PHY 375
Homework 2 solutions
(Due by beginning of class on Wednesday, April 18, 2012)

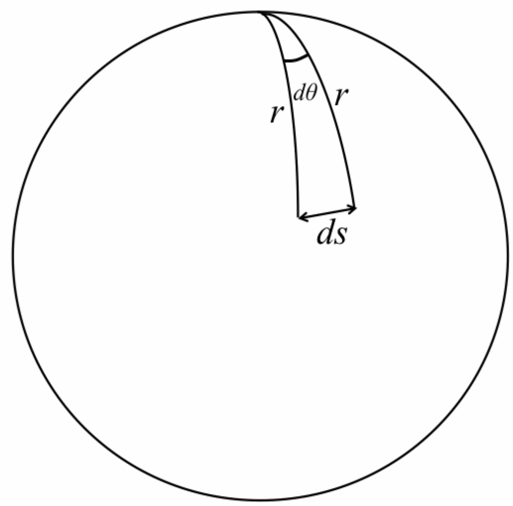
1. Suppose you are a two-dimensional being, living on the surface of a sphere with radius R. An object of width $ds \ll R$ is at a distance r from you (remember, all distances are measured on the surface of the sphere).

(a) What angular width $d\theta$ will you measure for the object?

Solution:

As discussed in lecture for 2-D surfaces, set up your location at the origin, which for convenience should be taken to be the north pole of the sphere.

This setup for the coordinate system is shown in the figure on the right. Recall that in setting up the metric for a 2-D surface, r is measured from the pole, and θ is the angle measured with respect to a great circle, so $d\theta$ as shown in the figure on the right matches this stipulation.



Now, since $ds \ll R$, whatever r may be, the distance from you to ds can be taken to be the same along the two sides of the triangle; another way of saying this is that you can take $dr = 0$.

The metric $ds^2 = dr^2 + R^2 \sin^2(r/R)d\theta^2$, with $dr = 0$, is

$$ds^2 = R^2 \sin^2\left(\frac{r}{R}\right) d\theta^2$$

Taking the square root, the angular width you will measure is

$$\boldsymbol{d\theta = \frac{ds}{R \sin(r/R)}}$$

(b) Examine and explain the behavior of $d\theta$ as $r \to \pi R$.

Solution: Notice that, at $r = \pi R/2$, we get $d\theta = ds/R$, the smallest value for $d\theta$.

So, for fixed ds, we find that $d\theta$ decreases with increasing r up to $r = \pi R/2$, whereas for $r > \pi R/2$, $d\theta$ increases with increasing r. This makes perfect sense if you look at the figure: if you keep ds fixed and bring it near the poles it will subtend a much larger angle $d\theta$ at the poles, whereas the same ds will subtend a smaller angle at the poles when it is taken near the equatorial region of the sphere.

2. The critical mass density of the Universe at the current epoch is given by

$$\rho_{c,0} = \frac{3}{8\pi G}\, H_0^2$$

where the Hubble parameter at the current epoch is $H_0 = (70 \pm 7)$ km s^{-1} Mpc^{-1}.

(a) Calculate the value of the critical mass density in the current epoch. Show your calculation clearly.

Solution: The critical mass density in the current epoch is given by:

$$\rho_{c,0} = \frac{3}{8\pi G}\, H_0^2 = \frac{3}{8\pi\left(6.67 \times 10^{-11}\ \mathrm{m^3\ kg^{-1}\ s^{-2}}\right)} \left(\frac{70,000\ \mathrm{m\ s^{-1}}}{\mathrm{Mpc}} \frac{1}{3.1 \times 10^{22}\ \mathrm{m/Mpc}}\right)^2$$

from which we obtain $\boldsymbol{\rho_{c,0} = 9.2 \times 10^{-27}\,\mathrm{kg\ m^{-3}}}$.

(b) Calculate the uncertainty in the value of the critical mass density in the current epoch. Show all steps clearly.

Solution:

If F is a function of variables $X_1, X_2, \ldots$, so that $F \equiv F(X_1, X_2, \ldots)$, then the uncertainty in F is found from:

$$\sigma_F = \sqrt{\left[(\partial F/\partial X_1)\,\sigma_{X_1}\right]^2 + \left[(\partial F/\partial X_2)\,\sigma_{X_2}\right]^2 + \ldots}$$

So

$$\begin{aligned}
\delta\rho_{c,0} &= \sqrt{\left(\frac{\partial \rho_{c,0}}{\partial H_0}\,\sigma_{H_0}\right)^2} \\
&= \frac{\partial \rho_{c,0}}{\partial H_0}\,\sigma_{H_0} \\
&= \frac{\partial}{\partial H_0}\left[\frac{3}{8\pi G}\,H_0^2\right]\sigma_{H_0} \\
&= \frac{3}{8\pi G}\left(2H_0\right)\sigma_{H_0} \\
&= \frac{3}{8\pi\left(6.67 \times 10^{-11}\,\mathrm{m^3\ kg^{-1}\ s^{-2}}\right)} \frac{(2)\,70,000\,\mathrm{m\ s^{-1}/Mpc}}{3.1 \times 10^{22}\ \mathrm{m/Mpc}} \left[\frac{7000\ \mathrm{m\ s^{-1}/Mpc}}{3.1 \times 10^{22}\ \mathrm{m/Mpc}}\right]
\end{aligned}$$

Therefore, the uncertainty is $\boldsymbol{\delta\rho_{c,0} = 1.8 \times 10^{-27}\,\mathrm{kg\ m^{-3}}}$.

(c) Since we just got through with March madness, let us suppose the universe is comprised of basketballs, each of mass 0.62 kg, and radius 0.12 m. If the basketballs were divided uniformly throughout the universe, what number density of basketballs would be required to make the mass density of the universe equal to the critical mass density in the current epoch that you calculated in part (a)?

Solution:

If m is the mass of a basketball, and n is the required number density of basketballs to make the mass density equal to the critical mass density of $\rho_{c,0} = 9.2 \times 10^{-27}$ kg m^{-3}, then $nm = \rho_{c_0}$, so that

$$n = \frac{9.2 \times 10^{-27}\,\text{kg m}^{-3}}{0.62\text{ kg}} = \mathbf{1.5 \times 10^{-26}\ m^{-3}}$$

(d) Given the number density of basketballs you found in part (c), how far would you be able to see, on average, before your line of sight intersected a basketball?

Solution:

Recall from HW 1 that this is obtained from

$$\frac{1}{n(\pi r^2)} = \frac{1}{(1.5 \times 10^{-26}\ \text{m}^{-3})\pi(0.12)^2} = \mathbf{1.5 \times 10^{27}\ m \equiv 48\ Gpc}$$

(e) Since we can see galaxies at distances of $c/H_0 \sim 4000$ Mpc, comment on whether the transparency of the universe on this length scale (of 4000 Mpc) places any useful limit on the number density of intergalactic basketballs.

Solution:

Since there are clear lines of sight up to 48 Gpc even with the critical density of basketballs present, transparency on the scale of 4000 Mpc places *no constraints whatsoever* on the number density of intergalactic basketballs.