PHY 375
Homework 1 solutions
(Due by beginning of class on Monday, April 9, 2012)

1. The Hubble "constant" H_0 can be used to obtain a rough estimate of the age of the Universe under a certain assumption.

(a) We discussed the assumption in class. What is it?

Solution:

The assumption is that the Universe has been expanding at constant velocity.

(b) In class, we wrote that under the current consensus value of $H_0 = 70$ km s^{-1} Mpc^{-1} and the assumption above, the approximate age of the Universe is 14×10^9 yr, or 14 Gyr. Due to severe underestimates of his measured distances to galaxies, Hubble originally measured $H_0 = 500$ km s^{-1} Mpc^{-1}. What would this value of H_0 give you for the approximate age of the Universe? *For full credit, you must show all your calculation steps clearly.*

Solution:

All we need to do is find H_0^{-1}, along with a straightforward conversion from Mpc to km.

Recall that 1 Mpc $\equiv 3.1 \times 10^{19}$ km

$$\begin{aligned} H_0 &= 500 \text{ km s}^{-1} \text{ Mpc}^{-1} \\ &= 500 \, \frac{\text{km}}{\text{s}} \, \frac{1}{\text{Mpc}} \\ &= 500 \, \frac{\text{km}}{\text{s}} \, \frac{1}{\text{Mpc}} \, \frac{\text{Mpc}}{3.1 \times 10^{19} \text{ km}} \end{aligned}$$

so that

$$H_0^{-1} = \frac{3.1 \times 10^{19} \text{ km}}{500 \text{ km s}^{-1}} = 6.2 \times 10^{16} \text{ s}$$

Converting to yr, we get

$$H_0^{-1} = \frac{6.2 \times 10^{16} \text{ s}}{3600 \times 24 \times 365 \text{ s/yr}} = 1.966 \times 10^9 \text{ yr} \equiv \boxed{\textbf{2 Gyr}}$$

2. Suppose that you are in an infinitely large, infinitely old universe in which standard Euclidean geometry holds true.

(a) The density of stars in this universe is $n_\star = 10^9$ Mpc^{-3} and the average radius of a star is equal to the Sun's radius: $R_\star = R_\odot = 7 \times 10^8$ m. How far, on average, could you see in any direction before your line of sight struck a star?

Solution:

The easiest way to solve this is by analogy with the mean free path of atoms/molecules in a gas. The treatment below is adapted from the chapter on Kinetic Theory of Gases by *Halliday & Resnick*.

The strategy in this derivation is to invert the situation. That is, instead of considering a ray of light (hence a point) moving along and striking a star surface of radius $R_\star$, think instead of the light as a circular surface of radius $R_\star$ and the star as a point. In other words, the light moves along as a circular disk, thereby defining a long cylinder of radius $R_\star$. So, the average distance between collisions (and, in our case, we need only one collision) is given by the length of this cylinder divided by the number of collisions in the cylinder. Since we are talking about light, the length of the cylinder in time t will be ct, whereas the number of collisions in the cylinder in time t will be the volume of the cylinder (V_{cylinder}) times the number of stars per unit volume ($n_\star$) in the cylinder. So,

$$\text{Distance before striking star} = \frac{\text{length of cylinder}}{\text{number of collisions}} = \frac{ct}{V_{\text{cylinder}} n_\star} = \frac{ct}{\pi R_\star^2 (ct) n_\star} = \frac{1}{n_\star \pi R_\star^2}$$

Let's calculate the denominator first; recall 1 Mpc $\equiv 3.1 \times 10^{22}$ m:

$$\begin{aligned} n_\star \pi R_\star^2 &= \left[10^9 \text{ Mpc}^{-3}\right] \pi \left(7 \times 10^8 \text{ m}\right)^2 \\ &= \left[10^9 \frac{1}{\text{Mpc}^3} \left\{\frac{1 \text{ Mpc}^3}{(3.1 \times 10^{22} \text{ m})^3}\right\}\right] \pi \left(7 \times 10^8 \text{ m}\right)^2 \\ &= 5.16726 \times 10^{-41} \text{ m}^{-1} \end{aligned}$$

where more digits than significant have been retained temporarily to avoid rounding errors.

Therefore

$$\text{Distance before striking star} = \frac{1}{n_\star \pi R_\star^2} = \frac{1}{5.16726 \times 10^{-41} \text{ m}^{-1}} = 1.9 \times 10^{40} \text{ m}$$

On average, therefore, you could see out to $\boxed{\mathbf{1.9 \times 10^{40}\ m}}$ before your line of sight struck a star.

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2. ***(Continued from previous page)***

(b) If the stars are clumped into galaxies with a density of $n_{\rm gal} = 1\ \text{Mpc}^{-3}$, and average radius $R_{\rm gal} = 2000$ pc, how far, on average, could you see in any direction before your line of sight struck a galaxy?

Solution:

As in part (a), let us calculate the denominator first, but this time with $n_{\rm gal} = 1\ \text{Mpc}^{-3}$ and $R_{\rm gal} = 2000$ pc ; recall, 1 pc $= 3.1 \times 10^{16}$ m, and 1 Mpc $\equiv 3.1 \times 10^{22}$ m:

$$\begin{aligned} n_{\rm gal}\,\pi R_{\rm gal}^2 &= \left[1\ \text{Mpc}^{-3}\right]\,\pi\left(2000\ \text{pc}\right)^2 \\ &= \left[1\ \frac{1}{\text{Mpc}^3}\ \left\{\frac{1\ \text{Mpc}^3}{(3.1\times10^{22}\ \text{m})^3}\right\}\right]\ \pi\ \left(2000\ \text{pc}\ \left\{3.1\times10^{16}\ \frac{\text{m}}{\text{pc}}\right\}\right)^2 \\ &= 4.05367\times10^{-28}\ \text{m}^{-1} \end{aligned}$$

where more digits than significant have been retained temporarily to avoid rounding errors.

So, distance before striking star $= \dfrac{1}{n_{\rm gal}\,\pi R_{\rm gal}^2} = \dfrac{1}{4.05367\times10^{-28}\ \text{m}^{-1}} = 2.5{\times}10^{27}$ m

On average, therefore, you could see out to $\boxed{\mathbf{2.5\times10^{27}\ m}}$ before your line of sight struck a galaxy.

(c) To make sense of your results, convert your answers in parts (a) and (b) to Mpc, and compare them to the approximate size of the Universe c/H_0, then comment on how this helps you with resolving Olbers' paradox.

Solution:

The distance that you could see, on average, before your line of sight struck a star was found in part (a); it is

$$1.9\times10^{40}\ \text{m} = 1.9\times10^{40}\ \text{m}\ \left[\frac{1\ \text{Mpc}}{3.1\times10^{22}\ \text{m}}\right] = \boxed{\mathbf{6.1\times10^{17}\ Mpc}}$$

The distance that you could see, on average, before your line of sight struck a galaxy was found in part (b); it is

$$2.5\times10^{27}\ \text{m} = 2.5\times10^{27}\ \text{m}\ \left[\frac{1\ \text{Mpc}}{3.1\times10^{22}\ \text{m}}\right] = \boxed{\mathbf{8.1\times10^{4}\ Mpc}}$$

Since $c/H_0 = 3\times10^5$ km/s/70 km/s Mpc^{-1} = 4300 Mpc, these distances are orders of magnitude larger than the approximate size of the Universe. Therefore, on average, it is rare that your line of sight will end in a star or galaxy, which resolves Olber's paradox — the night sky is dark because only a few lines of sight end on a star or galaxy.

3. Carry out the following calculations, showing your steps clearly.

(a) In 1950, astronomer Jan Oort proposed that long-period comets come from a vast distant spherical shell of icy bodies that may extend as far out as 100,000 AU from the Sun. Express this distance in light years (LY) and in parsec (pc).

Solution: Recall that 1 pc = 3.26 LY

$$100,000 \text{ AU} \left(\frac{150\times 10^6 \text{ km}}{1 \text{ AU}}\right)\left(\frac{1 \text{ LY}}{9.46\times 10^{12} \text{ km}}\right) = \boxed{\textbf{1.6 LY}} = \frac{1.6 \text{ LY}}{3.26 \text{ LY/pc}} = \boxed{\textbf{0.5 pc}}$$

(b) The NGC 3795 galaxy in the Ursa Major group has a redshift $z = 0.004036$. Find its recession velocity, and use this to find the distance to NGC 3795 in Mpc. For the Hubble constant, use the value $H_0 = 70$ km s^{-1} Mpc^{-1}.

Solution:

Since z is small, we can use $v = cz$ to obtain the recession velocity v.

$$v = cz = \left(3\times 10^5 \text{ km s}^{-1}\right) 0.004036 = \boxed{\textbf{1211 km s}^{-1}}$$

Using the Hubble law, $v = H_0\, d$, the distance will then be

$$d = \frac{v}{H_0} = \frac{1211 \text{ km s}^{-1}}{70 \text{ km s}^{-1} \text{ Mpc}^{-1}} = \boxed{\textbf{17.3 Mpc}}$$

(c) Quasars are extremely luminous sources that are at great distances from us.. The most distant quasar was discovered last year at a redshift of $z = 7.1$. Calculate the recession velocity of this quasar, and hence find the distance to it in Mpc.

Solution:

We will use the full relativistic Doppler effect formula to avoid faster than light recession velocity (but see posted lecture notes about why this step angers theoreticians, especially because faster than light motions are not a problem in general relativity; in fact, the preference is to keep distances in terms of z rather than convert like we're doing here).

$$1 + z = \sqrt{\frac{(c+v)}{(c-v)}} \quad \Rightarrow \left(1 + 7.1\right)^2 = \frac{c+v}{c-v}$$

So

$$\left(c - v\right)(8.1)^2 = c + v \quad \Rightarrow \quad \left(65.61 - 1\right)c = \left(65.61 + 1\right)v$$

Therefore

$$v = \frac{64.61\, c}{66.61} = \frac{64.61\left(3\times 10^5 \text{ km s}^{-1}\right)}{66.61} = \boxed{\textbf{2.9}\times\textbf{10}^{\textbf{5}}\textbf{ km s}^{-1}}$$

and the distance to this quasar would be

$$d = \frac{v}{H_0} = \frac{2.9\times 10^5 \text{ km s}^{-1}}{70 \text{ km s}^{-1} \text{ Mpc}^{-1}} = \boxed{\textbf{4143 Mpc}}$$