TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 360, Number 2, February 2008, Pages 959–987
S 0002-9947(07)04249-3
Article electronically published on June 25, 2007

QUANTUM SYMMETRIC L^p DERIVATIVES

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ABSTRACT. For $1 \leq p \leq \infty$, a one-parameter family of symmetric quantum derivatives is defined for each order of differentiation as are two families of Riemann symmetric quantum derivatives. For $1 \leq p \leq \infty$, symmetrization holds, that is, whenever the L^p kth Peano derivative exists at a point, all of these derivatives of order k also exist at that point. The main result, desymmetrization, is that conversely, for $1 \leq p \leq \infty$, each L^p symmetric quantum derivative is a.e. equivalent to the L^p Peano derivative of the same order. For $k = 1$ and 2, each kth L^p symmetric quantum derivative coincides with both corresponding kth L^p Riemann symmetric quantum derivatives, so, in particular, for $k = 1$ and 2, both kth L^p Riemann symmetric quantum derivatives are a.e. equivalent to the L^p Peano derivative.

1. Introduction

A real-valued function f has a Peano derivative of order k at $x \in \mathbb{R}$, i.e., $f \in t_k(x)$, if there are constants $f_0(x), f_1(x), \ldots, f_k(x)$ such that

$$f(x+h) = f_0(x) + f_1(x)h + \cdots + f_k(x)\frac{h^k}{k!} + o\left(h^k\right) \text{ as } h \to 0.$$

In particular, call $f_k(x)$ the kth Peano derivative of f at x. Thus $f \in t_k(x)$ means that f is well approximated near x by a kth degree polynomial. This is a very natural generalization of the geometric interpretation of the first derivative's existence at x meaning that the function is well approximated by a line near x. In fact, for a function continuous at x (which guarantees that $f_0(x) = f(x)$), having an ordinary derivative at x and belonging to $t_1(x)$ are equivalent conditions. For higher values of k, the definitions are different. For example,

$$f_2(x) = \lim_{h \to 0} \frac{f(x+h) - f_0(x) - f_1(x)h}{h^2/2}.$$

This difference is much more than a formality. Although Peano first proved a strong version of Taylor's theorem which asserts that the existence of the ordinary kth derivative of f at x implies that $f \in t_k(x)$, the converse implication is far from true: When $k \geq 2$, there is a set E of positive Lebesgue measure and a function having k Peano derivatives everywhere on E but not having k ordinary derivatives at any point of E.

Received by the editors July 22, 2005 and, in revised form, January 28, 2006.
2000 *Mathematics Subject Classification.* Primary 26A24; Secondary 26A27.
Key words and phrases. Generalized derivatives, quantum derivatives, L^p derivatives.
The first author's research was partially supported by NSF grant DMS 9707011 and a grant from the Faculty and Development Program of the College of Liberal Arts and Sciences, DePaul University.

It is our feeling that the Peano derivative is "more natural" than the corresponding ordinary derivative. As evidence for this claim we will look at a large number of generalized derivatives and find that most have already been shown to be a.e. equivalent to the corresponding Peano derivative. In fact we conjecture (and expect) that ultimately this a.e. equivalence will be proven to hold for *all* of our generalized derivatives. To be more explicit about this, every generalized derivative we will consider will be stronger than Peano differentiation in the sense that the existence of the kth Peano derivative at a point will imply the existence of every kth generalized derivative at that point. Conversely, a generalized derivative will be said to be a.e. equivalent to the kth Peano derivative if for every function, the set where the function is differentiable in that sense but is not Peano differentiable of the same order has measure zero. (Measure will always mean Lebesgue measure.)

The remarks above all refer to the ordinary or L^∞ derivatives. There is an analogous scheme for L^p derivatives. (The index p will always be a real parameter satisfying $1 \le p < \infty$.) For example, f has k Peano derivatives in L^p at x, $f \in t_k^p(x)$, if there are constants $f_{0p}(x), f_{1p}(x), \dots, f_{kp}(x)$ so that

$$\left(\frac{1}{h}\int_0^h \left| f(x+t) - \left\{ f_{0p}(x) + f_{1p}(x)\,t + \cdots + f_{kp}(x)\frac{t^k}{k!}\right\}\right|^p dt\right)^{1/p} = o\left(h^k\right)$$

as $h \to 0$. L^p Peano differentiation is strictly stronger than L^∞ Peano differentiation. It is clear that L^∞ Peano differentiation implies L^p Peano differentiation pointwise. However, there is a set E of positive Lebesgue measure and a function having k L^p Peano derivatives on E but not having even one L^∞ Peano derivative at any point of E.[A2]

We will consider four basic types of generalized derivatives:

(1) additive L^∞ derivatives such as the second Riemann derivative

$$R_2 f(x) = \lim_{h\to 0} \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2},$$

(2) additive L^p derivatives such as the second Riemann L^p derivative $R_{2p} f(x)$ which satisfies

$$\left(\frac{1}{h}\int_0^h \left| f(x+2t) - 2f(x+t) + f(x) - R_{2p} f(x)\, t^2\right|^p dt\right)^{1/p} = o\left(h^2\right)$$

as $h \to 0$,

(3) quantum L^∞ derivatives such as the second quantum Riemann derivative

$$QR_2 f(x) = \lim_{q\to 1} \frac{f\left(q^2 x\right) - (1+q)\, f(qx) + q f(x)}{(q-1)^2 x^2}, \tag{1.1}$$

and

(4) quantum L^p derivatives such as the second quantum Riemann L^p derivative $QR_2^p f(x)$ which satisfies

$$\left(\frac{1}{q-1}\int_1^q \left| f\left(t^2 x\right) - (1+t)\, f(tx) + t f(x) - QR_2^p f(x)\,(t-1)^2 x^2\right|^p \frac{dt}{t}\right)^{1/p} = o\left((q-1)^2\right) \text{ as } q \to 1.$$

1.1. **Additive L^∞ derivatives.**

1.1.1. *Definitions.* The additive L^∞ derivatives are of two types: the generalized Riemann kth derivatives which have the form

$$\lim_{h\to 0} h^{-k}\sum_{i=0}^{k+e} w_i f(x+u_i h) \tag{1.2}$$

where the weights $\{w_i\}$ and base points $\{u_i\}$ satisfy

$$\sum_{i=0}^{k+e} w_i u_i^j = \begin{cases} 0 & \text{if } j=0,1,\ldots,k-1, \\ k! & \text{if } j=k, \end{cases} \tag{1.3}$$

and the symmetric kth derivatives $S_k^+ f(x)$ which are defined inductively and satisfy

$$\frac{f(x+h)+(-1)^k f(x-h)}{2} = \sum_{\substack{j\equiv k \bmod 2\\ 0\le j\le k}} S_j^+ f(x)\frac{h^j}{j!} + o\left(h^k\right).$$

The conditions (1.3) arise naturally when each $f(x+u_i h)$ in equation (1.2) is Taylor expanded about x; see reference [A] for details. Notice that the existence of $S_k^+ f(x)$ requires only the existence of lower order symmetric derivatives of the same parity–e.g., the existence of the three lower order symmetric derivatives $S_0^+ f(x)$, $S_2^+ f(x)$, and $S_4^+ f(x)$ is a prerequisite for the existence of $S_6^+ f(x)$. Two important special cases of generalized kth Riemann derivatives are the kth Riemann derivative itself where $e=0, w_i = \binom{k}{i}(-1)^{k-i}$ and $u_i = i$ and the kth symmetric Riemann derivative with the same w_i but with $u_i = i-\frac{k}{2}$. Some of the additive L^∞ derivatives must have been investigated very early since Leibniz's $\frac{d^k f}{dx^k}$ notation is suggestive of a kth difference divided by the kth power of the differencing variable. This is highly suggestive of the Riemann derivative since if $k=2$, and we put $h=\Delta x$,

$$\frac{f(x+2h)-2f(x+h)+f(x)}{h^2} = \frac{\Delta^2 f(x,h)}{(\Delta x)^2} = \frac{\Delta^1\left(\Delta^1 f(x,h)\right)}{(\Delta x)^2},$$

where $\Delta^1 f(x,h) = f(x+h)-f(x)$.

The integer e, which is necessarily non-negative, is called the excess. The Riemann derivatives and the auxiliary derivatives used in the proofs below all have $e=0$. Generalized Riemann derivatives with $e>0$ arise for technical reasons in [A] and are often used in numerical analysis [AJJ], [AJ].

1.1.2. *Results.* The *consistency condition* asserts that whenever the kth Peano derivative exists at a point, then all the corresponding generalized derivatives also exist at that point and are equal to it. The consistency condition is a simple calculation for both the generalized kth Riemann derivatives and the symmetric kth derivatives. The a.e. converse is also true.[A] Historically, this was first proved for the kth Riemann derivative and the kth symmetric Riemann derivative.[MZ] As a corollary we also have the a.e. converse for the kth symmetric derivative, since a very short and simple calculation shows that whenever the kth Symmetric derivative exists at a point, then the kth symmetric Riemann derivative also exists at that point and is equal to it.

1.2. **Additive L^p derivatives.**

1.2.1. *Definitions.* The additive L^p derivatives are defined in a completely analogous way to the L^∞ ones. For instance $G_{kp}f(x)$ denotes the generalized Riemann L^p derivative of f at x if

$$\left\| \sum_{i=0}^{k+e} w_i f(x+u_i t) - G_{kp} f(x) t^k \right\|_p (h) = o\left(h^k\right),$$

where $\|g(t)\|_p(h)$ is defined to be $\left(\frac{1}{h}\int_0^h |g(t)|^p \, dt\right)^{1/p}$ and the $\{w_i\}$ and $\{u_i\}$ are as in the L^∞ case; and the symmetric L^p derivative $S_{kp}^+ f(x)$ is defined inductively by

$$\left\| \frac{f(x+t)+(-1)^k f(x-t)}{2} - \sum_{\substack{j \equiv k \bmod 2 \\ 0 \le j \le k}} \frac{S_{jp}^+ f(x)}{j!} t^j \right\|_p (h) = o\left(h^k\right).$$

1.2.2. *Results.* Essentially the same simple calculations as in the L^∞ case show consistency with the kth L^p Peano derivative for both the generalized Riemann L^p derivative and the symmetric L^p derivative. The a.e. converse is also true here.[A] The history of the converse here is slightly different. First Mary Weiss proved the a.e. converse for the symmetric L^p derivative[W]; the next and last step was the general theorem.

1.3. **Quantum derivatives.**

1.3.1. *Definitions.* The quantum L^∞ derivatives are of two types. The first are the generalized Riemann kth quantum derivatives which have the form

$$\lim_{q \to 1} \frac{\sum_{i=0}^{k+e} w_i(q) f(q^{u_i} x)}{(q-1)^k x^k}$$

where the weights $\{w_i(q)\}$, which are measurable functions of q, and the exponents $\{u_i\}$ satisfy

$$\lim_{q \to 1} \sum_{i=0}^{k+e} w_i(q) (q^{u_i})^j = \begin{cases} 0 & \text{if } j = 0, 1, \dots, k-1, \\ k! & \text{if } j = k. \end{cases}$$

Notice that there is an annoying detail here; the definition does not make sense when $x = 0$. Whenever discussing any quantum derivative of any kind, we shall always suppose implicitly that $x \neq 0$. Again it is not hard to establish consistency with the ordinary kth Peano derivative.[R] An important special case is the generalized Riemann kth quantum derivative,

$$QR_k f(x) = \lim_{q \to 1} \frac{\sum_{i=0}^{k} (-1)^k \begin{bmatrix} k \\ i \end{bmatrix}_q q^{(i-1)i/2} f\left(q^{k-i} x\right)}{q^{(k-1)k/2} (q-1)^k x^k},$$

where the q-binomial coefficient $\left[\begin{smallmatrix}k\\i\end{smallmatrix}\right]_q$ satisfies

$$\begin{bmatrix}k\\-1\end{bmatrix}_q=\begin{bmatrix}k\\\ell\end{bmatrix}_q=0 \text{ for all } \ell>k\geq 0, \text{ and for all } k\geq i\geq 0, \tag{1.4}$$

$$\begin{bmatrix}k\\i\end{bmatrix}_q=\frac{[k]_q!}{[i]_q!\,[k-i]_q!},\quad [i]_q!=\begin{cases}1 & \text{if } i=0,\\ [1]_q\,[2]_q\cdots[i]_q & \text{if } i>0;\end{cases} \tag{1.5}$$

and, temporarily (see also definition (1.7) below),

$$[k]_q=\frac{q^k-1}{q-1}, \text{ for } k=1,2,\ldots. \tag{1.6}$$

In particular, if $k=2$,

$$QR_2f(x)=\lim_{q\to 1}\frac{f\left(q^2x\right)-(1+q)\,f(qx)+qf(x)}{(q-1)^2\,x^2}$$

as was mentioned in formula (1.1) above. A somewhat symmetrical version of this, namely

$$\lim_{q\to 1}\frac{\sum_{i=0}^k(-1)^k\left[\begin{smallmatrix}k\\i\end{smallmatrix}\right]_q q^{(i-1)i/2}f\left(q^{k/2-i}x\right)}{q^{(k-1)k/2}\,(q-1)^k\,x^k},$$

was also studied in [ACR], where it was called the generalized symmetric Riemann kth quantum derivative. In this paper, we will define and study other generalized derivatives that are more deserving of that name.

Since this paper will focus on symmetric quantum derivatives, from this point onward we will redefine the quantity $[k]_q$ in a symmetrical way,

$$[k]_q=\frac{q^{k/2}-q^{-k/2}}{q^{1/2}-q^{-1/2}}, \text{ for } k=0,\pm 1,\pm 2,\ldots, \tag{1.7}$$

while keeping the relationships (1.4) and (1.5) in force so as to also give more symmetrical definitions to $\left[\begin{smallmatrix}k\\i\end{smallmatrix}\right]_q$ and $[i]_q!$. We will also define a family of symmetric quantum nth derivatives $S_n^af=S_n^af(x)$ inductively by

$$\frac{q^{-\frac{a}{2}}f(qx)+(-1)^n\,q^{\frac{a}{2}}f\left(q^{-1}x\right)}{2_{an}}=\sum_{\substack{k\equiv n \bmod 2\\0\leq k\leq n}}S_k^af(x)\,\frac{\Delta^k}{k!}+o\left(\Delta^n\right), \tag{1.8}$$

where $\Delta=\left(q^{1/2}-q^{-1/2}\right)x$, a is an integer and

$$2_{an}=\begin{cases}2, & \text{if } n\equiv a \bmod 2,\\ [2]=[2]_q=q^{\frac12}+q^{-\frac12} & \text{otherwise.}\end{cases}$$

Note that Taylor expanding the left hand sides about x shows that whenever $f\in t_n(x)$,

$$S_n^af(x)=\frac{1}{2_{an}}\left([2]_{q^{n-a}}\,f_n(x)+\frac{n\,[n-a-1]_q}{x}f_{n-1}(x)\right),$$

provided the jth Peano derivative f_j exists at x. Notice that the values $a=-1,0$, and 1 are distinguished by the relations

$$\begin{aligned} S_j^0 f(x) &= f_j(x), j=0,1, \\ S_j^1 f(x) &= f_j(x), j=0,2, \\ S_j^{-1} f(x) &= f_j(x), j=0. \end{aligned} \tag{1.9}$$

The relation $S_j^a f(x) = f_j(x)$ holds in general for no other pairs (a,j). We will denote by $S_j^a(x)$ the class of functions for which $S_j^a f(x)$ exists, so that $f \in S_j^a(x)$ if and only if $S_j^a f(x)$ exists. We will follow this convention for all of the different generalized derivatives to be considered.

The quantum symmetric derivative definitions are not canonical, unlike the situation in the additive case. We will discuss the question of whether they are "natural" in Subsection 2.2 below.

To discuss L^p quantum differentiation, we will need a little notation. Recall that $\|f(t)\|_p(h)$ means $\left(h^{-1}\int_0^h |f(t)|^p\,dt\right)^{1/p}$. Similarly, $\|f(t)\|_p(q)$ will be defined by

$$\|f(t)\|_p(q) := \left(\frac{1}{q-1}\int_1^q |f(t)|^p \frac{dt}{t}\right)^{1/p}.$$

The factor $1/t$ in this definition will be seen to be very convenient for our work, but has no effect on the various definitions and estimates occurring in this paper since q will always be close to one. Also there is an obvious notational clash between the classical additive notation and the quantum notation which differ only alphabetically. Mentioning this here should be sufficient to avoid confusion. To each L^∞ quantum symmetric derivative corresponds an L^p one in the obvious way. For example, $S_n^a f(x) = S_n^{a\infty} f(x)$ is the number satisfying

$$\left|\frac{q^{-\frac{a}{2}} f(qx) + (-1)^n q^{\frac{a}{2}} f\left(q^{-1}x\right)}{2_{an}} - \sum_{\substack{j\equiv n \bmod 2 \\ 0\le j\le n}} S_j^a f(x)\frac{\Delta^j}{j!}\right| = o(\Delta^n)$$

as $q\to 1$, while $S_n^{ap} f(x)$ is the number satisfying

$$\left\|\frac{t^{-\frac{a}{2}} f(tx) + (-1)^n t^{\frac{a}{2}} f\left(t^{-1}x\right)}{2_{an}} - \sum_{\substack{j\equiv n \bmod 2 \\ 0\le j\le n}} S_j^{ap} f(x)\frac{\Delta^j}{j!}\right\|_p (q) = o(\Delta^n)$$

as $q\to 1$.

The last derivatives that we need to define are the quantum symmetric Riemann derivatives. In the quantum case, for each integer a, the nth *a-quantum Riemann symmetric derivative of f at x* is defined to be the limit

$$R_n^a f(x) = R_n^{a\infty} f(x) = \lim_{q\to 1}\frac{\sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k\end{bmatrix} q^{-\left(\frac{n}{2}-k\right)a} f(q^{\frac{n}{2}-k}x)}{\left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)^n x^n}.$$

If f is n times differentiable at x, then f is a-quantum Riemann symmetric differentiable of order n at x and $f^{(n)}(x) = R_n^a f(x)$. Another family of nth a-quantum

Riemann symmetric derivatives of f at x, the nth *a-quantum pseudo-Riemann symmetric derivative of f at x*, is defined by

$$\widehat{R_n^a}f(x) = \lim_{q\to 1} \frac{\sum_{k=0}^{n}(-1)^k \left(\left[{n-1 \atop k}\right] + \left[{n-1 \atop k-1}\right]\right) q^{-\left(\frac{n}{2}-k\right)a} f(q^{\frac{n}{2}-k}x)}{\left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)^n x^n};$$

and a third family of nth a-quantum Riemann symmetric derivatives of f at x is the *constant coefficient nth a-quantum Riemann symmetric derivative of f at x*, defined by

$$CR_n^a f(x) = \lim_{q\to 1} \frac{\sum_{k=0}^{n}(-1)^k \binom{n}{k} q^{-\left(\frac{n}{2}-k\right)a} f(q^{\frac{n}{2}-k}x)}{\left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)^n x^n}.$$

Notice that the "constant coefficient" has truly constant coefficients only when $a = 0$. Replacing the L^∞ norm by the L^p norm similarly defines three more families, $R_n^{ap}(x)$, $\widehat{R_n^{ap}}(x)$, and $CR_n^{ap}(x)$, for each $p \in [1,\infty)$. Parallel definitions of the classes $S_n^{ap}(x)$ complete our menagerie of generalized quantum derivatives enjoying $q \leftrightarrow q^{-1}$ symmetry.

A new unsymmetrical nth quantum derivative with constant coefficients $\{w_i\}$ of the form $\tilde{C}_k f(x) = \lim_{q\to 1}\left(w_0 f(x) + \sum_{i=1}^{n} w_i f\left(q^{2^{i-1}}x\right)\right) / (q-1)^n$ will be defined in Section 3 below and play a role similar to the one played by the additive generalized derivative $\tilde{D}_k f(x)$ in [MZ].

1.4. **Results and open questions.** Consistency with the kth Peano derivative holds for all symmetric quantum kth derivatives in the sense that if $f \in t_k(x)$, then $f \in S_k^a(x)$ for every integer a. Consistency with each a-indexed quantum symmetric kth derivative holds for the three corresponding symmetric Riemann kth quantum derivatives. All of these consistency results are true pointwise, have purely algebraic proofs, and are true for the L^p cases with the same proofs. These results are spelled out in Section 2 below.

Converse theorems are much more difficult and at best can be true only a.e. First we discuss the L^∞ situation. The a.e. converse was established in 2002 for passage from the Riemann kth quantum derivative to the kth Peano derivative.[ACR] It will be shown below in Theorem 4 that the a.e. converse also holds for passage from any symmetric kth quantum derivative to the kth Peano derivative. The a.e. converse in general remains an open question. Several additional cases have been treated in Rios' thesis [R].

We also prove in Theorem 1 that the a.e. converse holds for every symmetric kth quantum L^p derivative. This also proves the a.e. converse for every symmetric Riemann kth quantum L^p derivative when $k = 1$ and $k = 2$, since there is no distinction between symmetric and symmetric Riemann in those two cases. All the remaining a.e. converse questions remain open. Our methods are not powerful enough to treat them, but we would be most astonished if the a.e. converse result ever failed. The simplest remaining open questions are the a.e. converse results for the third and fourth symmetric Riemann quantum L^p derivatives. This question will certainly require new ideas when p is finite.

Actually, both major results in this section can be strengthened from "If f has k Peano derivatives in the L^y sense, $1 \le y \le \infty$, then f has k derivatives of type X in the L^y sense. Conversely, if f has k derivatives of type X in the L^y sense on the measurable set E, then f has k Peano derivatives in the L^y sense at a.e. point of E" to "If f is kth order Peano bounded in the L^y sense on the measurable set E, then f has k derivatives of type X in the L^y sense at a.e. point of E. Conversely, if f is kth order bounded of type X in the L^y sense on the measurable set E, then f has k Peano derivatives in the L^y sense at a.e. point of E." To make this explicit, we will formulate the strong form of the last mentioned result.

Theorem 1. *If there are constants $f_{0p}(x), f_{1p}(x), \ldots, f_{(k-1)p}(x)$ so that*

$$\left\| f(x+t) - \left\{ f_{0p}(x) + f_{1p}(x)t + \cdots + f_{(k-1)p}(x)\frac{t^{k-1}}{(k-1)!} \right\} \right\|_p (h) = O\left(h^k\right)$$

as $h \to 0$ every point of the measurable set E, then f has both symmetric and alternate symmetric kth quantum derivatives in L^p at almost every point of E. Conversely, if for some integer a

$$\left\| \frac{t^{-\frac{a}{2}} f(tx) + (-1)^k t^{\frac{a}{2}} f\left(t^{-1}x\right)}{2_{ak}} - \sum_{\substack{j \equiv k \\ 0 \le j \le k-2}} S_j^{ap} f(x) \frac{\Delta(t)^j}{j!} \right\|_p (q) = O\left(\Delta^k\right)$$

for every point x of a measurable set E, then $f \in t_k^p(x)$ for almost every $x \in E$.

Remark 1. We have proved (or in some cases conjectured) that for every $k = 1, 2 \ldots,$ and p, $1 \le p \le \infty$, all L^p kth derivatives are almost everywhere equivalent. What about improving this result from almost everywhere to everywhere? It is our belief that everywhere equivalence never holds except when two definitions coincide. For example, the kth symmetric and the kth Riemann symmetric derivatives coincide when $k = 0, 1,$ or 2; and the kth quantum symmetric and the kth quantum Riemann symmetric derivatives coincide when $k = 0, 1,$ or 2. Some easy counterexamples can be found in [A1] and [R]; any other desired counterexample should be equally easy to construct.

Symmetrization, by which we mean the passage from Peano bounded to symmetric differentiable, is the easier side of Theorem 1. The first step, passage a.e. from Peano bounded to Peano differentiable, is as follows.

Lemma 1. *Let $p \in [1, \infty]$. If for every $x \in E$, we have*

$$\left\| f(x+t) - \left\{ f_{0p}(x) + t f_{1p}(x) + \cdots + \frac{t^{n-1}}{(n-1)!} f_{(n-1)p}(x) \right\} \right\|_p (h) = \omega(x, h),$$

where $\omega(x, h) = O(1)$ as $h \to 0$, then the Peano derivative $f_{np}(x)$ exists for almost every $x \in E$.

When $p = \infty$ this is [MZ, Lemma 7] . When $p \in [1, \infty)$, this follows from [CZ, Theorem 10]. The other step of symmetrization is taken in Section 2. Section 3 is devoted to the proof of the much harder reverse implication.

2. Symmetrization

Here we want to make some consistency connections between Peano, symmetric, and Riemann differentiation. What should be true at *each* point is that "Peano implies symmetric implies Riemann." These implications should be purely algebraic. In fact, all of this works, although the quantum versions are somewhat involved.

2.1. **From Peano to quantum symmetric.** Replacing $f(x+t)$ and $f(x-t)$ by their Taylor expansions in definition (1.8) shows that whenever $f \in t_n(x)$, then f has an nth symmetric derivative at x, and $S_n^+ f(x) = f_n(x)$. The converse is not true. For example, $f(x) = |x|$ at $x = 0$ is not differentiable, but it has symmetric derivatives of any odd order. It is well-known that if f is n times differentiable at x, then it is also $(n-1)$-times differentiable at x. This is not true for symmetric derivatives. For example the even function $\sqrt{|x|}$ has a symmetric third derivative at zero, but it has no symmetric second derivative at zero. What is true is that if f has an nth symmetric derivative at x, then f has an $(n-2)$ symmetric derivative at x. Indeed, if n is even, the polynomial in the right side must be an even function of h, that is, $S_1^+ f(x) = S_3^+ f(x) = ... = S_{n-1}^+ f(x) = 0$, and if n is odd, the same polynomial must be an odd function of h, that is, $S_0^+ f(x) = S_2^+ f(x) = ... = S_{n-1}^+ f(x) = 0$.

We will next see that similar properties hold in the quantum case, but finding an appropriate definition of an nth quantum symmetric derivative turned out to be highly non-trivial. Indeed, in the case of an n times differentiable function f at x, Taylor expansion about x shows that

$$q^{-\frac{a}{2}} f(qx) + (-1)^n q^{\frac{a}{2}} f(q^{-1}x) = \sum_{k=0}^{n} a_k x^k \frac{f^{(k)}(x)}{k!} + o\left((q-1)^n\right),$$

where

$$\begin{aligned} a_k &= q^{-\frac{a}{2}}(q-1)^k + (-1)^n q^{\frac{a}{2}}(q^{-1}-1)^k \\ &= (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^k (q^{\frac{k-a}{2}} + (-1)^{n+k} q^{\frac{a-k}{2}}). \end{aligned}$$

Since our treatment is based on polynomials in the infinitesimal $q^{\frac{1}{2}} - q^{-\frac{1}{2}}$, we will need the second factor $E(q) = q^{\frac{k-a}{2}} + (-1)^{n+k} q^{\frac{a-k}{2}}$ to be a Laurent polynomial in $q^{\frac{1}{2}}$. Therefore, throughout the paper, a will be an integer. There are four cases to consider:

- If $n \equiv k \equiv a \bmod 2$, then $E(q)$ is a symmetric polynomial in q and q^{-1}.
- If $n+k$ and $k-a$ are both odd, then $E(q)$ is $q^{\frac{1}{2}} - q^{-\frac{1}{2}}$ times a symmetric polynomial in q and q^{-1}.
- If $n \equiv k \bmod 2$ and k and a have opposite parities, then $E(q)$ is $q^{\frac{1}{2}} + q^{-\frac{1}{2}}$ times a symmetric polynomial in q and q^{-1}.
- If n and k have opposite parities and $k \equiv a \bmod 2$, then $E(q)$ is $q - q^{-1}$ times a symmetric polynomial in q and q^{-1}.

Since any symmetric polynomial in q and q^{-1} is a polynomial in the fundamental symmetric polynomials $q + q^{-1} = \left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)^2 + 2$ and $qq^{-1} = 1$, hence an even polynomial in $q^{\frac{1}{2}} - q^{-\frac{1}{2}}$, the above four cases can be reduced to two:

- If $n \equiv a \bmod 2$, then a_k is a polynomial in $q^{\frac{1}{2}} - q^{-\frac{1}{2}}$, for all k. The same is true for $q^{-\frac{a}{2}} f(qx) + (-1)^n q^{\frac{a}{2}} f(q^{-1}x)$.
- If n and a have opposite parity, then a_k is $q^{\frac{1}{2}} + q^{-\frac{1}{2}}$ times a polynomial in $q^{\frac{1}{2}} - q^{-\frac{1}{2}}$, for all k. The same is true for $q^{-\frac{a}{2}} f(qx) + (-1)^n q^{\frac{a}{2}} f(q^{-1}x)$.

This motivates our definitions of quantum symmetric derivatives: we say that a function f *has an a-quantum symmetric derivative of order n at x*, and write $f \in S_n^a(x)$, if there exists a polynomial $P(t) = \alpha_0 + \alpha_1 t + \cdots + \alpha_n t^n$, depending on

x, such that

$$\frac{q^{-\frac{a}{2}}f(qx)+(-1)^n q^{\frac{a}{2}}f(q^{-1}x)}{2_{an}} = P(q^{\frac{1}{2}}-q^{-\frac{1}{2}})+o(\left(q^{\frac{1}{2}}-q^{-\frac{1}{2}}\right)^n),$$

where

$$2_{an} = \begin{cases} 2, \text{ if } n \equiv a \bmod 2, \\ q^{\frac{1}{2}}+q^{-\frac{1}{2}} = [2], \text{ if } n \text{ and } a \text{ have opposite parity.} \end{cases}$$

Interchange q with q^{-1} to see that P is a polynomial of the same parity as n. An immediate consequence of the process we have gone through to develop the definition of $S_n^a f(x)$ is the following consistency result for quantum symmetric differentiation.

Proposition 1. *If $f \in t_n(x)$, then $f \in S_n^a(x)$ for every integer a. This is also true in L^p.*

The converse is not true for any a. For example,

$$f(x) = \begin{cases} x^{\frac{a+1}{2}}, \text{ if } x > 1, \\ x^{\frac{a-1}{2}}, \text{ if } x \le 1, \end{cases}$$

has a-quantum symmetric derivatives of any odd order at $x = 1$, but it is not differentiable at $x = 1$.

2.2. **Why these definitions?** Before now there were L^∞ quantum symmetric Riemann derivatives in the literature, for example in [GR], [ACR], and [R]; but quantum symmetric derivatives had not been studied, except for orders 1 and 2 where the two notions coincide. When we first tried to desymmetrize in L^p (pass from symmetric to Peano a.e.), we were able to treat the first derivative $a = 0$ case (which involves $\int_1^q \frac{f(tx)-f\left(t^{-1}x\right)}{2}\frac{dt}{t}$) fairly easily, but ran into difficulty with the second derivative $a = 0$ case (which seemed to involve $\int_1^q \frac{f(tx)+f\left(t^{-1}x\right)}{2}\frac{dt}{t}$). We resolved this difficulty by using $a = 1$ there. The calculations $\int_1^q t^{-\alpha}f(tx)\frac{dt}{t} = x^{-\alpha}(G(qx)-G(x))$ and $\int_1^q t^{-\beta}f\left(t^{-1}x\right)\frac{dt}{t} = x^{\beta}\left(H(x)-H\left(q^{-1}x\right)\right)$, where $G(x) = \int_1^x f(t)\frac{dt}{t^{1+\alpha}}$ and $H(x) = \int_1^x f(t)\frac{dt}{t^{1-\beta}}$ coincide only if $\beta = -\alpha$ indicate that from the viewpoint of integration, the most general candidate for an nth quantum symmetric difference should have the form $\frac{t^{-\alpha}f(tx)+(-1)^n t^{\alpha}f\left(t^{-1}x\right)}{2}$. More compelling is that this insures $t \longleftrightarrow -t$ antisymmetry in the odd order cases and $t \longleftrightarrow -t$ symmetry in the even order cases. That α should be a half integer does not seem crucial, but it seems sufficiently general and allows us to use the polynomial methods introduced in Subsections 2.3 and 2.4 below.

The relationships (1.9) give some support to the idea that "the correct" odd symmetric derivative is S_n^1 and "the correct" even symmetric derivative is S_n^0. But this evidence does not feel totally compelling to us.

Both of the early cases $n = 1, a = 0$ and $n = 2, a = 1$ were of the sort where a and n had opposite parity. It was much later when we realized that replacing the denominator 2 by $[2] = t + t^{-1}$ was the key to dealing with the cases where $a \equiv n \bmod 2$. That is why the integer a is not of constrained parity in Theorems 1 and 4 below.

Using $\frac{dt}{t}$ rather than dt in defining L^p quantum derivatives is optional since t is always close to 1 and hence bounded above and below but seems more natural for two reasons. First on the general principle that the situation is multiplicative, and second that calculations such as the one yielding formula (3.21) in the last section work so smoothly.

One annoying technical point (which causes no essential difficulty in practice) is that the first 0-quantum symmetric derivative is based on $f(qx) - f\left(q^{-1}x\right)$, while the first 0-quantum symmetric Riemann derivative is based on $f\left(q^{1/2}x\right) - f\left(q^{-1/2}x\right)$. (Of course the substitution $q \to q^{1/2}$ shows that the two concepts are actually identical.) This sad notational feature follows from the historical accident that the additive symmetric Riemann derivative is usually written with a step length of h, so that the first two are based on $f\left(x + \frac{1}{2}h\right) - f\left(x - \frac{1}{2}h\right)$ and $f(x+h) + f(x-h) - 2f(x)$, while the first two additive symmetric derivatives are usually based on $f(x+h) - f(x-h)$ and $f(x+h) + f(x-h)$.

The derivatives CR_n^a are used for technical reasons and are a kind of in-between object that is neither additive nor quantum. They play a fine role in the proof of the hardest theorem of this paper, Theorem 1, but they should probably be thought of only as tools and not useful definitions. A similar remark applies to the derivatives $\tilde{C}_k^a(f,x)$ defined in Section 3.

To support our goal of establishing consistency from quantum symmetric differentiation to quantum symmetric Riemann differentiation, we will first develop some algebraic machinery.

2.3. **Polynomials and quantum differences.** The set $\mathcal{F} = \{f | f : \mathbb{R} \longrightarrow \mathbb{R}\}$ of all real-valued functions is an $\mathbb{R}$-vector space under addition

$$(f+g)(x) = f(x) + g(x)$$

and scalar multiplication

$$(rf)(x) = rf(x),$$

for all $r, x \in \mathbb{R}$, and $f, g \in \mathcal{F}$. The set $\mathcal{L}(\mathcal{F})$ of all $\mathbb{R}$-linear mappings on $\mathcal{F}$ is an $\mathbb{R}$-algebra under addition

$$(L_1 + L_2)(f) := L_1(f) + L_2(f),$$

multiplication

$$(L_1 L_2)(f) := (L_1 \circ L_2)(f),$$

for all $L_1, L_2 \in \mathcal{L}(\mathcal{F})$ and $f \in \mathcal{F}$, and natural embedding of $\mathbb{R}$ in $\mathcal{L}(\mathcal{F})$ given by $1 \mapsto \mathrm{id}_{\mathcal{F}}$. For $q \in \mathbb{R} - \{0\}$, let L_q be the element of $\mathcal{L}(\mathcal{F})$ defined by

$$L_q(f)(x) = f(qx),$$

and let M_q be the element of $\mathcal{L}(\mathcal{F})$ defined by

$$M_q(f)(x) = qf(x).$$

Then both L_q and M_q are invertible elements of $\mathcal{L}(\mathcal{F})$. In addition, they satisfy $(L_q)^n = L_{q^n}$, $(M_q)^n = M_{q^n}$, for all $n \in \mathbb{Z}$, and $L_q \circ M_q = M_q \circ L_q$. It follows that the map $q \mapsto M_q$, $y \mapsto L_q$ extends to a unique algebra homomorphism $T : \mathbb{R}[q, q^{-1}, y, y^{-1}] \longrightarrow \mathcal{L}(\mathcal{F})$. This map is one-to-one. Indeed, if $P(q,y) = \sum_{i=-n}^{n} a_i(q) y^i \in \mathbb{R}[q, q^{-1}][y, y^{-1}]$ is so that $T(P) = 0$, then

$$T(P)(f)(x) = \sum_{i=-n}^{n} a_i(q) f(q^i x) = 0,$$

for all $f \in \mathcal{F}$, $x \in \mathbb{R}$. By taking

$$f(t) = f_i(t) = \begin{cases} 1, \text{ if } t = q^i x, \\ 0, \text{ otherwise}, \end{cases}$$

we deduce that $a_i(q) = 0$, for all i and q. Therefore $P = 0$, and this makes T a one-to-one map. Thus T is an algebra isomorphism from $\mathbb{R}[q, q^{-1}, y, y^{-1}]$ to the algebra of two-sided quantum differences $T(\mathbb{R}[q, q^{-1}, y, y^{-1}])$. In particular, we have $\mathbb{R}[q, y] \cong T(\mathbb{R}[q, y])$, the algebra of one-sided quantum differences, and $\mathbb{R}[q + q^{-1}, y + y^{-1}] \cong T(\mathbb{R}[q + q^{-1}, y + y^{-1}])$, the algebra of symmetric quantum differences.

In this paper computations on differences will be routinely reduced to computations on polynomials via T. This reduction is being used for two specific jobs:

(1) Writing an n-difference as a composition of smaller differences. For example, the two-difference $D(f)(x) = f(q^2 x) - (q + 1)f(qx) + qf(x)$ is a composition of two one-differences as follows: $D = T(y^2 - (q + 1)y + q) = T((y - 1)(y - q)) = T(y - 1) \circ T(y - q) = D_1 \circ D_2$, where $D_1(f)(x) = T(y - 1)(f)(x) = (L_q - \mathrm{id})(f)(x) = f(qx) - f(x)$ and where $D_2(f)(x) = T(y - q)(f)(x) = (L_q - q{\cdot}\mathrm{id})(f)(x) = f(qx) - qf(x)$.

(2) Checking that a certain difference is a linear combination of shifts of another difference. If a difference $D(f)(x)$ is written more explicitly as $D(f)(q, x)$, then a shift of it is any difference of the form $D'(f)(x) = D(f)(q, q^i x)$, for some integer i. In terms of polynomials, this means that $T^{-1}(D') = y^i T^{-1}(D)$. Moreover, a difference D' is a linear combination of shifts of another difference D if and only if $T^{-1}(D')$ divides $T^{-1}(D)$. For example, since $y-1$ divides $y^2-(q+1)y+q$, the difference $f(q^2x)-(q+1)f(qx)+qf(x)$ is a linear combination of shifts of the difference $f(qx) - f(x)$. Indeed, $f(q^2 x) - (q + 1)f(qx) + qf(x) = (f(q^2 x) - f(qx)) - q(f(qx) - f(x))$.

2.4. **The correspondence (*a).** If a is an arbitrary integer, then the map t_a, defined by $t_a(q) = q$ and $t_a(y) = q^{-\frac{a}{2}} y$, extends uniquely to an algebra isomorphism $t_a : \mathbb{R}[q, q^{-1}, y, y^{-1}] \to \mathbb{R}[q, q^{-1}, q^{-\frac{a}{2}} y, q^{\frac{a}{2}} y^{-1}] \subseteq \mathbb{R}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}, y, y^{-1}]$. The composition $T_a = T \circ t_a : \mathbb{R}[q, q^{-1}, y, y^{-1}] \to T(\mathbb{R}[q, q^{-1}, q^{-\frac{a}{2}} y, q^{\frac{a}{2}} y^{-1}])$ is also an algebra isomorphism. Throughout the paper, the correspondence determined by T_a and T_a^{-1} will be referred to as *the correspondence (*a)*. For example, the difference

$$q^{-\frac{3}{2}} f(qx) - (q + 1)f(x) + q^{\frac{3}{2}} f(q^{-1}x)$$

under (*-3) corresponds to the polynomial $y - (q + 1) + y^{-1}$, and under (*0) it corresponds to the polynomial $q^{-\frac{3}{2}} y - (q + 1) + q^{\frac{3}{2}}$.

2.5. **From quantum symmetric to quantum Riemann.** We prove the second part of the next proposition because we haven't seen the statement or proof written elsewhere and also because it gives an overview as to what must be done in the more technically complicated quantum cases to follow.

Proposition 2. *(i) If $f \in t_k(x)$, then $f \in S_k^+(x)$ and $S_k^+ f(x) = t_k(x)$.*
(ii) If $f \in S_k^+(x)$, then $f \in R_k(x)$ and $R_k^+ f(x) = S_k^+(x)$.
(iii) The L^p analogues of (i) and (ii) are also valid.

Proof. Part (i) follows immediately upon Taylor expansion of $f(x+h)$ and $f(x - h)$. For (ii), assume that the kth symmetric L^∞ derivative of f exists at x. Using $x = 0$

and $S_j = S_j^+ f(0)$ for convenience, we calculate

$$
\begin{aligned}
&2\sum_{n=0}^{k}(-1)^n\binom{k}{n}f\left(\left(\tfrac{k}{2}-n\right)h\right)\\
&=\sum_{n=0}^{k}(-1)^n\binom{k}{n}f\left(\left(\tfrac{k}{2}-n\right)h\right)+\sum_{n=0}^{k}(-1)^{k-n}\binom{k}{k-n}f\left(\left(n-\tfrac{k}{2}\right)h\right)\\
&=\sum_{n=0}^{k}(-1)^n\binom{k}{n}\left\{f\left(\left(\tfrac{k}{2}-n\right)h\right)+(-1)^k f\left(\left(n-\tfrac{k}{2}\right)h\right)\right\}\\
&=\sum_{n=0}^{k}(-1)^n\binom{k}{n}\left\{2\sum_{i=0}^{\lfloor k/2\rfloor}\frac{S_{k-2i}}{(k-2i)!}\left(\tfrac{k}{2}-n\right)^{k-2i}h^{k-2i}\right\}+o\left(h^k\right)\\
&=2\sum_{i=0}^{\lfloor k/2\rfloor}\frac{S_{k-2i}}{(k-2i)!}h^{k-2i}\left\{\sum_{n=0}^{k}(-1)^n\binom{k}{n}\left(\tfrac{k}{2}-n\right)^{k-2i}\right\}+o\left(h^k\right)\\
&=2S_k f(x)\,h^k+o\left(h^k\right),
\end{aligned}
$$

so that the L^∞ Riemann symmetric derivative also exists at x. The L^p argument is very similar. □

The next result shows that the above three notions of quantum Riemann symmetric are somewhat equivalent:

Theorem 2. *The following are equivalent:*

(i) f has a-Riemann symmetric derivatives of orders $n, n-2, n-4, \ldots$ at x, that is, $f \in \bigcap_{\{i:n-2i\geq 0\}} R^a_{n-2i}(x)$.

(ii) f has constant coefficients a-Riemann symmetric derivatives of orders $n, n-2, n-4, \ldots$ at x, that is, $f \in \bigcap_{\{i:n-2i\geq 0\}} CR^a_{n-2i}(x)$.

(iii) f has pseudo a-Riemann symmetric derivatives of orders $n, n-2, n-4, \ldots$ at x, that is, $f \in \bigcap_{\{i:n-2i\geq 0\}} \widehat{R^a_{n-2i}}(x)$.

Proof. The proof is algebraic, based on the polynomial correspondence (*a) defined in Subsection 2.4. Recall that under this correspondence, say when $i=0$, the numerators in parts (i), (ii), and (iii) correspond to Laurent polynomials

$$
\begin{aligned}
D_n(q,y) &= y^{-\frac{n}{2}}\prod_{i=0}^{n-1}\left(y-q^{\frac{n-1}{2}-i}\right),\\
D_n(1,y) &= y^{-\frac{n}{2}}\prod_{i=0}^{n-1}(y-1)=\left(y^{\frac{1}{2}}-y^{-\frac{1}{2}}\right)^n, \text{ and}\\
\left(y^{\frac{1}{2}}-y^{-\frac{1}{2}}\right)D_{n-1}(q,y) &= y^{-\frac{n}{2}}(y-1)\prod_{i=0}^{n-2}\left(y-q^{\frac{n-2}{2}-i}\right), \text{ respectively.}
\end{aligned}
$$

We only prove the equivalence of (i) and (ii), say in the odd case. The even case and the equivalence of (ii) and (iii) are similar. Indeed, since

$$\begin{aligned}
D_{2n+1}(q,y) &= \left(y^{\frac12}-y^{-\frac12}\right)\prod_{i=1}^{n}[y^{-1}(y-q^i)((y-q^{-i}))] \\
&= \left(y^{\frac12}-y^{-\frac12}\right)\prod_{i=1}^{n}(y+y^{-1}-q^i-q^{-i}) \\
&= \left(y^{\frac12}-y^{-\frac12}\right)\prod_{i=1}^{n}[(y^{1/2}-y^{-1/2})^2-(q^{i/2}-q^{-i/2})^2] \\
&= \left(y^{\frac12}-y^{-\frac12}\right)\prod_{i=1}^{n}[(y^{1/2}-y^{-1/2})^2-(q^{1/2}-q^{-1/2})^2[i]^2] \\
&= \sum_{k=0}^{n}u_{n,k}(q)(q^{1/2}-q^{-1/2})^{2k}(y^{1/2}-y^{-1/2})^{2n+1-2k} \\
&= \sum_{k=0}^{n}u_{n,k}(q)(q^{1/2}-q^{-1/2})^{2k}D_{2n+1-2k}(1,y),
\end{aligned}$$

where $u_{n,k}(q)$ are symmetric Laurent polynomials in q, and $u_{n,0}(q)=1$. In particular, division by $(q^{1/2}-q^{-1/2})^{2n+1}$ yields

$$\frac{D_{2n+1}(q,y)}{(q^{1/2}-q^{-1/2})^{2n+1}}=\sum_{k=0}^{n}u_{n,k}(q)\frac{D_{2n+1-2k}(1,y)}{(q^{1/2}-q^{-1/2})^{2n+1-2k}}.$$

With the notation $y_s=\frac{D_{2s+1}(q,y)}{(q^{1/2}-q^{-1/2})^{2s+1}}$ and $z_s=\frac{D_{2s+1}(1,y)}{(q^{1/2}-q^{-1/2})^{2s+1}}$, the above relation written for $s=0,1,2,...,n$, yields the system

$$[y_n,y_{n-1},...,y_0]^T=A\,[z_n,z_{n-1},...,z_0]^T,$$

where A is the upper-triangular matrix $(a_{ij})_{0\le i,j\le n}$ and $a_{ij}=u_{n-i,j-i}(q)$, for $j\ge i$. With its diagonal elements being 1, matrix A is invertible. Therefore, knowing the vector $\mathbf{y}=[y_n,y_{n-1},...,y_0]^T$ is equivalent to knowing the vector $\mathbf{z}=[z_n,z_{n-1},...,z_0]^T$. Going back to the function f via the correspondence $(*a)$, and taking limit as $q\to 1$, this means that (i) is equivalent to (ii), as desired. $\square$

The following example illustrates the main idea of the computations that we will make later on in this section. Suppose $f\in S_4^0$. Then

$$f(qx)+f(q^{-1}x)-2f(x)=\alpha_2\left(q^{\frac12}-q^{-\frac12}\right)^2+\alpha_4\left(q^{\frac12}-q^{-\frac12}\right)^4+o\left(\left(q^{\frac12}-q^{-\frac12}\right)^4\right).$$

Changing q into q^2, this relation becomes:

$$f(q^2x)+f(q^{-2}x)-2f(x)=\alpha_2\left(q-q^{-1}\right)^2+\alpha_4\left(q-q^{-1}\right)^4+o\left(\left(q^{\frac12}-q^{-\frac12}\right)^4\right).$$

Multiplication of the first equation by $\left(q^{1/2}+q^{-1/2}\right)^2=[2]^2$ and subtraction from the second will eliminate the α_2-term from the right side to produce

$$\begin{aligned}&f(q^2x)-\left(q+2+q^{-1}\right)f(qx)+2\left(q+1+q^{-1}\right)f(x)\\&-\left(q+2+q^{-1}\right)f(q^{-1}x)+f(q^{-2}x)\\&=\alpha_4\left([2]^4-[2]^2\right)\left(q^{\frac{1}{2}}-q^{-\frac{1}{2}}\right)^4+o\left(\left(q^{\frac{1}{2}}-q^{-\frac{1}{2}}\right)^4\right).\end{aligned}$$

Since the left side is $\sum_{k=0}^{4}(-1)^k(\left[{3\atop k}\right]+\left[{3\atop k-1}\right])f(q^{2-k}x)$, the above relation implies that

$$\lim_{q\to1}\frac{\sum_{k=0}^{4}(-1)^k(\left[{3\atop k}\right]+\left[{3\atop k-1}\right])f(q^{2-k}x)}{\left(q^{1/2}-q^{-1/2}\right)^4}=\alpha_4\left(2^4-2^2\right).$$

In other words, we proved that f has a quantum pseudo-Riemann symmetric derivative of order 4, that is, $f\in\widehat{R_4^0}$. In general, for larger n, and any integer a, we will need to remove more than one α-term from the right side of the quantum symmetric relation. We will therefore need two technical results on systems of linear equations in Laurent polynomials with Vandermonde coefficient matrices. The first deals with the symmetric case, and the second with the anti-symmetric case.

Lemma 2. *(i) There exist $\beta_1,\beta_2,...,\beta_{n-1},\beta_n=1$ rational functions in q, such that*

$$\sum_{j=1}^{n}\beta_j(q^{\frac{j}{2}}-q^{-\frac{j}{2}})^{2i}=0,\quad\textit{for all } i=1,2,...,n-1.$$

(ii) The same β's satisfy

$$\sum_{j=1}^{n}\beta_j(q^{\frac{ij}{2}}-q^{-\frac{ij}{2}})^2=0,\ \textit{for all } i=1,2,...,n-1.$$

(iii) The Laurent polynomial $\phi_{2n}(y)=\sum_{j=1}^n\beta_j(y^j+y^{-j}-2)$ factors as

$$\begin{aligned}\phi_{2n}(y)&=y^{-n}\prod_{i=0}^{n-1}(y-q^i)(y-q^{-i})\\&=\left(y^{\frac{1}{2}}-y^{-\frac{1}{2}}\right)\sum_{i=0}^{2n-1}(-1)^i\begin{bmatrix}2n-1\\i\end{bmatrix}y^{\frac{2n-1}{2}-i}\\&=\sum_{i=0}^{2n}(-1)^i(\begin{bmatrix}2n-1\\i\end{bmatrix}+\begin{bmatrix}2n-1\\i-1\end{bmatrix})y^{n-i}.\end{aligned}$$

(iv) The coefficients β_j in parts (i) – (iii) are given by the expression

$$\beta_{n-j}=(-1)^j(\begin{bmatrix}2n-1\\j\end{bmatrix}+\begin{bmatrix}2n-1\\j-1\end{bmatrix})=(-1)^j\begin{bmatrix}2n\\j\end{bmatrix}\cdot\frac{q^{\frac{n-j}{2}}+q^{-\frac{n-j}{2}}}{q^{\frac{n}{2}}+q^{-\frac{n}{2}}},$$

for $j=0,1,...,n-1$.

(v) The Laurent polynomial

$$\delta_{2n}(y) = (q^{\frac{n}{2}} + q^{-\frac{n}{2}}) \sum_{j=1}^{n} \beta_j \left(\frac{y^j + y^{-j}}{q^{\frac{j}{2}} + q^{-\frac{j}{2}}} - 1 \right)$$

can be written as

$$\delta_{2n}(y) = \sum_{i=0}^{2n} (-1)^i \begin{bmatrix} 2n \\ i \end{bmatrix} y^{n-i} = y^{-n} \prod_{i=0}^{2n-1} (y - q^{\frac{2n-1}{2} - i}).$$

Proof. (i) We denote $[j] = [j]_{q^{1/2}} = \frac{q^{j/2} - q^{-j/2}}{q^{1/2} - q^{-1/2}}$, and divide the ith equation by $(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{2i}$. The system is then equivalent to

$$\sum_{j=1}^{n-1} \beta_j [j]^{2i} = -[n]^{2i}, \text{ for all } i = 1, 2, ..., n-1.$$

This new system has a Vandermonde coefficient matrix $([j]^{2i})_{i,j}$ and, by Cramer's rule, it has a unique solution $\beta_1, \beta_2, ..., \beta_{n-1}$. (ii) Since

$$\begin{aligned} (u^{\frac{1}{2}} - u^{-\frac{1}{2}})^{2m} &= -(-1)^m \binom{2m}{m} + \sum_{l=0}^{m} (-1)^{m-l} \binom{2m}{m-l} ((u^{\frac{l}{2}} - u^{-\frac{l}{2}})^2 + 2) \\ &= \sum_{l=1}^{m} (-1)^{m-l} \binom{2m}{m-l} (u^{\frac{l}{2}} - u^{-\frac{l}{2}})^2, \end{aligned}$$

the sets of linearly independent and symmetric in u, u^{-1}-Laurent polynomials

$$\left\{ (u^{\frac{1}{2}} - u^{-\frac{1}{2}})^{2m} : m = 1, 2, ..., i \right\} \text{ and } \left\{ (u^{\frac{m}{2}} - u^{-\frac{m}{2}})^2 : m = 1, 2, ..., i \right\}$$

are two bases of the same space of Laurent polynomials. We can then write $(u^{\frac{i}{2}} - u^{-\frac{i}{2}})^2 = \sum_{m=1}^{i} \gamma_{m,i} (u^{\frac{1}{2}} - u^{-\frac{1}{2}})^{2m}$, for all $i = 1, 2, ..., n-1$, and consequently

$$\begin{aligned} \sum_{j=1}^{n} \beta_j (Q^{\frac{ij}{2}} - Q^{-\frac{ij}{2}})^2 &= \sum_{j=1}^{n} \beta_j \sum_{m=1}^{i} \gamma_{m,i} (Q^{\frac{j}{2}} - Q^{-\frac{j}{2}})^{2m} \\ &= \sum_{m=1}^{i} \gamma_{m,i} \sum_{j=1}^{n} \beta_j (Q^{\frac{j}{2}} - Q^{-\frac{j}{2}})^{2m} = \sum_{m=1}^{i} \gamma_{m,i} \cdot 0 = 0. \end{aligned}$$

The first expression in part (iii) follows from a degree argument, since by part (ii) ϕ_n has roots 1 and $q^{\pm i}$, for $i = 1, 2, ..., n-1$, and the y-symmetry forces 1 to be a double root. The remaining two expressions are easy applications of the quantum binomial formula. Part (iv) comes from identification of y-power coefficients and by easy binomial coefficients computations. The first expression in part (v) comes easily from part (iv), and the second from the quantum binomial formula. □

Lemma 3. *(i) There exist $\beta_1, \beta_2, ..., \beta_{n-1}, \beta_n = 1$ rational functions in q, such that*

$$\sum_{j=1}^{n} \beta_j (q^{\frac{2j-1}{2}} - q^{-\frac{2j-1}{2}})^{2i-1} = 0, \text{ for all } i = 1, 2, ..., n-1.$$

(ii) The same β's satisfy

$$\sum_{j=1}^{n} \beta_j (q^{\frac{(2i-1)(2j-1)}{2}} - q^{-\frac{(2i-1)(2j-1)}{2}}) = 0, \text{ for all } i = 1, 2, ..., n-1.$$

(iii) The Laurent polynomial $\phi_{2n-1}(y) = \sum_{j=1}^{n} \beta_j (y^{\frac{2j-1}{2}} - y^{-\frac{2j-1}{2}})$ *can be written as*

$$\begin{aligned}\phi_{2n-1}(y) &= y^{-\frac{2n-1}{2}}(y-1)\prod_{i=1}^{n-1}(y-q^{2i-1})(y-q^{-2i+1}) \\ &= \left(y^{\frac{1}{2}} - y^{-\frac{1}{2}}\right)\sum_{i=0}^{2n-2}(-1)^i \begin{bmatrix} 2n-2 \\ i \end{bmatrix}_{q^2} y^{n-1-i} \\ &= \sum_{i=0}^{2n-1}(-1)^i(\begin{bmatrix} 2n-2 \\ i \end{bmatrix}_{q^2} + \begin{bmatrix} 2n-2 \\ i-1 \end{bmatrix}_{q^2})y^{\frac{2n-1}{2}-i}.\end{aligned}$$

(iv) The coefficients β_j *in parts (i) – (iii) are given by the expression*

$$\beta_{n-j} = (-1)^j(\begin{bmatrix} 2n-2 \\ j \end{bmatrix}_{q^2} + \begin{bmatrix} 2n-2 \\ j-1 \end{bmatrix}_{q^2}) = (-1)^j \begin{bmatrix} 2n-1 \\ j \end{bmatrix}_{q^2} \cdot \frac{q^{\frac{2n-1}{2}-j} + q^{-\frac{2n-1}{2}+j}}{q^{\frac{2n-1}{2}} + q^{-\frac{2n-1}{2}}},$$

for $j = 0, 1, ..., n-1$.

(v) The Laurent polynomial

$$\delta_{2n-1}(y) = (q^{\frac{2n-1}{2}} + q^{-\frac{2n-1}{2}})\sum_{j=1}^{n}\beta_j \cdot \frac{y^{\frac{2j-1}{2}} - y^{-\frac{2j-1}{2}}}{q^{\frac{2j-1}{2}} + q^{-\frac{2j-1}{2}}}$$

can be written as

$$\delta_{2n-1}(y) = \sum_{i=0}^{2n-1}(-1)^i \begin{bmatrix} 2n-1 \\ i \end{bmatrix}_{q^2} y^{\frac{2n-1}{2}-i} = y^{-\frac{2n-1}{2}} \prod_{i=-(n-1)}^{n-1}(y-q^{2i}).$$

Proof. The proof is quite similar to that of Lemma 2 and is included to emphasize the different nature of the polynomial root $q = 1$. (i) Dividing the ith equation by $(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{(2i-1)}$, and with the notation $[j] = [j]_{q^{1/2}} = \frac{q^{j/2} - q^{-j/2}}{q^{1/2} - q^{-1/2}}$, the system is equivalent to

$$\sum_{j=1}^{n-1}\beta_j[2j-1]^{2i-1} = -[2n-1]^{2i-1}, \text{ for all } i = 1, 2, ..., n-1.$$

This system has a solution $\beta_1, \beta_2, ..., \beta_{n-1}$, since the columns of the coefficient matrix are scalar multiples of a Vandermonde matrix. Part (ii) follows from the sets $\left\{y^{2i-1} - y^{-2i+1} : i = 1, 2, ..., n-1\right\}$ and $\left\{(y - y^{-1})^{2i-1} : i = 1, 2, ..., n-1\right\}$ being two bases of the same space of Laurent polynomials in y. The first expression in part (iii) follows from a degree argument, since by part (ii) ϕ has roots 1 and $q^{\pm(2i-1)}$, for $i = 1, 2, ..., n-1$. The remaining two are easy applications of the quantum binomial formula. Part (iv) comes from identification of y-power coefficients and by easy binomial coefficients computations. The first expression in part (v) comes easily from part (iv), and the second from the quantum binomial formula. □

We remark that the polynomials $\delta_n(y)$ and $\phi_n(y)$ of the above two lemmas correspond under the isomorphism ($*a$) to the numerators of the Riemann symmetric derivatives $R_n^a f(x)$ and $\widehat{R_n^a} f(x)$, respectively. We are now ready to prove the main result of this section.

Theorem 3 (Symmetric implies Riemann symmetric)**.** *If $f \in S_n^a$, then $f \in \widehat{R_n^a} \cap R_n^a \cap CR_n^a$. In other words, if f has an a-symmetric quantum derivative of order n at a point x, then f has an a-Riemann symmetric, a pseudo a-Riemann symmetric, and a constant coefficient a-Riemann symmetric derivative of order n at x. Similarly, if $f \in S_n^{ap}$, then $f \in \widehat{R_n^{ap}} \cap R_n^{ap} \cap CR_n^{ap}$.*

Proof. It suffices to show that $f \in S_N^a$ implies either $f \in \widehat{R_N^a}$ or $f \in RS_N^a$. Indeed, this comes from Theorem 2, since $f \in S_N^a$ easily implies that $f \in S_{N-2i}^a$, for each i such that $0 \leq 2i \leq N$. We distinguish 4 cases, depending on whether N is even (in which case we write $N = 2n$) or N is odd (in which case we write $N = 2n-1$) and whether a is even or odd.

Case 1: N is even. First let $f \in S_{2n}^a$, where a is even. Then there exist real numbers $\alpha_2, \alpha_4, ..., \alpha_{2n}$ such that

$$\begin{aligned} q^{-\frac{a}{2}} f(qx) + q^{\frac{a}{2}} f(q^{-1}x) - 2f(x) &= \alpha_2 \left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)^2 + \alpha_4 \left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)^4 + ... \\ &+ \alpha_{2n} \left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)^{2n} + o\left(\left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)^{2n}\right). \end{aligned}$$

In particular, this is true for q^j in place of q, and we have a system of relations:

$$\begin{aligned} q^{-j\frac{a}{2}} f(q^j x) + q^{j\frac{a}{2}} f(q^{-j}x) - 2f(x) &= \alpha_2 \left(q^{\frac{j}{2}} - q^{-\frac{j}{2}}\right)^2 + \alpha_4 \left(q^{\frac{j}{2}} - q^{-\frac{j}{2}}\right)^4 + ... \\ &+ \alpha_{2n} \left(q^{\frac{j}{2}} - q^{-\frac{j}{2}}\right)^{2n} + o\left(\left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)^{2n}\right), \end{aligned}$$

for $j = 1, 2, ..., n$. We have used the fact that for q near 1, $q^{j/2} - q^{-j/2} = [j]\left(q^{1/2} - q^{-1/2}\right) \sim j\left(q^{1/2} - q^{-1/2}\right)$, so that the conditions $o\left(\left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)^{2n}\right)$ and $o\left(\left(q^{\frac{j}{2}} - q^{-\frac{j}{2}}\right)^{2n}\right)$ are equivalent. We multiply the jth relation by β_j of Lemma 2 and add these up for $j = 1, 2, ..., n$ to deduce that

$$\begin{aligned} &\sum_{i=0}^{2n} (-1)^i \left(\begin{bmatrix} 2n-1 \\ i \end{bmatrix} + \begin{bmatrix} 2n-1 \\ i-1 \end{bmatrix} \right) q^{-2(n-i)a} f(q^{n-i}x) \\ &= \alpha_{2n} \left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)^{2n} \sum_{j=1}^{n} \beta_j [j]^{2n} + o\left(\left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)^{2n}\right). \end{aligned}$$

The expression in the left side comes from Lemma 2(iii) and the correspondence (*a). The expression in the right side comes from Lemma 2(i). Thus $f \in \widehat{R_{2n}^a}$. The other subcase when $N = 2n$ and a is odd is done in a similar manner, except for the fact that Lemma 2(iv) is being used to deduce that $f \in R_{2n}^a$.

Case 2: N is odd. First let $f \in S_{2n-1}^a$, where a is odd. Then there exist real numbers $\alpha_1, \alpha_3, ..., \alpha_{2n-1}$ such that

$$\begin{aligned} q^{-\frac{a}{2}} f(qx) - q^{\frac{a}{2}} f(q^{-1}x) &= \alpha_1 \left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right) + \alpha_3 \left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)^3 + ... \\ &+ \alpha_{2n-1} \left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)^{2n-1} + o\left(\left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)^{2n-1}\right). \end{aligned}$$

As in the even case, we multiply this equation with q^{2j-1} in place of q by the β_j of Lemma 3, and add all these relations up for $j = 1, 2, ..., n-1$ to deduce that

$$\sum_{j=1}^{n-1} \beta_j [q^{-(2j-1)\frac{a}{2}} f(q^{2j-1}x) - q^{(2j-1)\frac{a}{2}} f(q^{-2j+1}x)]$$
$$= \sum_{i=1}^{n} \alpha_{2i-1} \sum_{j=1}^{n-1} \beta_j \left(q^{\frac{2j-1}{2}} - q^{-\frac{2j-1}{2}}\right)^{2i-1} + o\left(\left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)^{2n-1}\right).$$

Using Lemma 3(iii) and the correspondence $(*a)$ in the left side, and Lemma 3(i) in the right side, this expression becomes:

$$\sum_{i=0}^{2n-1} (-1)^i (\begin{bmatrix} 2n-2 \\ i \end{bmatrix}_q + \begin{bmatrix} 2n-2 \\ i-1 \end{bmatrix}_q) q^{-(2n-1-2i)\frac{a}{2}} f(q^{2n-1-2i}x)$$
$$= \alpha_{2n-1} \left(q^{1/2} - q^{-1/2}\right)^{2n-1} \sum_{j=1}^{n-1} \beta_j [2j-1]^{2n-1} + o\left(\left(q^{1/2} - q^{-1/2}\right)^{2n-1}\right).$$

By taking $q^{1/2}$ in place of q, this clearly implies that $f \in \widehat{R^a_{2n-1}}$. The other subcase when a is even is done in a similar manner, except for the fact that Lemma 3(iv) is being used to deduce that $f \in R^a_{2n-1}$.

The L^p proof is a line by line transposition of the L^∞ one just given. □

3. Desymmetrization

The first desymmetrization theorem involves L^∞ derivatives. It is the converse to Proposition 1.

3.1. From quantum symmetric to Peano, the L^∞ case.

Theorem 4. *If f has an a-quantum symmetric derivative of order k at x, $f \in S^a_k(x)$, for every x in a set E of positive measure, then f has a Peano derivative of order k, $f \in t_k(x)$, at a.e. $x \in E$.*

Proof. Theorem 3 tells us that f has a 0-constant coefficient kth quantum Riemann symmetric constant coefficient derivative, $f \in CR^0_k(x)$, at a.e. $x \in E$. We can now copy quite closely an argument appearing in [MZ].

In L^∞, additive translation $(\sum w_i f(x + u_i h) \mapsto \sum w_i f(x + (u_i + \alpha) h))$ preserves additive generalized Riemann differentiation a.e. Also, a finite linear combination of additive translations of

$$\sum\nolimits_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(x + ih) \tag{3.1}$$

can be found which is equal to the difference associated with the additive derivative $\tilde{D}_k f(x)$.[MZ] The mapping $(+) : f(x + \beta h) \mapsto y^\beta$ extends to a unique algebra homomorphism into $\mathcal{L}(\mathcal{F})$. In particular the image of (3.1) is $(y-1)^k$ and if $P_k(y) = \bar{\alpha}_0 + \bar{\alpha}_1 y + \bar{\alpha}_2 y^2 + \bar{\alpha}_3 y^4 + \cdots + \bar{\alpha}_k y^{2^{k-1}}$ is the image of the difference associated with $\tilde{D}_k f(x)$, then the linear combination fact just mentioned is equivalent to the existence of a polynomial Q such that $P_k(y) = Q(y)(y-1)^k$ [MZ, §12].

Now define $\tilde{C}_k^a(f,x)$ by forming differences $\tilde{c}_k$ inductively as

$$\tilde{c}_1^a f(q,x) = q^{-a} f(qx) - f(x),$$
$$\tilde{c}_i^a f(q,x) = \tilde{c}_{i-1}^a f\left(q^2,x\right) - 2^{i-1}\tilde{c}_{i-1}^a f(q,x), i = 2,\dots,k,$$

and then letting

$$\tilde{C}_k^a(f,x) = \lim_{q\to 1}\frac{\tilde{c}_k^a f(q,x)}{(q-1)^k}.$$

In L^∞, there is a quantum analogue of the additive translation theorem which asserts in particular that the multiplicative translation $(\sum_{i=0}^k (-1)^{k-i}\binom{k}{i}q^{-ia}f\left(q^ix\right) \mapsto \sum_{i=0}^k (-1)^{k-i}\binom{k}{i}q^{-ia}f\left(q^{i+\alpha}x\right))$ preserves kth order boundedness a.e.[ACR]. The correspondence $(*{-2a})$ maps $\sum_{i=0}^k (-1)^{k-i}\binom{k}{i}q^{-ia}f\left(q^ix\right)$ into $(y-1)^k$ and $\tilde{c}_k^a f(q,x)$ into $P_k(y)$, so from the very same relation $P_k(y) = Q(y)(y-1)^k$ found in [MZ], it follows that we have a.e.

$$\tilde{c}_k^a f(q,x) = O\left((qx-x)^k\right). \tag{3.2}$$

□

Lemma 4. *If $f\in t_n(x)$, then for any half-integer a, $\tilde{C}_n^a f(x)$ exists and has the form $\sum_{i=0}^n \alpha_i(a)\frac{f_i(x)}{x^{n-i}}$.*

Proof. Step 1: We consider three sequences of polynomials in y and q. The first sequence, $\widetilde{c}_n^a = \widetilde{c}_n^a(y,q)$, corresponds under (*0) to the sequence of differences $\widetilde{c}_n^a f(q,x)$ defined above. The second sequence, $\widetilde{\delta}_n = \widetilde{\delta}_n(y,q)$, is defined by

$$\widetilde{\delta}_1(y,q) = y-1,$$
$$\widetilde{\delta}_n(y,q) = \widetilde{\delta}_{n-1}(y^2,q^2) - \lambda_{n-1}\widetilde{\delta}_{n-1}(y,q),\ n\geq 2,$$

where $\lambda_n = \lambda_n(q) = \left(q^{2^{n-1}}+1\right)\Pi_{i=0}^{n-2}\left(q^{2^{n-1}}+q^{2^i}\right)$. This corresponds under (*0) to the sequence of differences $\widetilde{\Delta}_n(q,x;f)$ defined in [ACR, Proposition 2]. The third sequence is $\widetilde{d}_n := \widetilde{\delta}_n(y,1)$; here λ_n becomes 2^n. All three sequences are defined for $n=0$ by setting $\widetilde{c}_0^a = \widetilde{\delta}_0 = \widetilde{d}_0 = 1$.

Step 2: We claim that there exist $\alpha_i = \alpha_i(q,n,a)$ Laurent polynomials in q and $\beta_i = \beta_i(q,n)$ polynomials in q such that

$$\widetilde{c}_n^a = \sum_{i=0}^n \alpha_i\widetilde{d}_i(q-1)^{n-i},\ n\geq 0, \tag{3.3}$$

and

$$\widetilde{\delta}_n = \sum_{i=0}^n \beta_i\widetilde{d}_i(q-1)^{n-i},\ n\geq 0. \tag{3.4}$$

The case $n=0$ is clear. Assume, by way of induction, that $n\geq 1$ and

$$\widetilde{c}_{n-1}^a(y,q) = \sum_{i=0}^{n-1}\alpha_i(q)\widetilde{d}_i(y,q)(q-1)^{n-1-i}$$

and calculate

$$\begin{aligned}
\widetilde{c}_n^a(y,q) &= \widetilde{c}_{n-1}^a(y^2,q^2) - 2^{n-1}\widetilde{c}_{n-1}^a(y,q) \\
&= \sum_{i=0}^{n-1} \alpha_i(q^2)\widetilde{d}_i(y^2,q^2)(q^2-1)^{n-1-i} - 2^{n-1}\alpha_i(q)\widetilde{d}_i(y,q)(q-1)^{n-1-i} \\
&= \sum_{i=0}^{n-1} \alpha_i(q^2)\left[\widetilde{d}_{i+1}(y,q) + 2^i\widetilde{d}_i(y,q)\right](q^2-1)^{n-1-i} \\
&\quad - 2^{n-1}\alpha_i(q)\widetilde{d}_i(y,q)(q-1)^{n-1-i} \\
&= \sum_{i=0}^{n-1} \alpha_i(q^2)(q+1)^{n-1-i}\widetilde{d}_{i+1}(y,q)(q-1)^{n-1-i} \\
&+ \sum_{i=0}^{n-1} \left[\alpha_i(q^2)2^i(q+1)^{n-1-i} - 2^{n-1}\alpha_i(q)\right]\widetilde{d}_i(y,q)(q-1)^{n-1-i}.
\end{aligned}$$

The result follows for n, since the last expression under brackets is a Laurent polynomial in q that has $q=1$ as a root, i.e., it is divisible by $q-1$, which allows $q-1$ to be factored out. The existence of the second system (3.4) is established in a similar way.

Step 3: The actual proof. Since $\beta_0 = 1$, the triangular system of linear equations defined by relations (3.4) is invertible. Therefore we can write the $\widetilde{c}_n^a$'s as functions of the $\widetilde{\delta}_i$'s in the form

$$\widetilde{c}_n^a(y,q) = \sum_{i=0}^{n} a_i(q)\widetilde{\delta}_i(y,q)(q-1)^{n-i}, \text{ for } n \geq 0.$$

This corresponds under (*0) to

$$\widetilde{c}_n^a f(q,x) = \sum_{i=0}^{n} a_i(q)\widetilde{\Delta}_i(q,x;f)(q-1)^{n-i}, \text{ for } n \geq 0.$$

Divide by $x^n(q-1)^n$ and take the limit as $q \to 1$. By [ACR, Lemma 1], for each i,

$$\lim_{q\to 1} \frac{\widetilde{\Delta}_i(q,x;f)}{(qx-x)^i} = \frac{\mu_i}{i!} f_i(x),$$

where $\mu_i = 2^{i-1}\prod_{j=0}^{i-2}\left(2^{i-1}-2^j\right)$. This completes the proof of the lemma. □

Lemma 5. *If* $\tilde{c}_k(q) = \tilde{c}_k^a f(q,x) = O\left((qx-x)^k\right)$ *for all* $x \in E$*, then* f *has* k *Peano derivatives at a.e.* $x \in E$*.*

Proof. Fix f and $x \in E$. For every $q \in [2/3, 3/2]$, we have

$$\left|\tilde{c}_{k-1}\left(q^2\right) - 2^{k-1}\tilde{c}_{k-1}\left(q^{1/2^0}\right)\right| \le M\left|q^{1/2^0} - 1\right|^k, \tag{3.5}$$

$$\left|\tilde{c}_{k-1}\left(q^{1/2^0}\right) - 2^{k-1}\tilde{c}_{k-1}\left(q^{1/2^1}\right)\right| \le M\left|q^{1/2^1} - 1\right|^k,$$

$$\cdots$$

$$\left|\tilde{c}_{k-1}\left(q^{1/2^{n-1}}\right) - 2^{k-1}\tilde{c}_{k-1}\left(q^{1/2^n}\right)\right| \le M\left|q^{1/2^n} - 1\right|^k.$$

Multiply the ith equation by $2^{(k-1)(i-1)}$, $i = 2, \ldots, n$, and add to the first equation, noting that the left hand sides telescope, to get

$$\left|\tilde{c}_{k-1}\left(q^2\right) - 2^{(k-1)n}\tilde{c}_{k-1}\left(q^{1/2^n}\right)\right| \le CM\left|q-1\right|^k.$$

On the right side we have used the inequalities $|q^\epsilon - 1| \le 2\epsilon\,|q-1|$ for $q \in [2/3, 3/2]$, and $1+\sum_{i=2}^n 2^{(k-1)(i-1)}\left(2\cdot 1/2^{i-1}\right)^k \le 1+2^k\sum_{i=2}^\infty 2^{-(i-1)} = 1+2^k = C$. Rewrite the last inequality as

$$\left|\tilde{c}_{k-1}\left(q^2\right) - \frac{\tilde{c}_{k-1}\left(q^{1/2^n}\right)}{\left(q^{1/2^n}x - x\right)^{k-1}}\left\{x\frac{q^{1/2^n}-1}{1/2^n}\right\}^{k-1}\right| \le CM\left|q-1\right|^k. \tag{3.6}$$

Observation 1. For almost every $x \in E$, if $\tilde{c}_k f(q,x) = O\left((qx-x)^k\right)$, then $\tilde{c}_{k-1} f(q,x) = O\left((qx-x)^{k-1}\right)$. Proof: By Lemma 5 of [ACR] we may assume that f is bounded in a neighborhood of x. So inequality (3.6) tells us that

$$\frac{\tilde{c}_{k-1}\left(q^{1/2^n}\right)}{\left(q^{1/2^n}x - x\right)^{k-1}} \text{ for } n = 0, 1, \ldots$$

is uniformly bounded for q in the annulus $\left[b^{-1}, b^{-1/2}\right] \cup \left[b^{1/2}, b\right]$ for some $b > 1$. As q varies through the annulus and n varies through the non-negative integers, $q^{1/2^n}$ takes on all values in $\left[b^{-1}, b\right] \setminus \{1\}$.

Observation 2. If $f \in t_{k-1}(x)$ and $\tilde{C}_k^+(x) = \limsup_{q\to 1}\left|\frac{\tilde{c}_k(q)}{(qx-x)^k}\right| < \infty$ on E, then f is Peano bounded of order k at a.e. $x \in E$. Proof: Because of Lemma 4, we may assume $f_0(x) = \cdots = f_{k-1}(x) = 0$. Letting $n \to \infty$ in inequality (3.6), taking into account that $\lim_{n\to\infty}\frac{\tilde{c}_{k-1}\left(q^{1/2^n}\right)}{\left(q^{1/2^n}x-x\right)^{k-1}} = 0$ by virtue of Lemma 4, and observing that the quantity in curly brackets tends to $x\ln q$ gives $\tilde{c}_{k-1}\left(q^2\right) = O\left(|q-1|^k\right) = O\left(\left|q^2-1\right|^k\right)$, that is, $\tilde{c}_{k-1}(q) = O\left(|q-1|^k\right)$. Because of Observation 1, we may repeat the whole process $k-2$ more times to ultimately get that $f(qx) = q^a\tilde{c}_1(q) = O\left(|q-1|^k\right)$ which by change of variable tells us that $f(x+h) = O\left(h^k\right)$, that is, f is k Peano bounded.

By Observation 2 and Lemma 1, f_k exists for a.e. $x \in E$. □

Corollary 1. *If f has an a-quantum constant coefficient Riemann symmetric derivative of order k at x, $f \in CR_k^a(x)$, for every x in a set E of positive measure, then f has a Peano derivative of order k, $f \in t_k(x)$, at a.e. $x \in E$.*

Proof. This follows immediately from the proof of Theorem 4 since the first sentence of that proof transferred the hypothesis $f \in S_k^a(x)$ to the condition $f \in CR_k^a(x)$ and the original hypothesis was never used again. $\square$

Corollary 2. *If f is a-quantum symmetric bounded of order k for every x in a set E of positive measure, then f has a Peano derivative of order k, $f \in t_k(x)$, at a.e. $x \in E$.*

Proof. Just as in the last corollary, observe that at one point in the proof of Theorem 4, we only conclude relation (3.2) when more is actually true, and then nothing more than (3.2) is used for the rest of the proof. But the argument that produced (3.2) works line by line from the weaker assumption of *a-quantum symmetric boundedness.* $\square$

3.2. **From quantum symmetric to Peano, the L^p case.** In our proof of the converse part of Theorem 1, we will use the following three lemmas.

Lemma 6. *Let P be a degree $k-1$ polynomial of a certain parity. Then whether 2_ϵ be $[2] = t^{1/2}+t^{-1/2}$ or 2, $\int_1^q P\left(t^{1/2} - t^{-1/2}\right) 2_\epsilon \frac{dt}{t}$ can be written as $Q\left(q^{1/2} - q^{-1/2}\right) + O\left(\left(q^{1/2} - q^{-1/2}\right)^{k+2}\right)$ where Q is a polynomial of degree k of parity opposite to P.*

Proof. In the first case, use the change of variable $u = t^{\frac{1}{2}} - t^{-\frac{1}{2}}, du = \frac{1}{2}\left(t^{\frac{1}{2}} + t^{-\frac{1}{2}}\right)\frac{dt}{t}$ to evaluate $\int [2]P(t^{\frac{1}{2}} - t^{-\frac{1}{2}})\frac{dt}{t}$. The result is immediate and there is no error term.

In the other case, let $t(d) = \left(1 + d\left(\frac{1}{2}d + \frac{1}{2}\sqrt{d^2+4}\right)\right)^{1/2}$. Then for small d, $d = t(d)^{1/2} - t(d)^{-1/2}$, so we may expand

$$\frac{1}{[2]_t} = \frac{1}{t(d)^{\frac{1}{2}} + t(d)^{-\frac{1}{2}}} = R(d) + O\left(d^{k+1}\right), \tag{3.7}$$

where R is the Maclaurin polynomial of degree $\le k$ and is independent of f. Since $t(d)\,t(-d) = 1$, R is even. We have

$$\begin{aligned}\int_1^q P(t^{\frac{1}{2}} - t^{-\frac{1}{2}})\frac{dt}{t} &= \int_1^q P(t^{\frac{1}{2}} - t^{-\frac{1}{2}})\frac{1}{[2]}[2]\frac{dt}{t} \\ &= \int_1^q P(t^{\frac{1}{2}} - t^{-\frac{1}{2}})R\left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right)[2]\frac{dt}{t} + O\left(\left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)^{k+2}\right).\end{aligned}$$

Now PR is a polynomial of the same parity as P and may be integrated to a polynomial of the opposite parity and degree at least k as in the first case. $\square$

Lemma 7. *Let f be integrable near x and let α be a real number. If $G(x) = G(x,\alpha) = \int_1^x f(t)\,t^\alpha dt$ is k Peano bounded at x, then so is $F(x) = \int_1^x f(t)\,dt$.*

Proof. Apply integration by parts to

$$F(x+h) - F(x) = \int_x^{x+h} t^{-\alpha}\left(t^\alpha f(t)\,dt\right)$$

by letting $U = t^{-\alpha}, dV = t^{\alpha} f(t)\, dt$ to get

$$\begin{aligned} F(x+h) - F(x) &= t^{-\alpha} G(t)\,|_x^{x+h} + \alpha \int_x^{x+h} t^{-\alpha-1} G(t)\, dt \\ &= (x+h)^{-\alpha} G(x+h) - x^{-\alpha} G(x) \\ &+ \alpha \int_0^h (x+s)^{-\alpha-1} G(x+s)\, ds. \end{aligned}$$

As soon as $|h| < |x|$, the functions $(x+h)^{-\alpha}$ and $(x+s)^{-\alpha-1}$ are infinitely differentiable and hence have arbitrarily long Taylor expansions at x. It is now clear that the right side can be expanded out to order h^{k-1} with error $O\left(h^k\right)$. □

Lemma 8. *Fix an integer a and a non-negative integer k. If for every $x \in E$,*

$$\int_1^q \left| t^{-\frac{a}{2}} f(tx) + (-1)^k t^{\frac{a}{2}} f\left(t^{-1} x\right) \right|^p \frac{dt}{t} = O\left((q-1)^k \right),$$

and if $f(x) = 0$ on E, then for a.e. $x \in E$,

$$\int_1^q |f(tx)|^p \frac{dt}{t} = O\left((q-1)^k \right). \tag{3.8}$$

By uniformizing, we may assume that there are constants $M > 0$ and $\delta \in (0,1)$ such that for all $x \in E$ and $|q-1| < \delta$,

$$\int_1^q \left| t^{-\frac{a}{2}} f(tx) + (-1)^k t^{\frac{a}{2}} f\left(t^{-1} x\right) \right|^p \frac{dt}{t} \le M (q-1)^k. \tag{3.9}$$

It suffices to show that relation (3.8) holds at every point of density of E. To simplify notation assume that $x = 1$ is a point of density of f. We will show that for $q \in (1,2)$ sufficiently close to 1, we have

$$\int_{1/q}^q |f(t)|^p \frac{dt}{t} \le 2^{\frac{|a|p}{2}+2} M (q-1)^k. \tag{3.10}$$

Proof. For specificity, suppose that

$$\int_1^q |f(t)|^p \frac{dt}{t} \ge \frac{1}{2} \int_{1/q}^q |f(t)|^p \frac{dt}{t}.$$

(The treatment of the contrary case $\int_{1/q}^1 |f(t)|^p \frac{dt}{t} > \frac{1}{2} \int_{1/q}^q |f(t)|^p \frac{dt}{t}$ is similar.) For any x in the interval $\left[1, \sqrt{q}\right]$ we may write

$$\int_1^q |f(t)|^p \frac{dt}{t} \le \int_x^q |f(t)|^p \frac{dt}{t} + \int_{x^2/q}^x |f(t)|^p \frac{dt}{t}. \tag{3.11}$$

Letting respectively $t = xs, \frac{dt}{t} = \frac{ds}{s}$ and $t = xs^{-1}, \frac{dt}{t} = -\frac{ds}{s}$ gives

$$\int_x^q |f(t)|^p \frac{dt}{t} + \int_{x^2/q}^x |f(t)|^p \frac{dt}{t} = \int_1^{q/x} |f(xs)|^p \frac{ds}{s} + \int_1^{q/x} \left| f\left(xs^{-1}\right) \right|^p \frac{ds}{s} = A + B. \tag{3.12}$$

Let F be the complement of E, $A(x,q) = [x,q] \cap \{t : x^2/t \in F\}$, and $B(x,q) = [x^2/q, x] \cap \{t : x^2/t \in F\}$. Then

(3.13)
$$A = \int\limits_{\substack{1 \le s \le q/x \\ xs^{-1} \in F}} |f(xs)|^p \frac{ds}{s} + \int_1^{q/x} \left| f(sx) + (-1)^k s^a f\left(s^{-1}x\right) \right|^p \frac{ds}{s}$$

$$\le \int\limits_{A(x,q)} |f(t)|^p \frac{dt}{t} + \sup_{s \in \left[1, \frac{q}{x}\right]} s^{\frac{ap}{2}} \int_1^{q/x} \left| s^{-\frac{a}{2}} f(sx) + (-1)^k s^{\frac{a}{2}} f\left(s^{-1}x\right) \right|^p \frac{ds}{s} \tag{3.14}$$

and

$$B = \int\limits_{\substack{1 \le s \le q/x \\ xs \in F}} \left| f\left(xs^{-1}\right) \right|^p \frac{ds}{s} + \int_1^{q/x} \left| s^{-a} f(sx) + (-1)^k f\left(s^{-1}x\right) \right|^p \frac{ds}{s}$$

(3.15)
$$\le \int\limits_{B(x,q)} |f(t)|^p \frac{dt}{t} + \sup_{s \in \left[1, \frac{q}{x}\right]} s^{-\frac{ap}{2}} \int_1^{q/x} \left| s^{-\frac{a}{2}} f(sx) + (-1)^k s^{\frac{a}{2}} f\left(s^{-1}x\right) \right|^p \frac{ds}{s}.$$

Combining relations (3.11) to (3.15) and taking

$$\sup_{s \in \left[1, \frac{q}{x}\right]} s^{\frac{ap}{2}} + \sup_{s \in \left[1, \frac{q}{x}\right]} s^{-\frac{ap}{2}} \le 2^{\frac{|a|p}{2}} + 2^{\frac{|a|p}{2}} = 2^{\frac{|a|p}{2}+1}$$

into account gives

(3.16)
$$\int_1^q |f(t)|^p \frac{dt}{t} \le \int\limits_{C(x,q)} |f(t)|^p \frac{dt}{t} + 2^{\frac{|a|p}{2}+1} \int_1^{q/x} \left| s^{-\frac{a}{2}} f(sx) + (-1)^k s^{\frac{a}{2}} f\left(s^{-1}x\right) \right|^p \frac{ds}{s}$$

where $C(x,q) = \left[x^2/q, q\right] \cap \{t : x^2/t \in F\}$.

If $x \in E$ and if

$$\int_{C(x,q)} |f(t)|^p \frac{dt}{t} \le \frac{1}{2} \int_1^q |f(t)|^p \frac{dt}{t}, \tag{3.17}$$

then by inequalities (3.9) and (3.16), relation (3.10) will follow. Thus it suffices to show that there exists an $x \in \left[1, \sqrt{q}\right]$ for which relation (3.17) holds.

For each $x \in \left[1, \sqrt{q}\right]$, let C_x be the curve $\left\{\left(t, x^2/t\right) : x^2/q \le t \le q\right\}$. Each of these curves lies in the square S with lower left corner $(1/q, 1/q)$ and upper right corner (q,q) and joins a point on the top edge of S to a point on the right edge of S. The curves are all disjoint. The Jacobian of the transformation $u = t, v = x^2/t$ is $2x/t$ and for all q, x, and t as above, $t/(2x) \le q/2 \le 1$, so for any non-negative measurable function $s(u,v)$ we have

$$\int_1^{\sqrt{q}} \left(\int_{x^2/q}^q s\left(t, x^2/t\right) dt \right) dx \le \int_{q^{-1}}^q \int_{q^{-1}}^q s(u,v)\, du dv. \tag{3.18}$$

Since 1 is a point of density of E, if we let χ_F be the characteristic function of the complement of E and define $\epsilon = \epsilon(q)$ by

$$\epsilon = \frac{1}{q-q^{-1}} \int_{q^{-1}}^{q} \chi_F(v)\, dv,$$

then $\epsilon \to 0$ as $q \to 1$. Next let $s(u,v) = \chi_F(v) |f(u)|^p /u$ and compute

$$\int_{q^{-1}}^{q} \int_{q^{-1}}^{q} s(u,v)\, dudv = \epsilon\left(q-q^{-1}\right) \int_{q^{-1}}^{q} |f(u)|^p \frac{du}{u} \tag{3.19}$$

$$\leq 2\epsilon\left(q-q^{-1}\right) \int_{1}^{q} |f(u)|^p \frac{du}{u}.$$

Next observe that

$$\int_{C(x,q)} |f(t)|^p \frac{dt}{t} = \int_{x^2/q}^{q} \chi_F\left(x^2/t\right) |f(t)|^p \frac{dt}{t} = \int_{x^2/q}^{q} s\left(t, x^2/t\right) dt. \tag{3.20}$$

Relations (3.18)-(3.20) lead to

$$\int_{E\cap\left[1,\sqrt{q}\right]} \left\{ \int_{C(x,q)} |f(t)|^p \frac{dt}{t} \right\} dx \leq \int_{1}^{\sqrt{q}} \left\{ \int_{C(x,q)} |f(t)|^p \frac{dt}{t} \right\} dx$$

$$\leq 2\epsilon\left(q-q^{-1}\right) \int_{1}^{q} |f(u)|^p \frac{du}{u}.$$

So there must be an $x \in E \cap \left[1, \sqrt{q}\right]$ for which

$$|E \cap [1, \sqrt{q}]| \int_{C(x,q)} |f(t)|^p \frac{dt}{t} \leq 2\epsilon\left(q-q^{-1}\right) \int_{1}^{q} |f(u)|^p \frac{du}{u},$$

and, because of the inequality

$$\begin{aligned} |E \cap [1, \sqrt{q}]| &= |[1, \sqrt{q}]| - |F \cap [1, \sqrt{q}]| \\ &\geq (\sqrt{q} - 1) - \left(q - q^{-1}\right)\epsilon \\ &= \left(q-q^{-1}\right)\left(\frac{q}{\left(\sqrt{q}+1\right)(q+1)} - \epsilon\right), \end{aligned}$$

there holds for this x the inequality

$$\int_{C(x,q)} |f(t)|^p \frac{dt}{t} \leq \left\{ \frac{2\epsilon}{\frac{q}{\left(\sqrt{q}+1\right)(q+1)} - \epsilon} \right\} \int_{1}^{q} |f(u)|^p \frac{du}{u}.$$

As $q \searrow 1$, $\epsilon \to 0$, so the quantity in curly brackets is less than $1/2$ for q sufficiently close to 1. This establishes relation (3.17) and, consequently, completes the proof of the lemma. $\square$

We turn now to proving the converse part of Theorem 1.

Proof. Our hypothesis is that at every point x of a measurable set E,

$$\left\| \frac{t^{-\frac{a}{2}} f(tx) + (-1)^k t^{\frac{a}{2}} f\left(t^{-1}x\right)}{2_{ak}} - P\left(t^{1/2} - t^{-1/2}\right) \right\|_p (q) = O\left((q-1)^k\right)$$

where P is a polynomial of degree $k-2$ when $k \geq 2$, and $P = 0$ when $k = 1$. Since 2_{ak} is bounded we rewrite this as

$$\left\| t^{-\frac{a}{2}} f(tx) + (-1)^k t^{\frac{a}{2}} f\left(t^{-1}x\right) - 2_{ak} P\left(t^{1/2} - t^{-1/2}\right) \right\|_p (q) = O\left((q-1)^k\right).$$

The polynomial P has only terms of odd (resp. even) order if k is odd (resp. even). Holder's inequality implies the existence of a constant C_p so that for q close to 1,

$$\left| \frac{1}{q-1} \int_1^q g(t) \frac{dt}{t} \right| \leq C_p \|g(t)\|_p (q),$$

so from our original hypothesis it follows that

$$\int_1^q t^{-\frac{a}{2}} f(tx) \frac{dt}{t} + (-1)^k \int_1^q t^{\frac{a}{2}} f\left(t^{-1}x\right) \frac{dt}{t} - \int_1^q P\left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right) 2_{ak} \frac{dt}{t} = O\left(d^{k+1}\right),$$

where $d = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$. Now let

$$G(x) = G(x,a) := \int_1^x f(t) \frac{dt}{t^{a/2+1}}.$$

The substitution $u = tx$ transforms the first integral into $G(qx) - G(x)$ and the substitution $u = t^{-1}x$ transforms the second into $G(x) - G\left(q^{-1}x\right)$. So we have the relation

$$G(qx) + (-1)^{k+1} G\left(q^{-1}x\right) = \left[1 - (-1)^k\right] G(x) + \int_1^q P\left(t^{1/2} - t^{-1/2}\right) 2_{ak} \frac{dt}{t} + O\left(d^{k+1}\right). \tag{3.21}$$

The integral may be taken to be a polynomial Q of degree $k-1$ in d of parity opposite to P (Lemma 6). In other words, G is $k+1$ ($a = 0$) symmetric quantum bounded in L^∞ and so by Corollary 2 has a $k+1$ L^∞ Peano derivative a.e. on E. Integration by parts shows that $F(x) := \int_1^x f(t)\,dt$, the indefinite integral of f, also has $k+1$ L^∞ Peano derivatives a.e. on E (Lemma 7).

The final step is to modify an argument of Mary Weiss in a routine way to conclude that f has a.e. one fewer L^p Peano derivatives than its indefinite integral F has L^∞ ones. From the original hypothesis of

$$\frac{1}{q-1} \int_1^q \left| \frac{t^{-\frac{a}{2}} f(tx) + (-1)^k t^{\frac{a}{2}} f\left(t^{-1}x\right)}{2_{ak}} - P(t) \right|^p \frac{dt}{t} = O\left((q-1)^{kp}\right)$$

for every $x \in E$, we wish to conclude that for a.e. $x \in E$, there is a polynomial $Q(t)$ such that

$$\frac{1}{h} \int_0^h |f(x+t) - f(x) - Q(t)|^p \, dt = o\left(h^{kp}\right).$$

As we showed above, we may assume that F, the indefinite integral of f has $k+1$ Peano derivatives on E. By a theorem of Marcinkiewicz [Zygmund, Trig. Series, vol. II, p. 73], there is a perfect set $\Pi \subset E$, with measure arbitrarily close to that of E and functions G and L so that $F = G + L$, $G \in C^{k+1}$ and $L(x) = 0$ when $x \in \Pi$. Since F has $k+1$ Peano derivatives in Π, F is differentiable there, and hence L is also. Since Π is perfect, $L'(x) = 0$ when $x \in \Pi$. So writing $G'(x) = g(x)$ and $L'(x) = \ell(x)$, we have

$$f(x) = g(x) + \ell(x), \tag{3.22}$$

valid in the set where $F'(x)$ exists and equals $f(x)$ (and so a.e.). Here $g(x) \in C^k$ and $l(x) = 0$ for $x \in \Pi$. Since $g \in C^k$, the estimate

$$\frac{1}{q-1}\int_1^q \left| t^{-\frac{a}{2}} g(tx) + (-1)^k t^{\frac{a}{2}} g\left(t^{-1}x\right) - 2_{ak} P(t,f) \right|^p \frac{dt}{t} = O\left((q-1)^{kp}\right)$$

holds everywhere, and by the decomposition (3.22), on Π we also have

$$\int_1^q \left| t^{-\frac{a}{2}} \ell(tx) + (-1)^k t^{\frac{a}{2}} \ell\left(t^{-1}x\right) \right|^p \frac{dt}{t} = O\left((q-1)^{kp+1}\right).$$

Applying Lemma 8, we also have for $x \in \Pi$ that

$$\int_1^{1+h} |\ell(tx)|^p \frac{dt}{t} = O\left(h^{kp+1}\right).$$

Since t is close to 1, this implies

$$\int_1^{1+h} |\ell(tx)|^p \, dt = O\left(h^{kp+1}\right).$$

Letting $u = tx - x$ and $H = hx$

$$\int_0^H |\ell(x+u)|^p \, du = O\left(H^{kp+1}\right),$$

which, by a theorem of Calderón and Zygmund, implies that ℓ has a kth Peano L^p derivative a.e. in Π. Since both g and ℓ do, so does f. $\square$

References

[A] J. M. Ash, *Generalizations of the Riemann derivative,* Trans. Amer. Math. Assoc., **126**(1967), 181–199. MR0204583 (34:4422)

[A1] J. M. Ash, *Symmetric and quantum symmetric derivatives of Lipschitz functions,* J. Math. Anal. Appl., **288**(2003), 717-721. MR2020192 (2004j:26006)

[A2] J. M. Ash, *An L^p differentiable non-differentiable function,* Real Analysis Exchange, **30** (2004/05), no. 2, 747–754. MR2177431 (2006g:26012)

[ACR] J. M. Ash, S. Catoiu, and R. Ríos-Collantes-de-Terán, *On the nth quantum derivative,* J. London Math. Soc., **66**(2002), 114-130. MR1911224 (2003h:26009)

[AJ] J. M. Ash and R. Jones, *Optimal numerical differentiation using three function evaluations,* Math. Comp., **37** (1981), 159-167. MR0616368 (84a:65008)

[AJJ] J. M. Ash, S. Jansen and R. Jones, *Optimal numerical differentiation using n function evaluations , Estratto da Calcolo,* **21**(1984), 151-169. MR0799618 (86k:65017)

[CZ] A. P. Calderón and A. Zygmund, Local properties of solutions of elliptic partial differential equations, Studia Math., **20**(1961), 171–225. MR0136849 (25:310)

[GR] G. Gasper and M. Rahman, Basic hypergeometric series. Encyclopedia of Mathematics and its Applications, 96. Cambridge Univ. Press, Cambridge, 2004. MR2128719 (2006d:33028)

[MZ] J. Marcinkiewicz and A. Zygmund, *On the differentiability of functions and summability of trigonometric series,* Fund. Math. **26**(1936), 1–43.

[R] R. Ríos-Collantes-de-Terán, *Conjuntos de unicidad de sistemas de funciones independientes. Quantum derivadas.,* Thesis, Departamento de Análisis Matemático de la Universidad de Sevilla, 2001.

[W] M. Weiss, *On symmetric derivatives in L^p,* Studia Math., **24**(1964), 89–100. MR0162094 (28:5295)

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