sections, ignoring the contribution of radiation, is reasonable. The minor correction to the estimate of age by including the effects of radiation is drowned out by the 10% uncertainty in the value of H_0.

Having looked at all these contributions, we are now ready to look at the contents, history, and future of our actual Universe.