WebHelp: Mathematical Induction

Mathematical induction is used to prove a sequence of statements indexed by the positive integers. For example, if

\[ S(n) : 1 + 2 + \cdots + n = \frac{n(n+1)}{2}, \]

mathematical induction can be used to prove that \( S(n) \) is true for all positive integers \( n \). In other words, mathematical induction can be used to prove that

\[
S(1) : 1 = \frac{1 \cdot 2}{2},
S(2) : 1 + 2 = \frac{2 \cdot 3}{2},
S(3) : 1 + 2 + 3 = \frac{3 \cdot 4}{2},
\vdots
S(n) : 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2},
S(n + 1) : 1 + 2 + 3 + \cdots + n + (n + 1) = \frac{(n + 1)(n + 2)}{2},
\vdots
\]

are all true. [Notice that \( S(n + 1) \) is obtained from \( S(n) \) by everywhere substituting \( n + 1 \) for \( n \).]

In its simplest form (we discuss the strong form of mathematical induction at the end of this WebHelp), mathematical induction requires two steps

- **Basis Step.** Prove that \( S(1) \) is true.
- **Inductive Step.** For every $n$, assume that $S(n)$ is true and prove that $S(n+1)$ is true.

To see why the Basis and Inductive Steps prove that $S(n)$ is true for all $n$, consider any specific value of $n$, for example, $n = 5$. Is $S(5)$ true? Well, because of the Basis Step, $S(1)$ is true. The Inductive Step says that if $S(1)$ is true, then $S(2)$ is true. $S(1)$ is true! Therefore $S(2)$ is true. The Inductive Step says that if $S(2)$ is true, then $S(3)$ is true. $S(2)$ is true! Therefore $S(3)$ is true! The Inductive Step says that if $S(3)$ is true, then $S(4)$ is true. $S(3)$ is true! Therefore $S(4)$ is true. The Inductive Step says that if $S(4)$ is true, then $S(5)$ is true. $S(4)$ is true! Therefore $S(5)$ is true! We could use a similar argument for any value of $n$, therefore $S(n)$ is true for every $n$.

The Basis Step is usually straightforward. In the previous example, the Basis Step is to prove

$$1 = \frac{1 \cdot 2}{2},$$

which is certainly true! In the previous example, the Inductive Step is to assume that

$$1 + 2 + \cdots + n = \frac{n(n + 1)}{2}$$

is true, and then prove that

$$1 + 2 + \cdots + n + (n + 1) = \frac{(n + 1)(n + 2)}{2}$$

is true. *We recommend that you always write out both case $n$ and case $n+1$ before proceeding further with the Inductive Step.*

The key to proving the Inductive Step is to “uncover” case $n$ within case $n + 1$. Although the meaning of “uncover” depends on the context, it is *always* the case that the success of the Inductive Step rests on uncovering case $n$ within case $n + 1$.

For our example, case $n$ involves

$$1 + 2 + \cdots + n,$$

which appears within case $n + 1$:

$$1 + 2 + \cdots + n + (n + 1).$$

This is case $n$.

Since we are assuming that case $n$ is true, that is, that

$$1 + 2 + \cdots + n = \frac{n(n + 1)}{2},$$

we may substitute

$$\frac{n(n + 1)}{2}$$
for
\[1 + 2 + \cdots + n\]
in case \(n + 1\) to obtain
\[1 + 2 + \cdots + n + (n + 1) = \frac{n(n + 1)}{2} + (n + 1).
\]
[Notice that the term \((n + 1)\) was not replaced by anything and, so, is just copied from the left side to the right side of the equation.] The Inductive Step is completed by using algebra to get the right side into the correct form, namely,
\[
\frac{(n + 1)(n + 2)}{2}.
\]
We have
\[1 + 2 + \cdots + n + (n + 1) = \frac{n(n + 1)}{2} + (n + 1) = \underbrace{\text{factor out } n + 1}_{\text{factor out } n + 1} = (n + 1) \left( \frac{n}{2} + 1 \right) = \frac{(n + 1)(n + 2)}{2}.
\]
The Inductive Step is finished and the proof by mathematical induction is complete.

Summary
To give a proof that \(S(n)\) is true for every positive integer \(n\) using mathematical induction in its simplest form:

- Prove directly that \(S(1)\) is true.
- Assume that \(S(n)\) is true and prove that \(S(n + 1)\) is true. As an aid, write out \(S(n)\) and \(S(n + 1)\) explicitly, remembering that \(S(n + 1)\) is obtained from \(S(n)\) by everywhere replacing \(n\) by \(n + 1\).

To help with the latter step, look for \(S(n)\) within \(S(n + 1)\).

Strong Form of Mathematical Induction
In the strong form of mathematical induction, the preceding Inductive Step

- Inductive Step. For every \(n\), assume that \(S(n)\) is true and prove that \(S(n + 1)\) is true.

is replaced by

- Inductive Step for Strong Form of Mathematical Induction. For every \(n\), assume that \(S(k)\) is true for all \(k < n\) and prove that \(S(n)\) is true.
The Basis Step is unchanged. In the strong form of mathematical induction, to prove that $S(n)$ is true we may assume the truth of $S(k)$ for all $k$ that precede $n$, namely $1, 2, \ldots, n - 1$. In the simple form of mathematical induction, to prove that $S(n + 1)$ is true, we assume the truth of $S(k)$ only for the $k$ that immediately precedes $n + 1$, namely $n$. In other words, in the strong form of mathematical induction, to prove that $S(n)$ is true, we assume that $S(1), S(2), \ldots, S(n - 1)$ are all true. In the simple form of mathematical induction, to prove that $S(n + 1)$ is true, we assume only that $S(n)$ is true. The strong form of mathematical induction is useful when the truth of all the preceding cases helps prove case $n$. 