Risk Formulæ for Proportional Betting

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Introduction

A canon of the theory of betting is that the optimal procedure is to bet proportionally to one's advantage, adjusted by variance (see [Ep, Th, www.bjmath.com] for discussion and more references). This is the well-known "Kelly Criterion". It results in maximum expected geometric rate of bankroll growth, but entails wild swings, which are not for the faint of heart. A more risk-averse strategy used by many is to scale things back and bet a fraction of the Kelly bet. This is done commonly by blackjack teams (see www.bjmath.com) and futures traders, e.g. [Vi], where the Kelly fraction is referred to as "optimal f".

In this article we examine what happens when we bet a fraction of Kelly in terms of the risk of losing specified proportions of one's bank. We employ a diffusion model, which is a continuous approximation of discrete reality. This model is appropriate when the bets made are "small" in relation to the bankroll. The resulting formulae are limiting versions of discrete analogs and are often much simpler and more elegant. This is the theoretical set-up used for the Kelly theory.

The main result presented gives the probability that one will win a specified multiple of one's bankroll before losing to specified fraction as a function of the fraction of Kelly bet. This formula (2.1) was reported in [Go]. There it is derived from a more complicated blackjack-specific stochastic model. See also [Th] for related results. Our approach results in the same formula, but more assumes from the outset a standard "continuous random walk with drift" model. We do not have a historical citation, but it is certainly true our results here are almost as old as Stochastic Calculus itself, and predate any mathematical analyses of blackjack. It is our hope that our exposition of these results will bring some greater unity and clarity to them in the mathematics of blackjack and related communities.

The mathematical results quoted here require Stochastic Calculus. To derive them from first principles would entail serious graduate study in probability theory. Less ambitious readers can just believe the results and skip to section 2 (or even section 4), or try to view them through the lens of the discrete analog that we will introduce first in section 1. Our main reference is the relatively elementary text [KT], which is accessible to those with a solid background in advanced Calculus and Probability Theory (advanced undergraduate or entry-level graduate Mathematics). There are many other more advanced texts on Stochastic Calculus, e.g. [Øk]. Computations of limits and algebraic manipulations are left as easy exercises for those who passed freshman calculus.

In proportional betting you never lose everything, since you are betting a fraction (less than one) of your bank. It seems sensible to ask then, not about ruin, but rather about the risk of ever losing a specified fraction of your bank (2.1 and 2.2). We believe the present approach to be the most direct and sensible approach to the risk of 'unhappiness'.

In the first section we present the basic diffusion for proportional betting, describe its relation to the discrete analog, state growth rates, and give the main drawdown formula. We continue and interpret the formula in terms of risk of unhappiness in a couple of ways (2.2 and 2.3). The first is an elegant formula for the probability that you will *ever* reach a specified fraction $a$of your bankroll. The answer turns out to be simply a power of a. We then comment on exit times in 2.4 for the sake mathematical completeness and as a reality check.

Some of the results here may be also obtained by manipulating "Risk of Ruin" formulas[1], where bets are not readjusted according to the bankroll. Indeed, in section 3 we discuss ruin formulas using a general risk formula for linear Brownian motion, tying things in with proportional betting. The general formula in section 2 is a geometric version of the general risk formula in section 3. We explain how Kelly fractional betting, risk of ruin, bankroll requirements and linear Brownian motion are related in 3.2-3.4.

We close in the 4th section with some numerical calculations, which

[1]Thanks to "MathProf", a frequent contributor to the blackjack sites bjmath.com and bj21.com for pointing this out.

affirm (subjectively, of course) the comfort of conventional Kelly fractions in the range of one-quarter to one-third.

1 The diffusion equation

Notation

$B = B(t)$ *bankroll at time* t
μ *expected return for unit bet*
σ *standard deviation for unit bet*
$v = \sigma^2$ *variance for unit bet*
k *fraction of Kelly bet*
$r(k - k^2/2)(\mu^2/v)$ *the expected continuous growth rate (see below)*

1.1 An informal discrete analog

Suppose you make a bet every unit of time Δt. Your k times Kelly bet is $(k\mu/v)B$. Your expected return for this bet is then $\Delta B = (k\mu^2/v)B$ and your standard deviation is $\sigma(k\mu/v)W$ where W is a standard normal random variable. So your return is

$$\Delta B = \frac{k\mu^2}{v} B\Delta t + \sigma \frac{k\mu}{v} BW.$$

The very rough idea now is to convert the deltas to infinitesimals by infinitely subdividing bets; the random term W is replaced by random variable that converges to a random process. If the random term were not present (by a very elementary differential equation) we would obtain exponential growth at rate $(k\mu^2/v)$.

The diffusion equation below is a continuous analog, which is a good approximation if the amount bet is a small fraction of the bankroll. This will be the case if μ^2/v is small. In blackjack, this bet is on the order of 10^{-4} (since the advantage is a little over 1% and the variance is a little more than 1).

1.2 The diffusion and growth rate

The k times Kelly bet is the k times expected return divided by the variance *$(k\mu/v)B$*. The expected return from this bet is therefore $(k\mu^2/v)B$. The diffusion equation is (cancelling σ's in the variance term):

$$dB = \frac{k\mu^2}{\sigma^2} Bdt. + \frac{k\mu}{\sigma} BdW.$$

where W is a Wiener process. It will be important at the end of section 3 to note that this diffusion only depends on k and the ratio μ/σ. The diffusion is equivalent to one with $\sigma = 1$, Ito's Lemma (a fundamental result for stochastic differential equations, e.g. [Øk]), and a little algebra tells us that

$$d(lnB) = (k - \frac{k^2}{2})\frac{k\mu^2}{\sigma^2}dt + (k\mu/\sigma)dW$$

This is an example of "Geometric Brownian Motion". The $-k^2/2$ term is a "variance penalty".

Some well-known facts that are now apparent: The expected growth rate is

$$r = (k - \frac{k^2}{2})\frac{k\mu^2}{\sigma^2}$$

which is maximized when $k = 1$, the pure Kelly bet. In this case the rate is $\mu^2/(2\sigma^2)$, so the variance cuts your growth rate in half. The intercepts at $r = 0$ are obtained when we overbet with $k = 2$ or we don't bet at all at $k = 0$. In general, our growth rate r is down a factor of $(2k - k^2)$ from the optimal $\mu^2/(2\sigma^2)$.

The practical question is: Where on this parabolic arc (with $0 < k \leq 1$) do you want to be? Note that $k > 1$ is always suboptimal since we can always do better with less risk. For example $k = 3/2$ and $k = 1/2$ give the same growth rates.

2 Drawdown formulae

2.1 The general formula for proportional betting

Theorem 1 *Suppose we bet k times Kelly as in section 1, and our initial bankroll is one (unit). Then the probability $P(a, b)$ that we reach $b > 1$before reaching $a < 1$ is*

$$P(a, b) = \frac{1 - a^{1-\frac{2}{k}}}{b^{1-\frac{2}{k}} - a^{1-\frac{2}{k}}}.$$

This formula follows from Chapter 4 of [KT] and follows from the linear version below (3.2). Notice that the advantage μ and the standard deviation σ disappear. Having a bigger advantage just speeds up time (see Exit Times below)! The σ's cancel out, as the k times Kelly bet 'normalizes' variance. If $a = .5$ and $b = 2$ we get the often quoted probability that you double before being halved 2/3 of the time at full Kelly. This means that 1/3 of the time you get halved before doubling. As we stated in the introduction, Kelly betting is not for the faint of heart.

2.2 The ultimate risk of unhappiness

The serious long-term blackjack player does not worry about winning too much, but worries about losing. We think of getting cut down to a fraction a as an alternate concept of "ruin" or "unhappiness". We model this by letting b go to infinity, and obtain the *probability P(a) of ever reaching B=a:*

$$P(a) = a^{2/k-1}$$

This is a pretty nice simplification, isn't it? At full Kelly, $k = 1$, the probability of hitting the fraction a of one's bank is simply a. At half-Kelly the probability is a^3.

Of course if you take the limit as a goes to zero in the formula for $P(a, b)$, we can check that (as expected) $P(a, b) \to 1$. So with proportional betting you always reach your goal, if you can weather the storm.

2.3 A symmetrized risk formula

The probability that you hit $b = 1/a$ before losing to a is $P(a, b)$. Some algebra reduces the ensuing expression to

$$P(a, \frac{1}{a}) = \frac{a^{1-\frac{2}{k}}}{1 + a^{1-\frac{2}{k}}}$$

Thus the likelihood of tripling before losing 2/3 (i.e., $a = 1/3$) at full Kelly is .75.

It is curious that, for any positive $k < 2$, the limit of P as $a \to 0$ and (as $b \to \infty$) is 1, while the limit as $a \to 1$ is .5 (variance overwhelms drift in the short run).

2.4 Exit times

Assume the set-up as in the Theorem with fixed k. The expected exit time from [KT] is

$$E(T) = \frac{1}{r}(q(\ln(b) - \ln(a)) + \ln(a))$$

where $q = P(a, b)$ as in the main formula 2.1 and $r = (k - k^2/2)(\mu^2/v)$. Here the random variable T is the exit time and $E(T)$ is its expected value. It simplifies to

$$E(T) = \frac{1}{r}\ln(b^q/a^{q-1})$$

This is the mean time before you either win and hit $B = b$ or lose to a.

The ultimate expected exit time: If you willing to weather any storm, until you reach your goal of $B = b$, then taking the limit as $a \to 0$ yields $q = P(a, b) \to 1$. The expected time it takes to reach b is

$$E(T) = \frac{1}{r} \ln(b)$$

This reality check should be no surprise to astute readers of section 1.

3 Drawdown Formulæ with No Bet Resizing

3.1 Ruin versus other measure of risk

In the blackjack literature and online community (e.g. [Sch]; bj21.com green chip area, bjmath.com) there is interest in risk of ruin for a fixed betting schemes ("units") for various games. Risk is sometimes parameterized by "risk of ruin" instead of other drawdowns such as we discussed above. There has been debate about whether to resize often or just stay with a pre-established betting unit for a given "trip". Resizing betting units as often as practically possible (e.g. to the nearest green chip) is the most sensible one, since any but the most foolhardy will in fact resize after enough bankroll movement. The results above can then be used to approximate or give bounds for various probabilities.

Still, many advantage players traditionally think in terms of risk of ruin, assuming (contrary to reality) that they will not ever resize. Thus they have a theoretical risk of ruin, which we think of as an *instantaneous risk of ruin*. It is instantaneous because it will change (in our continuous model) as the bankroll changes.

3.2 The Risk Formula for Brownian Motion with Linear Drift

The following result gives general exit probabilities for the linear analog of the Geometric Brownian Motion above. It can be found in standard texts, e.g. [KT], and implies the geometric drawdown formulae above as well as instantaneous risk of ruin formulae (below). Here $X = X(t)$ is the bankroll at time t and X_0 is the starting bankroll.

Theorem 2 *For Brownian motion $dX = rdt + sdW$ where r is the (constant) linear drift rate s is the (constant) standard deviation, W is the standard Wiener process, and $a < X_0 < b$, the probability that X hits b before a*

is

$$\frac{e^{f(X_0)} + e^{f(a)}}{e^{f(b)} + e^{f(r)}}$$

where $f(x) = -2xr/s^2$ *for all* x*.*

The proof of this follows from an "infinitesimal first step analysis", a Taylor series expansion and the resulting differential equation. It also follows from the Optimal Stopping Theorem [Øk].

3.3 Risk of Ruin

This formula can be interpreted by viewing X as the bankroll in a game with win rate r and variance s^2 (per unit of time). By taking the limit of this expression above (3.2) as a goes to zero and b goes to infinity we quickly arrive at the often quoted ruin formula (see [Sch]) for Brownian motion with linear drift.

Corollary 3 *The probability that* X *will ever hit zero is*

$$\exp(\frac{-2rX_0}{s^2})$$

where r *is the linear win rate,* s *is the standard deviation.*

3.4 Instantaneous Risk of Ruin with Initial Fractional Kelly Bet

The bettors who initially bet a k times Kelly bet, *but do not resize*, have a linear drift rate $r = (k\mu^2/v)B$ and standard deviation $\sigma(k\mu/v)B$ where $B = X_0$ is the initial bankroll (see section 1). A little easy algebra using the previous corollary gives the following ruin formula, which has appeared at various times on bjmath.com and bj21.com. It gives the "k times Kelly-equivalent risk of ruin".

Corollary 4 *The probability that* X *will ever hit zero is*

$$\exp(-2/k)$$

For example, at $k = 1$, the non-resizer has a risk of ruin of e^{-2} or about 13.5%, the so-called "Kelly-equivalent risk of ruin". It follows the bettor that constantly resizes so that his instantaneous risk of ruin (assuming no

k	risk of ruin
1.0	13.53%
.5	1.83%
.4	0.67%
.35	0.33%
.30	0.13%
.25	0.03%
.20	0.00%

subsequent resizing) is e^{-2} is precisely a Kelly bettor. Of course a similar statement holds for other values of k, as we show in the table below.

We can then ask, "What risk of ruin is right for me (or my team)?" The very small probabilities make this subjective question perhaps hard to fathom. However, we know empirically from our experience with blackjack teams that $k = .5$ is considered too risky, and most team settle in the .25 to .35 range. Thus we have an *implied* risk of ruin from blackjack in the range of .03% to about 0.3%. It should be noted that these numbers are gleaned from professional blackjack teams. It has been suggested that individuals might have realistically have a *much* smaller k (e.g. see K. Janacek on bjmath.com), and we concur with this. It should be noted that the value of k specifies a utility function, which characterizes risk tolerance (see e.g. [Ep]).

4 Some Numbers

We look more closely at some special cases of the formula to see how Kelly fractions affect risk. Here we introduce the variable $x = 1/k$, the inverse of the Kelly fraction. Thus $x = 1$ and $x = 2$ correspond to full Kelly and half-Kelly, respectively.

4.1 To Halve and to Halve Not

Below we tabulate the function $f(x) = 1 - a^{2x-1}$, which is the risk that you *never* reach the value a, as a varies from .5 to .8. For $a = .5$ it appears that this risk of being halved gets very small and doesn't change much as x increases above 4. This indicates (quite subjectively of course) that there is little reason for blackjack players to be more conservative than quarter-Kelly. Some futures traders suggest $k = 1/6$, a conservative fraction perhaps due to the fact that traders are not usually sure of their edge (among other infelicities).

The risk of never being unhappy

a \ x	**1**	**2**	**3**	**4**	**5**	**6**
.5	.50	.88	.97	.99	.99	1.00
.6	.40	.78	.92	.97	.99	1.00
.7	.30	.66	.83	.92	.96	.98
.8	.20	.48	.67	.79	.87	.91

a=0.5,...,0.8; $x=\frac{1}{k}$=1,...,6;

4.2 "They never looked back"

There are those successful bettors who win and expand their bankrolls, with no worries about sliding back into the red. There is a lot of folklore about those winners who "never looked back". The table below demonstrates how that once you win big, you are very unlikely to ever go down to half of the original bank at Kelly values $k \leq 1/3$ *(i.e.* $x = 3, 4, 5 \ldots$*).* Since the probabilities change little (for increasing b, $x \geq 2$), this means that if you are going to get halved, it is extremely likely to happen before you double. It is perhaps intuitively obvious, but we find the speed of convergence to the limiting values notable. We give the probabilities $P(a, b)$ for various Kelly fractions $k = 1/x$, for $a = .5$ and a range of $b's$ in the table below. Notice that the limiting row is the $a = .5$ row in the table above.

The probability of reaching goal b before being halved

b \ x	**1**	**2**	**3**	**4**	**5**	**6**
2	.67	.89	.97	.99	.99	1.00
3	.60	.88	.97	99	1.00	1.00
4	.57	.88	.97	.99	1.00	1.00
∞	.50	.88	.97	.99	1.00	1.00

b=2,3,4,∞; x=1,...,6

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