Uniqueness and non uniqueness for harmonic functions with zero nontangential limits

J. Marshall Ash*, DePaul University, Chicago, IL 60614
Russell Brown†, University of Kentucky, Lexington, KY 40506

1. Introduction.

Definitions. By D we mean the open unit disc which is centered at the origin in the complex plane and by T we mean its boundary, *i.e.*, its circumference.

The question we will address here is to what extent does the limiting behavior of a harmonic function on T determine its values on D. A simple result in this direction follows from the maximum principle: If a harmonic function has limit 0 at each point of T, then the function is 0 on D (see Thm. 1 in Section 2). Emboldened by this, one might conjecture that if a harmonic function merely has radial limit 0 at each point of T then the function is 0 on D. Unfortunately, this is not so. In fact, consider the function $u_1(r,\theta) := Im\left(\frac{z}{(1-z)^2}\right) = \sum_{n=1}^{\infty} n \sin(n\theta) r^n$, which is harmonic on D. If $e^{i\theta} \neq 1$,

$$\lim_{z\to e^{i\theta}} u_1(z) = Im\left(\frac{e^{i\theta}}{(1-e^{i\theta})^2}\right) = Im(\frac{1}{4}\csc^2\frac{\theta}{2}) = 0,$$

and $\lim_{r\to 1^-} u_1(r,0) = \lim_{r\to 1^-} \sum 0 = 0$.

Although u_1 is unpleasant, it is, in a strong sense, the worst that can happen. Given a harmonic function u and a positive number r, let $m(r) = m(r,u) := \sup_{|z|\leq r} |u(z)|$. The classical Theorem 3 of Section 2 asserts that if a harmonic function has radial limit 0 at each point of T and if $m(r) = o((1-r)^{-2})$, then the function is 0 on D. That $m(r,u_1)$ is exactly $O((1-r)^{-2})$ is a reflection of the sharpness of this result.

Definition. For any $0 < \alpha < \pi$, let C_α be the circumference $|z| = \sin\frac{\alpha}{2}$. By $\bar{\Omega}_\alpha$ we mean the closed region bounded between the two tangents from $z = 1$ to C_α and by the more distant arc of C_α between the points of contact. Then $\Omega_\alpha := \bar{\Omega}_\alpha \backslash \{1\}$ and the Stolz region $\Omega_\alpha(w)$ is the region Ω_α rotated through an angle $\arg(w)$ around $z = 0$. Note that the angle between the two straight edges of Ω_α is α. When there is no confusion, we shall write $\Omega(w)$ instead of $\Omega_\alpha(w)$.

Definition. We say that the nontangential limit of u at w is s and write

$$\lim\ n.t._{z\to w} u(z) = s$$

if, for each choice of α, $0 < \alpha < \pi$,

$$\lim_{\substack{z\to w \\ z\in\Omega_\alpha(w)}} u(z) = s.$$

The function u_1 does not have a limit as $z \to 1$ while staying within $\Omega_\alpha(1)$ no matter how small $\alpha > 0$ is chosen, so it seems reasonable to conjecture that if a harmonic

* Partially supported by a leave of absence granted by the DePaul University Research Council.

† Supported in part by the National Science Foundation.

function has nontangential limit 0 at every point of T, then the function is 0 on D. Even this is not so. In section 3 we present two examples of nontrivial harmonic functions on D which have nontangential limit 0 at each point of T. The first is somewhat complicated, but has less rapid growth at the boundary. The second was communicated to us by Walter Rudin.

The examples of Section 3 show that some growth condition is necessary. By choosing modes of convergence intermediate between the unconditional limit of Theorem 1 and the radial limit of Theorem 3, and pairing them with corresponding growth rates intermediate between the vacuous one of Theorem 1 and the very restrictive one of Theorem 3, we will interpolate a scale of theorems indexed by a real parameter α between Theorem 1 and Theorem 3. These theorems, Theorem 2α, Section 2, form a more quantitative version of this corollary:

Corollary 1. *If a harmonic function has nontangential limit 0 at each point of T and if there is a real number N so that $m(r) = o((1-r)^{-N})$ as $r \to 1$, then the function is 0 on D.*

Neither this corollary nor Theorem 2α are as sharp as a result of F. Wolf (see [W], page 65, last sentence and page 66, first sentence) which allows $m(r)$ to be larger, but our method of proof is different from Wolf's.

Although the examples of Section 3 show that some growth condition is necessary for Theorem 2α, they are somewhat disappointing in that they do not give any insight as to whether even the growth rate required by Wolf's version of Theorem 2α is really necessary. A second corollary, Corollary 2, Section 2, applies Theorem 2α to trigonometric series.

2. Results

Theorem 1. *Let u be harmonic on D. If*

$$\lim_{\substack{z\to w\\ |z|<1}} u(z) = 0$$

for each $w \in T$, then $u(z) = 0$ for all $z \in D$.

Proof. The maximum of a harmonic function which is continuous on the closure of D is attained on T. QED

Theorem 2α. *Let u be harmonic on D. Let $\alpha \in [0, \pi)$. If*

$$\lim_{\substack{z\to w\\ z\in\Omega_\alpha(w)}} u(z) = 0$$

for each $w \in T$, and if

$$m(r) = o\left(\frac{1}{(1-r)^{\frac{2\pi}{\pi-\alpha}}}\right)$$

then $u(z) = 0$ for all $z \in D$.

Corollary 2γ. *Let $\frac{a_0}{2} + \sum(a_n \cos n\theta + b_n \sin n\theta)$ be a trigonometric series satisfying $|a_n| + |b_n| = o(n^\gamma)$, as $n \to \infty$. Assume that for each $w \in T$, (writing $z = re^{i\theta}$) we have*

$$\lim_{\substack{z \to w \\ z \in \Omega_\alpha(w)}} \left\{ \frac{a_0}{2} + \sum (a_n \cos n\theta + b_n \sin n\theta) r^n \right\} = 0,$$

where $\alpha = \pi \left(\frac{\gamma - 1}{\gamma + 1} \right)$. Then all a_n and all b_n are 0.

Proof of Corollary 2γ. The sum in curly brackets, call it u, is harmonic in D. It is easy to see that $m(r) = \sum_{n=1}^{\infty} o(n^\gamma) r^n = o\left(\frac{1}{(1-r)^{\gamma+1}} \right)$. (Use formulas III.1.9 and III.1.15 of [Z] for this.) Note that if $\alpha := \pi \left(\frac{\gamma - 1}{\gamma + 1} \right)$, then $\frac{2\pi}{\pi - \alpha} = \gamma + 1$ and apply Theorem 2α with this α to get that $u(r, \theta) = 0$ on D. For any fixed $r < 1$, the series defining $u(\theta) = u(r, \theta)$ converges uniformly to zero. By I.4.10 of [Z], all of the a_n and b_n are 0. QED

Theorem 3. *(F. Wolf [W], V. Shapiro [S], B. E. J. Dahlberg [D], compare S. Verblunsky [Z], IX.8.1, [V]) Let u be harmonic on D. If $\lim_{r \to 1^-} u(rw) = 0$ for each $w \in T$, and if $m(r) = o\left(\frac{1}{(1-r)^2} \right)$, then $u(z) = 0$ for all $z \in D$.*

Remark. The various proofs of Theorem 3 have different embellishments, such as allowing small exceptional sets with additional hypotheses. Of course Theorem 2α also admits some of these extensions without much additional effort, but we resist that temptation here.

Remarks. Our proof of Theorem 2α follows Dahlberg's proof of Theorem 3. In fact, if you set $\alpha = 0$ in the proof of Theorem 2α given below, you will have essentially Dahlberg's proof of Theorem 3. [D] Similarly, if you set $\gamma = 1$ in Corollary 2, Verblunsky's uniqueness theorem for Abel summable trigonometric series follows. [Z],IX.8.1, [V] Finally, Theorem 1 may be thought of as Theorem 2α with $\alpha = \pi$.

Definition. To an arc I of T associate the curvilinear triangle $S(I) := \{tw \in D : w \in I$ and $0 \le t < 1\}$. For future reference, we note that $\bar{S}(\bar{I}) = S(\bar{I}) \cup \bar{I}$.

Proof of Theorem 2α. Let u be harmonic on D and fix α in $[0, \pi)$. Assume that

$$\lim_{\substack{z \to w \\ z \in \Omega_\alpha(w)}} u(z) = 0$$

for each $w \in T$. Our goal is to show that $u(z) = 0$ for every $z \in D$. Let $\mathcal{O} := \{w \in T : \limsup_{z \to w} u(z) \le 0\}$. It suffices to show that $\mathcal{O} = T$. For by symmetry it would then be immediate that $\{w \in T : \lim_{z \to w} u(z) \ge 0\} = T$ also, so that $\{w \in T : \lim_{z \to w} u(z) = 0\} = T$. The goal would be reached, since this is exactly the hypothesis of Theorem 1. Letting $u^+(z) := \max\{u(z), 0\}$ as usual, it is clear that $\mathcal{O} = \{w \in T : \lim_{z \to w} u^+(z) = 0\}$. Collecting the known properties of u^+, we will restate what must be done as Lemma 1. Thus, modulo the proof of Lemma 1 we are done. QED

Lemma 1. *If $p(z)$ is*

(1) subharmonic, continuous, and non-negative on D,

(2) $$\lim_{\substack{z\to w \\ z\in\Omega_\alpha(w)}} p(z) = 0 \quad \text{for each } w \in T,$$

and if $m(r) := \sup\{p(z) : |z| \le r\}$ *satisfies*

(3) $$m(r) = o\left(\frac{1}{(1-r)^{\frac{2\pi}{\pi-\alpha}}}\right);$$

then $\mathcal{O} := \{w \in T : \lim_{z\to w} p(z) = 0\} = T$.

Proof of Lemma 1. We first show that $\mathcal{O}$ is open. Let $w_o = e^{i\theta}$ be a point of $\mathcal{O}$. Then there is a neighborhood of w_o, say of the form $\{(r,\varphi) : 1-\delta < r < 1 \text{ and } |\varphi - \theta| \le \delta\}$ for some $\delta > 0$, on which p is bounded. But p, being continuous on the compact set $\{(r,\varphi) : 0 \le r \le 1-\delta \text{ and } |\varphi-\theta| \le \delta\}$ is also bounded there. Hence p is bounded on $S(I)$ where I $:= \{e^{i\varphi} \in T : |\varphi - \theta| \le \delta\}$. To proceed with the proof of Lemma 1 we will need:

Lemma 2. *Let p satisfy (1) and be bounded on $S(I)$ for some closed interval $I \subset T$. Suppose $\lim_{r\to 1^-} p(rw) = 0$ for each $w \in I$, then $\lim_{z\to w} p(z) = 0$ for each w interior to I.*

Proof of Lemma 2. Let $M := \sup\{p(z) : z \in S(I)\}$. Let F be a conformal map of the unit disc onto $S(I)$ and let J be the closed interval of T satisfying $F(J) = I$. Define a function v on D by $v(\zeta) = p(F(\zeta))$. Then v still enjoys property (1) and $0 \le v \le M$ on D. Then v has a least harmonic majorant h ([T], pp. 172–173). Since the constant function M is itself a harmonic majorant of v, $h \le M$. Since h is a bounded harmonic function on D, $h = PI(H)$ for some function H on T. We are using the notation $PI(H)$ to denote the Poisson integral of H:

$$\frac{1}{2\pi}\int_0^{2\pi} H(e^{i\varphi})\frac{1-r^2}{1-2r\,\cos(\theta-\varphi)+r^2}\,d\varphi.$$

Also, $\lim_{\zeta\to\eta}$ *n.t.* $h(\zeta) = H(\eta)$ almost everywhere [T], pp. 172-173. Since $\Omega(w)$ contains the radius terminating at w, it follows from (2) that

$$\lim_{\substack{\zeta\to\eta \\ \zeta\in C(\eta)}} v(\zeta) = 0$$

for each η interior to J where for $\eta = F(w)$, $C(\eta) := F^{-1}(\{rw : 0 \le r \le 1\})$ is a curve orthogonal to T at η. It follows that H must be 0 almost everywhere on J. In particular, H is essentially 0 on J and hence essentially continuous there, so that $\lim_{\zeta\to\eta} h(\zeta) = 0$ at all points interior to J ([T], p. 130). But then v is squeezed between 0 and h so $\lim_{\zeta\to\eta} v(\zeta) = 0$ everywhere on the interior of J, which is to say that $\lim_{z\to w} p(z) = 0$ everywhere on the interior of I. This proves Lemma 2. QED

Returning to the proof of Lemma 1, note that since the radius terminating at w is contained in $\Omega(w)$ for all choices of Ω, from Lemma 2 we can conclude that every point of T within δ of w_o is in $\mathcal{O}$. Thus each point of $\mathcal{O}$ is interior to $\mathcal{O}$, so $\mathcal{O}$ is an open subset of T.

Let $p^*(w) := \sup\{p(z) : z \in \Omega(w)\}$. Define $F_j := \{w \in T : p^*(w) \leq j\}$. Then F_j is closed. For if $\{w_k\}$ is a sequence of points in F_j tending to w, and if $z \in \Omega(w)$, then for each k there is a point $z_k \in \Omega(w_k)$ with $|z_k - z| < |w_k - w|$. Since $p(z_k) \leq p^*(w_k) \leq j$ and since p is continuous at z, $p(z) \leq j$. Since z was arbitrary, $p^*(w) \leq j$, $w \in F_j$, so F_j is closed.

Our goal is to show $\mathcal{O} = T$; so, letting $K := T \backslash \mathcal{O}$, we must show the closed set K to be empty. Assume not. From (2) it follows that $\cup F_j = T$, so that $\cup(F_j \cap K) = K$. By the Baire Category Theorem, there is an interval $I \subset T$ and an integer j so that the nonempty set $K \cap I$ is contained in $K \cap F_j$, *i.e.*, $K \cap F_j$ contains a portion of K [Z], I.12.1 If $I = (a, b)$, let $M = \max\{p^*(a),\, p^*(b),\, j\}$. To prove Lemma 1, we will show

$$p(z) \leq M \quad \text{for every } z \in S(\bar{I}). \tag{4}$$

From Lemma 2 it will then follow that $\lim_{z \to w} p(z) = 0$ for every $w \in I$ and hence for at least one $w \in K$, contrary to the definition of K. Thus Lemma 1, and, consequently, Theorem 2α will be proved. Write $I \backslash K$ as a countable union of open intervals and let $(c,\, d)$ be one of these intervals.(If c and d are points of T, by (c, d) we will mean the shorter arc of T lying between c and d. Note $|c - d| = 2 \sin\frac{\theta}{2}$ when the arclength of (c, d) is θ.) By Lemma 3α below it follows that $p \leq M$ on T_{cd}. Since $S(\bar{I})$ is covered by the union of these triangles together with a union of regions $\Omega(w)$ where $w \in F_j$, (4) follows.

QED

Lemma 3α. *Suppose p satisfies (1), (3), $p^*(c) \leq M$, $p^*(d) \leq M$, and $\lim_{z \to w} p(z) = 0$ for all w in the arc (c, d). Then $p(z) \leq M$ for all $z \in S([c, d])$.*

Proof. Let T_{cd} be the curvilinear triangle bounded by the arc (c, d) and the line segments pc and pd where p is the point of intersection of the straight side of $\Omega(c)$ closest to d and the straight side of $\Omega(d)$ closest to c. We need only show that $p(z) \leq M$ for $z \in T_{cd}$. Let F be a conformal mapping of T_{cd} onto the unit disc D. Then the angle at c of $\frac{\pi - \alpha}{2}$ is straightened out into an angle of π by F. In other words, neglecting a translation and a rotation, for $z \in T_{cd}$ near c, F is asymptotically $z^{\frac{2\pi}{\pi-\alpha}}$ (which maps e^{i0} to e^{i0} and $e^{i(\frac{\pi-\alpha}{2})}$ to $e^{i\pi}$). Hence if $|\zeta - F(c)| = \delta$, then $|F^{-1}(\zeta) - c| \simeq \delta^{\frac{\pi-\alpha}{2\pi}}$; so letting $v(\zeta) := p(F^{-1}(\zeta))$, it follows from (3) and Lemma 4 below that $|v(\zeta)| = o(\frac{1}{\delta})$. Similarly if $|\zeta - F(d)| = \delta$, then $v(\zeta) = o(\frac{1}{\delta})$. For $\eta \in (c, d)$, $\lim_{\zeta \to \eta} v(\zeta) = 0$, and for $\eta \in T \backslash [c, d]$, $v(\eta) = u(F^{-1}(\eta))$, where $F^{-1}(\eta) \in \Omega(c) \cup \Omega(d)$, so $v(\eta) \leq M$. Thus we may apply the Phragmén-Lindelöf Lemma 5 below to complete the proof of Lemma 3α. QED

Lemma 4. *Let $p(z)$ satisfy (1), $m(r) = o((1-r)^{-N})$ for some positive real number N, and $\lim_{z \to w} p(z) = 0$ for all w in the arc (c_1, c_2). Then, for both $i = 1$ and $i = 2$, as z tends to c_i, we have*

$$p(z) = o(|z - c_i|^{-N}). \tag{5}$$

Proof. We observe that if B is a disk centered at z and v a function harmonic in B and continuous on $\bar{B}$ and if $p \in (0,1)$, then

$$|v(z)| \leq C_p \left(\frac{1}{|B|} \int\int_B |v(x,y)|^p dxdy \right)^{\frac{1}{p}}. \tag{6}$$

This result is stated and proved on pages 172–173 of [FS]. For our purposes we need to lighten the hypothesis for inequality (6) from harmonic to subharmonic and nonnegative.

This is straightforward, so we will limit ourselves to a few remarks on p. 173 of [FS]. In line 3, " $= r$" should be " $= r^2$"; in line 6, append "provided $p > 1 - \theta$"; and in line 11, change the second upper limit of integration to 1. The only real change involves establishing the estimate $m_\infty(\rho) \leq A(1 - \rho r^{-1})^{-n} m_1(r)$ on line 8 for a nonnegative subharmonic function p. The notation m_p is from [FS]. To see this, introduce the harmonic function h which agrees with p on the origin-centered spherical surface of radius r and note (i) the estimate holds for h, (ii) $m_\infty(\rho, p) \leq m_\infty(\rho, h)$ by the maximum principle, and (iii) $m_1(r,h) = m_1(r,p)$ by the definition of h.

Let I $:= (c_1, c_2)$ and define

$$v(z) := \begin{cases} p(z), & \text{if } z \in S(I) \\ 0, & \text{if } z \notin S(I). \end{cases}$$

Then $v(z)$ is actually subharmonic and nonnegative on the infinite wedge $W := \{rw : w \in I, 0 \leq r < \infty\}$. This is easy to check: for example, if $w \in I$ and B is a ball about w small enough to be contained in W, then $v(w) = 0 \leq |B|^{-1} \int_{B \cap D} p = |B|^{-1} \int_B v$. We will deduce the required estimate (5) only at c_1, since the argument at c_2 is symmetric. Write $z = c_1 - \delta e^{i\varphi}$, so that the vector from c_1 to z makes an angle of φ, $0 < \varphi < \frac{\pi}{2}$, with the vector from c_1 to the origin. If $0 < \varphi \leq \frac{\pi}{4}$, then it is geometrically evident that there is an absolute constant C such that $|z - c_1|^{-1} \leq C(1 - |z|)^{-1}$, whence (5) is immediate. So assume $\frac{\pi}{4} < \varphi < \frac{\pi}{2}$, which insures that B, a disc of radius $\delta/8$ (say) about z is contained in W. Let A be the polar rectangle $\{(r,\theta) : 1 - 2\delta \leq r < 1, \arg(z) - \frac{\delta}{8} \leq \theta \leq \arg(z) + \frac{\delta}{8}\}$. Clearly $B \cap D \subset A$. Applying the inequality (6) we have, for any $q > 0$,

$$\begin{aligned} v(z)^q \leq C_q \frac{1}{|B|} \int\int_B v(x,y)^q \, dxdy &= C_q \frac{1}{|B|} \int\int_{B \cap D} p(x,y)^q \, dxdy \\ &\leq C_q \frac{1}{|B|} \int\int_A p(x,y)^q \, dxdy. \end{aligned} \tag{7}$$

Now set $q := 1/2N$, change to polar coordinates, note $|B| = O(\delta^2)$, note that the hypothesis on $m(r)$ can be rewritten as $p(r,\theta) = o((1-r)^{-N})$, and estimate the Jacobian r by 1. All this substituted into inequality (7), raised to the $2N$ th power yields

$$v(z) = O\left(\delta^{-2} \int_{1-2\delta}^{1} o((1-r)^{-\frac{1}{2}}) \, dr \int_{\arg(z)-\delta/8}^{\arg(z)+\delta/8} d\theta \right)^{2N} = o(\delta^{-2+1/2+1})^{2N} = o(\delta^{-N}).$$

QED

Lemma 5. *(Phragmén-Lindelöf Lemma) Let $p(z)$ satisfy (1) and suppose that*

$$\limsup_{z \to w} p(z) \le M$$

for all $w \in T \backslash \{c_1, ..., c_n\}$. Suppose also that for each j, $1 \le j \le n$, we have

(8) $$|p(z)| = o(|z - c_j|^{-1}) \text{ as } z \in D \text{ tends to } c_j.$$

Then $p(z) \le M$ in D.

Proof. Let $D(c, \delta) := \{z : |z - c| < \delta\}$; $I(c, \delta) := D(c, \delta) \cap T$, so that I is the arc (c^-, c^+) of T where $\arg(c^{\pm}) = \arg(c) \pm \arcsin(\frac{\delta}{2})$; and ω^z be the harmonic measure for D so that if $E \subset T$, then $\omega^z(E)$ is the Poisson integral of the characteristic function of E. If $e^{i\theta_0} \in T$, $z = re^{i\theta} \in D$, and $|z - e^{i\theta_0}| = \delta$, we have the estimate

$$\begin{aligned}
\omega^z(I(e^{i\theta_0}, \delta)) &:= \frac{1}{2\pi} \int_{c^-}^{c^+} \frac{1 - r^2}{1 - 2r \ \cos(\theta - \varphi) + r^2} d\varphi \\
&\ge c \int_{\delta/4}^{\delta/4} \frac{1 - r}{(1 - r)^2 + (\theta - \theta_o - \varphi)^2} d\varphi \\
&= c \left\{ \arctan \left(\frac{\theta - \theta_0 + \delta/4}{1 - r} \right) - \arctan \left(\frac{\theta - \theta_0 - \delta/4}{1 - r} \right) \right\}.
\end{aligned}$$

Now $1 - r \le \delta$, so if $\theta \ge \theta_o$, the first arctan exceeds $\arctan(\frac{\delta/4}{\delta})$, while the second arctan is negative, so that $c \arctan(\frac{1}{4})$ is a lower bound. The case of $\theta \le \theta_o$ is symmetrical. In other words, there is a positive absolute constant c_o so that

(9) $$\omega^z(I(e^{i\theta_0}, \delta)) \ge c_o \text{whenever } z \in D \text{ and} |z - e^{i\theta_0}| = \delta.$$

Fix $\delta > 0$; for each j, $1 \le j \le n$, let $u_j(z, \delta) := \omega^z(I(c_j, \delta))/c_o$; and define the domain $D_\delta := D \backslash (\cup_j D(c_j, \delta))$. Using the estimate (9) and applying the maximum principle for subharmonic functions in D_δ , we have

(10) $$p(z) \le M + \sum_{j=1}^{n} u_j(z, \delta) M_j(\delta)$$

for z in D_δ , where $M_j(\delta) := \sup\{p(z) : z \in \partial D(c_j, \delta) \cap D$. Now let $z = re^{i\theta} \in D$ and estimate as above to get

$$u_j(z, \delta) \le C\{\arctan \left(\frac{\theta - \theta_j + \delta/4}{1 - r} \right) - \arctan \left(\frac{\theta - \theta_j - \delta/4}{1 - r} \right)\},$$

where θ_j is the argument of c_j.

Finally, freeze z and let $\delta \to 0$. We see that there is a constant $C_j(z)$, depending only on j and z, so that $u_j(z, \delta) \le C_j(z) \cdot \delta$ as $\delta \to 0$. Thus, taking hypothesis (8) into account, we see that $u_j(z, \delta) M_j(\delta) = O(\delta) o(\frac{1}{\delta}) = o(1)$ as $\delta \to 0$. Substituting these n relations into inequality (10) and letting $\delta \to 0$ establishes Lemma 5. QED

3. Examples

Theorem 4. *There is a function which is not identically 0, which is harmonic on D, and which has non-tangential limit 0 at every point of T.*

Proof. Let

$$f(z) := \int_0^\infty \frac{e^{zt}}{t^t} dt$$

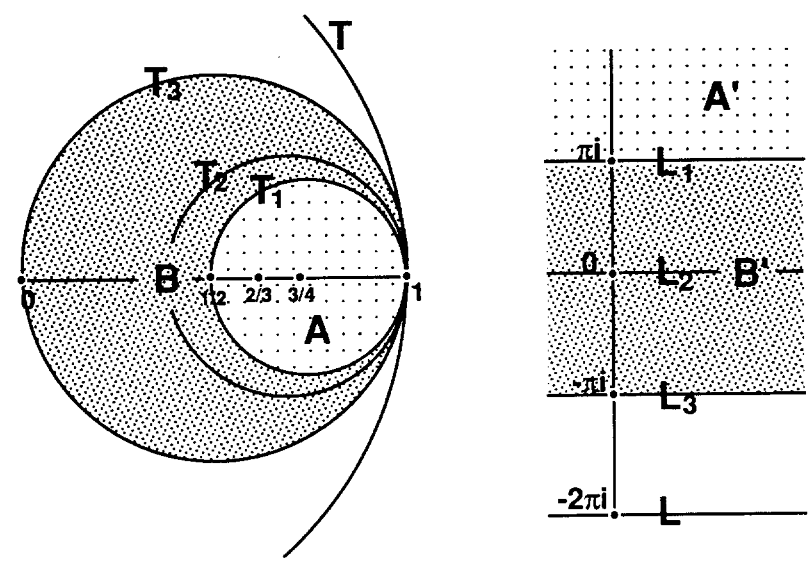
Then f satisfies

(11) there is a constant A so that $|f(z)| \leq A$ if $|Im(z)| \geq \pi$,

(12) f is entire and hence in particular continuous at each finite z, and

(13) for x real, $f(x) \simeq \sqrt{2\pi} e^{\{e^{(x-1)}+\frac{1}{2}(x-1)\}}$.

(See [BN], pp.140-143, for properties (11) and (12). See the estimate following this proof for (13).) Let $S(z) := \pi i(\frac{1+z}{1-z}) - 2\pi i$. Direct calculation shows that $S((1-a)+ae^{i\theta}) = -\frac{\pi}{a}\cot\frac{\theta}{2} + (\frac{1}{a} - 3)\pi i$. Setting $a = 1$ shows that S maps $T = \partial D$ onto the line $L := \{x - 2\pi i : -\infty < x < \infty\}$. Similarly setting $a = \frac{1}{2}, \frac{1}{3}$, and $\frac{1}{4}$ shows that S maps each circle T_i onto the line L_i, $i = 1, 2, 3$, as shown,



Also S maps the dotted disc A onto the dotted half plane A' and the dark shaded region B onto the similarly shaded strip B' as shown. Now let $g(z) := (1-z)f(S(z))$. From (12) it follows that

(14) g is continuous at every point of T except $z = 1$.

Since (11) holds and $A' \subset \{z : |Im\ z| \geq \pi\}$, it follows that

(15) g has non-tangential limit 0 as z tends to 1.

Finally, let $h(z) := g(z) - PI(g|_T)(z)$, where $(PI)(f)$ denotes the Poisson integral of f. From (12) we see that $g|_T$ is continuous on $T\backslash\{1\}$. Motivated by (15), define $(g|_T)(1) := 0$. From (11) and $L \cap \{|Im(z)| < \pi\} = \emptyset$, we have

$$\lim_{\substack{w\in T \\ w\to 1}} g(w) = 0$$

so that $g|_T$ is continuous on all of T. Thus at every point w of T, the non-tangential limit as $z \to w$ of $PI(g|_T)(z)$ is $g(w)$.[Z], III.7.9 From (14) and (15) it follows that h has non-tangential limit 0 everywhere on T.

Now the Poisson kernel is positive and has integral 1 so that $\sup_{z\in D} |PI(g|_T)(z)| \leq \sup_{w\in T} |g(w)|$, which is finite since $g|_T$ is a continuous function on a compact set. To get $h \neq 0$ we will show g to be unbounded on D. On T_2 we have the estimates for $\theta > 0$ small,

$$g(\frac{2}{3} + \frac{1}{3}e^{-i\theta}) = (\frac{1}{3} - \frac{1}{3}e^{-i\theta}) \int_0^\infty \frac{e^{(\frac{\pi}{2}\cot\frac{\theta}{2})t}}{t^t} dt$$
$$\simeq \frac{\sqrt{2\pi}}{3} i\theta e^{\{e^{\frac{\pi}{2}\cot\frac{\theta}{2}-1} + \frac{1}{2}(\frac{\pi}{2}\cot\frac{\theta}{2}-1)\}}$$

But if $d :=$ dist $\{\frac{2}{3} + \frac{1}{3}e^{-i\theta}, T\}$, $d = 1 - \frac{1}{3}\sqrt{5 + 4\cos\theta} \simeq (\frac{\theta}{3})^2$ so that $\theta \simeq 3\sqrt{d}$. Also, $\sqrt{2\pi} > \frac{5}{2}$ and $\frac{\pi}{2}\cot\frac{\theta}{2} \simeq \frac{\pi}{\theta}$, so for θ small, $\frac{\pi}{2}\cot\frac{\theta}{2} - 1 \approx \frac{\pi}{3\sqrt{d}} - 1 > \frac{1}{\sqrt{d}}$. Thus if $z = \frac{2}{3} + \frac{1}{3}e^{-i\theta}$ and if $d = 1 - |z|$, Im $g(z) > \frac{5}{2}\sqrt{d}e^{\{e^{1/\sqrt{d}} + 1/(2\sqrt{d})\}}$. QED

Remarks. In particular, if a real valued harmonic example is desired, Im h will do. Note that $PI(g|_T)$ cannot be analytic even though g is analytic on D, for then h would be an analytic function with nontangential boundary values 0 on a set of positive measure, contrary to a well known theorem of Privalov.[Z], XIV.1.9

Estimate.

$$\int_0^\infty \frac{e^{xt}}{t^t} dt \simeq \sqrt{2\pi} e^{\{e^{x-1} + \frac{(x-1)}{2}\}} \quad \text{as } x \to +\infty.$$

Proof. Let $t =: e^x s$. Then

$$I := \int_0^\infty (\frac{e^x}{t})^t dt = e^x \int_0^\infty (\frac{1}{s})^{e^x s} ds = e^x \int_0^\infty e^{e^x(s\log\frac{1}{s})} ds.$$

Let $u := s - e^{-1}$. Then

$$I = e^x \int_{-e^{-1}}^\infty e^{e^x[\ln(u) + e^{-1}]} du = e^x \cdot e^{e^{x-1}} \int_{-e^{-1}}^\infty e^{e^x h(u)} du$$

where $h(u) := (u + e^{-1}) \ln \frac{1}{(u+e^{-1})} - e^{-1}$ is increasing on $[-e^{-1}, 0]$, zero at $u = 0$, and decreasing on $[0, \infty)$. Also $h''(0) = -e$, so using the estimate $h(u) = -e\frac{u^2}{2} + o(u^2)$ near 0 and the estimate $e^x h(u) < -(e^x - 1)(-h(d)) + h(u)$ for $|u| > d$ it is easy to show (See [De], pp. 63-65) that

$$\int_{-e^{-1}}^\infty e^{e^x h(u)} du \simeq \int_{-\infty}^\infty e^{(e^x)\frac{h''(0)}{2}u^2} du.$$

Since

$$\int_{-\infty}^{\infty} e^{-(e^{x+1})\frac{u^2}{2}}du = \sqrt{\frac{2\pi}{e^{x+1}}},$$

$$I \simeq e^x e^{e^{x-1}}\sqrt{2\pi}e^{-\frac{1}{2}x-\frac{1}{2}} = \sqrt{2\pi}e^{\{e^{x-1}+\frac{1}{2}(x-1)\}} = \sqrt{2\pi e^{x-1}}e^{e^{x-1}}.$$

More precisely, ([De], pp. 66-69), $I = \sqrt{2\pi}e^{\{e^{x-1}+\frac{1}{2}(x-1)\}} + O\left(e^{\{e^{x-1}-\frac{1}{2}x\}}\right)$ or even

$$I = \sqrt{2\pi}\left[e^{\{e^{x-1}+\frac{1}{2}(x-1)\}} - \frac{1}{24}e^{\{e^{x-1}-\frac{1}{2}x-\frac{3}{2}\}}\right] + O(e^{\{e^{x-1}-\frac{3}{2}x\}}).$$

QED

Second proof of Theorem 4.(W. Rudin) For $\alpha > 0$, let Γ_α be the directed path in the complex plane obtained by traveling leftward along the ray from $(+\infty, -i\pi)$ to $(\alpha, -i\pi)$, then up the line segment from $(\alpha, -i\pi)$ to $(\alpha, +i\pi)$, and then rightward along the ray from $(\alpha, +i\pi)$ to $(+\infty, +i\pi)$. Let $P_\alpha := \{z : Re(z) < \alpha\}$. For $z \in P_\alpha$ define

$$f_\alpha(z) := \frac{1}{2\pi i}\int_{\Gamma_\alpha}\frac{e^{e^w}}{w-z}dw.$$

If $\alpha < \beta$, then, by Cauchy's theorem, $f_\alpha = f_\beta$ in P_α ; so there is an entire function f such that $f = f_\alpha$ in P_α. This f appears in exercise 11, chapter 16 of [R] and has the following properties:

(16) $f(x)$ is real for $-\infty < x < \infty$,
(17) $f(re^{i\theta}) \to 0$ as $r \to \infty$, uniformly in $0 < \delta \le \theta \le 2\pi - \delta$, for all $\delta > 0$, and
(18) $f \not\equiv 0$ (because $f(x) = e^{e^x} + O(1)$ as $x \to +\infty$).

Now define $u(z) + iv(z) := f(i\frac{1+z}{1-z})$. Then v is harmonic at all $z \neq 1$. Relation (16) implies that $v(e^{i\theta}) = 0$ for all $e^{i\theta} \neq 1$, v has nontangential limit 0 at 1 (from inside D) because of (17), but $v \not\equiv 0$ because of (18). QED

References.

[BN] J. Bak and D.J. Newman, *Complex Analysis*, Springer, New York, 1982.

[D] B. E. J. Dahlberg, On the radial boundary values of subharmonic functions, *Math. Scand.* **40** (1977), 301-317.

[De] N.G. DeBruijn, *Asymptotic Methods in Analysis*, Dover, New York, 1981.

[FS] C. Fefferman and E. M. Stein, H^p spaces of several variables, *Acta Math.* **129** (1972), 137-192.

[R] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, New York, 1987.

[S] V. Shapiro, The uniqueness of functions harmonic in the interior of the unit disk, *Proc. Lond. Math. Soc.* **13** (1963), 639-652.

[T] M. Tsuji, *Potential Theory in Modern Function Theory*, Maruzen, Tokyo, 1959.

[V] S. Verblunsky, On the theory of trigonometric series, (I) *Proc. Lond. Math. Soc.* **34** (1932), 441-456; (II) *ibid.*457-491.

[W] F. Wolf, The Poisson integral. A study in the uniqueness of harmonic functions, *Acta Math.* **74** (1941), 65-100.

[Z] A. Zygmund, *Trigonometric Series*, Vols.I, II, Cambridge Univ. Press, Cambridge, 1959.