International Mathematical Forum, Vol. 7, 2012, no. 4, 153 - 166

On the Boundary Behavior of Special Classes of $C^{\infty}-$Functions and Analytic Functions

J. M. Ash

Department of Mathematics, De Paul University
Chicago, IL 60614-3504, USA
mash@math.depaul.edu

M. T. Karaev

Suleyman Demirel University
Isparta (MYO) Vocation School
Department of Technical Programs
32260 Isparta, Turkey
garayev@fef.sdu.edu.tr

Abstract. We give a concrete example of a diagonal operator acting in the Hardy space $H^2(\mathbb{D})$ for which the Berezin symbol has radial limits at no point of the boundary $\partial\mathbb{D}$. We use the Berezin symbol technique in the discussion of several old problems from the classical book of I.I. Privalov [18] related with the Taylor coefficients and boundary behavior of analytic functions. In particular, we give in terms of Taylor coefficients $\left\{\widehat{f}(n)\right\}$ and Berezin symbols necessary and sufficient conditions ensuring existence of radial boundary values of the functions $f(z) = \sum_{n\geq 0} \widehat{f}(n) z^n$from the classes $l_A^p(\mathbb{D})$, $0 < p \leq \infty$.Lohwater and Piranian type example of a function analytic on the unit disc $\mathbb{D}$ which has radial limits at no point of the boundary$\partial\mathbb{D}$ is also presented. The proof depends on the "high-indices" Tauberian theorem of Hardy and Lilttlewood which states that Abel summability of a lacunary trigonometric series at a point implies convergence of that series. Some other questions are also discussed.

Mathematics Subject Classification: Primary 32 D15, 47B47; Secondary 44A15, 42A55

Keywords: Berezin symbol, analytic function, radial limits, diagonal operator, Abel summability

1. Introduction

Let $\mathbb{C}$ be the complex plane and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in $\mathbb{C}$. Let $\mathbb{T}$ denote the boundary of D, so $\mathbb{T} = \{e^{it} : t \in [0, 2\pi)\}$. A functional Hilbert space is a Hilbert space $\mathcal{H} = \mathcal{H}(\mathbb{D})$ of complex-valued functions on a disk $\mathbb{D}$ which has the property that point evaluations are continuous (i.e., for each $\lambda \in \mathbb{D}$, the map $f \to f(\lambda)$ is a continuous linear functional on $\mathcal{H}$). Then the Riesz representation theorem ensures that for each $\lambda \in \mathbb{D}$ there is a unique element k_λ of $\mathcal{H}$ such that $f(\lambda) = \langle f, k_\lambda \rangle$ for all $f \in \mathcal{H}$,where $\langle \cdot, \cdot \rangle$ denotes an inner product in $\mathcal{H}$. Because k_λ reproduces the value of functions in $\mathcal{H}$ at λ, it is called the reproducing kernel. The normalized reproducing kernel $\widehat{k}_\lambda$ is defined by $\widehat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|}$. It is well known (see, for instance , Halmos [5, Problem 37]) that if $\{e_n\}$ is an orthonormal basis for a functional Hilbert space $\mathcal{H}$, then the reproducing kernel of $\mathcal{H}$ is given by $k_\lambda(z) = \sum_n \overline{e_n(\lambda)} e_n(z)$.

For A a bounded linear operator on $\mathcal{H}$, the Berezin symbol of A, denoted $\widetilde{A}$, is the complex-valued function on $\mathbb{D}$ defined by

$$\widetilde{A}(\lambda) = \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle.$$

The range of the Berezin symbol $\widetilde{A}$ is called the Berezin set of the operator A, and will be denoted as $Ber(A)$. The Berezin number $berA$ of A is defined as $ber(A) = \sup\{|\lambda| : \lambda \in Ber(A)\}$(see, also [11],[12]). Clearly, $Ber(A) \subset W(A)$, where $W(A)$ is the numerical range of operator A defined as $W(A) = \{\langle Af, f\rangle : \|f\|_{\mathcal{H}} = 1\}$. The Berezin symbol associates smooth functions with operators on functional Hilbert spaces $H = H(\mathbb{D})$ of analytic functions in $\mathbb{D}$, namely, for each bounded operator A on H, the Berezin symbol $\widetilde{A}$ is a bounded real-analytic function on $\mathbb{D}$. This implies that $\widetilde{A}$ is infinitely differentiable on $\mathbb{D}$. The boundary behavior of the Berezin symbol $\widetilde{A}$ can be very irregular. For example, it is not known if the Berezin symbol of a bounded operator on H must have radial limits almost everywhere on the unit circle $\mathbb{T}$. For $H = L_a^2(\mathbb{D})$ (Bergman space, $L_a^2(\mathbb{D})$, consists of the analytic functions f on $\mathbb{D}$ such that $\int_{\mathbb{D}} |f|^2 dA < \infty$, where dA denotes area measure, normalized so that the area of $\mathbb{D}$ equals 1), this problem was formulated by Zorboska in [23]. In [13] Karaev solved Zorboska's problem in the negative, showing the existence of a concrete class of diagonal operators on $L_a^2(\mathbb{D})$ for which the Berezin symbol does not have radial boundary values anywhere on the unit circle $\mathbb{T}$. A similar result is also obtained in the case of Hardy space $H^2(\mathbb{D})$ in reference [13] and also in a paper of Englis [4]. In Section 2 of the present article, we give a more concrete example of a diagonal operator acting on the Hardy space $H^2(\mathbb{D})$ for which the Berezin symbol has radial limits at no point of the boundary $\mathbb{T}$ (see Theorem 1). In Section 3, we use the Berezin symbols technique to give a partial solution to a problem posed in the classical book of Privalov [18, Chapter II,§10,11] (see also Duren [3, Chapter 6]). That problem is to relate Taylor coefficients and boundary behavior of analytic functions. In particular, we give in terms of Taylor coefficients $\left\{\widehat{f}(n)\right\}$ and Berezin symbols necessary and sufficient conditions ensuring existence of radial boundary values of the functions $f(z) = \sum_{n\geq 0} \widehat{f}(n) z^n$ from the classes $l_A^p := l_A^p(\mathbb{D}), 0 < p \leq \infty$, which consist of the analytic functions $f(z) = \sum_{n\geq 0} \widehat{f}(n) z^n$ on $\mathbb{D}$ with $\left\{\widehat{f}(n)\right\} \in l^p$; the norm in l_A^p being defined by $\|f\|_{l_A^p} = \left\|\left\{\widehat{f}(n)\right\}\right\|_{l^p}$. It is easy to see that $l_A^p(\mathbb{D}) \subset H^p(\mathbb{D}) \subset l_A^{\infty}(\mathbb{D})$. Moreover, we describe the z-invariant subspaces E of l_A^p in terms of Berezin symbols of weighted shift operator associated with the Taylor coefficients of the functions in E. We also show that the Berezin set provides information about the range of the function $(1-|z|) f$ with $f \in l_A^{\infty}$ (see Section 4).

In Section 5, an example of a function analytic on the unit disc $\mathbb{D}$ which has radial limits at no point of the boundary $\mathbb{T}$ is presented. The proof depends on the "high-indices" Tauberian theorem of Hardy and Littlewood [7], which states that Abel summability of a lacunary trigonometric series at a point implies convergence of that series. Notice that since almost nowhere does not mean nowhere, our example has strictly worse behavior than the bad behavior proven to occur on pages 148-149 of Privalov's book [18].

2. A Berezin symbol without radial limits

For $1 \leq p < \infty$, the Hardy class $H^p = H^p(\mathbb{D})$ is the set of all functions f analytic on $\mathbb{D}$ such that

$$\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^p d\theta < \infty. \tag{1}$$

The p-th root of the left hand side of inequality (1) defines a complete norm on H^p. For more information on these spaces, see [18], [3], and [9]. In the case of $p = 2$, H^2 is the familiar Hardy space of all functions analytic on $\mathbb{D}$ with square-summable Taylor series coefficients.

Recall that the sequence $\{a_n\}_{n=0}^{\infty}$ of complex numbers a_n is Abel convergent to a if the limit

$$\lim_{t\to 1^-} (1-t) \sum_{n=0}^{\infty} a_n t^n$$

exists and is equal to a.

The following result gives a counterexample to the problem formulated above in Section 1.

Theorem 1. Let $a_n := n^{-ic}$, where $c \in \mathbb{R}\setminus\{0\}$, and let $D_{\{a_n\}}$ be a diagonal operator with diagonal elements a_n with respect to the standard orthonormal basis $\{z^n\}_{n\geq 0}$ of the Hardy space H^2. Then the Berezin symbol $\widetilde{D}_{\{a_n\}}$ of the operator $D_{\{a_n\}}$ has radial limits at no point of the boundary $\mathbb{T}$.

Proof. Let us denote

$$s_k = \sum_{n=1}^{k} n^{-1-ic}.$$

It is easy to verify that the series $\sum_{n=1}^{\infty} n^{-1-ic}$ is not convergent and $n^{-1-ic} = O\left(\frac{1}{n}\right)$. Then it follows from the Littlewood Tauberian theorem that $\sum_{n=1}^{\infty} n^{-1-ic}$ is not Abel convergent, that is $\{s_k\}$ is not an Abel convergent sequence. On the other hand, it can be showed that (see Hardy[6, p.163]) $s_k + \frac{a_k}{ic}$ tends to a finite limit as k tends to infinity. It follows from this that $\{a_n\}$ cannot be an Abel convergent sequence, for if it were we would get the contradiction that $\{s_k\}$ is Abel convergent. Clearly, $\{a_n\}_{n\geq 0}$ (we put $a_0 := 0$) is a bounded sequence, and therefore the diagonal operator $D_{\{a_n\}}$ is bounded in H^2. Then we have (see[14]):

$$\begin{aligned}\widetilde{D}_{\{a_n\}}(\lambda) &= \left\langle D_{\{a_n\}}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle = \left(1-|\lambda|^2\right)\left\langle D_{\{a_n\}}\sum_{n=0}^{\infty}\overline{\lambda}^n z^n, \frac{1}{1-\overline{\lambda}z}\right\rangle \\ &= \left(1-|\lambda|^2\right)\left\langle \sum_{n=0}^{\infty}\overline{\lambda}^n a_n z^n, \frac{1}{1-\overline{\lambda}z}\right\rangle = \left(1-|\lambda|^2\right)\sum_{n=0}^{\infty} a_n |\lambda|^{2n},\end{aligned}$$

where $\widehat{k}_\lambda = \left(1-|\lambda|^2\right)\left(1-\overline{\lambda}z\right)^{-1}$ is the normalized reproducing kernel of H^2. Thus,

$$\widetilde{D}_{\{a_n\}}(\lambda) = \widetilde{D}_{\{a_n\}}\left(|\lambda|^2\right)$$

(i.e., $\widetilde{D}_{\{a_n\}}$ is a radial function), and

$$\widetilde{D}_{\{a_n\}}\left(|\lambda|^2\right) = \left(1-|\lambda|^2\right)\sum_{n=0}^{\infty} a_n |\lambda|^{2n},\ \lambda \in \mathbb{D}, \tag{3}$$

or

$$\widetilde{D}_{\{a_n\}}(t) = (1-t)\sum_{n=0}^{\infty} a_n t^n, 0 \leq t < 1, \tag{4}$$

where $t = |\lambda|^2$. Since $\{a_n\} = \{n^{-ic}\}$ is not an Abel convergent sequence, it follows from (2)–(4) that the Berezin symbol $\widetilde{D}_{\{a_n\}}$ has no radial limits anywhere on the unit circle $\mathbb{T}$. This completes the proof.

3. Berezin symbols and boundary behavior of analytic functions from the class $l_A^{\infty}(\mathbb{D})$

Following Privalov [18] and Duren [3], we consider a function $f(z) = \sum_{n\geq 0} \widehat{f}(n) z^n$ belonging to a certain H^p space and ask what can be said about its Taylor coefficients $\left\{\widehat{f}(n)\right\}$?

It is also interesting to ask how an H^p function can be recognized by the behavior of its Taylor coefficients. Ideally, one would like to find a condition on the $\widehat{f}(n)$ which is both necessary and sufficient for f to be in H^p. For $p = 2$, of course, the problem is completely solved: $f \in H^2$ if and only if $\sum_{n=0}^{\infty} \left|\widehat{f}(n)\right|^2 < \infty$. For $p = \infty$, the problem of coefficients was solved by I. Schur in 1919 (see, [18, Chapter 2]). Some classical results about the Taylor coefficients of functions in the Hardy space $H^p$and Bergman spaces $L_a^p(\mathbb{D})$ are also known (see, for instance [18], [3], [8], [22], [20]). Some recent results about Taylor coefficients of entire functions in the Fock spaces F_{α}^p have been obtained by Tung [20]. But the general situation is much more complicated, and no complete answer is available. This section contains some scattered information in terms of Berezin symbols about analytic functions f with bounded Taylor coefficients $\left\{\widehat{f}(n)\right\}$. We will show that the boundary behavior of the Berezin symbols of diagonal operators on the Hardy space H^2 can be also used in the study of radial boundary values of analytic functions on the unit disc $\mathbb{D}$. Our main result in this section is the following.

Theorem 2. If $f(z) = \sum_{n\geq 0} \widehat{f}(n) z^n \in l_A^{\infty}(\mathbb{D})$, then f has radial limits almost everywhere on the unit circle $\mathbb{T}$ if and only if

$$\widetilde{D}_{\left\{\widehat{f}(n)e^{in\theta}\right\}}\left(\sqrt{t}\right) = O(1-t) \text{ as } t \to 1^{-}$$

for almost all $\theta \in [0, 2\pi)$.

Proof. Let $f \in l_A^{\infty}(\mathbb{D})$ be an analytic function in $\mathbb{D}$ with sequence of Taylor coefficients $\left\{\widehat{f}(n)\right\}$. Let us denote, as usual, $r = |\lambda|$ and $\theta = \arg(\lambda)$ for any $\lambda \in \mathbb{D}$. Then $f(\lambda) = \sum_{n=0}^{\infty} \widehat{f}(n)\lambda^n$. Write $\lambda = re^{i\theta}$, and regard $\theta \in [0, 2\pi)$ as fixed. The Taylor series then becomes a power series on $[0, 1)$, of the form $\sum_{n=0}^{\infty} \widehat{f}(n) e^{in\theta} r^n$. Let $D_{\left\{\widehat{f}(n)e^{in\theta}\right\}}$ be a diagonal operator defined by

$$D_{\left\{\widehat{f}(n)e^{in\theta}\right\}} z^n = \widehat{f}(n) e^{in\theta} z^n, \ n \geq 0,$$

on the Hardy space $H^2(\mathbb{D})$. Since $\left\{\widehat{f}(n)\right\}$ is a bounded sequence (because $f \in l_A^{\infty}(\mathbb{D})$) and θ is fixed number, the operator $D_{\left\{\widehat{f}(n)e^{in\theta}\right\}}$ is bounded. Then by using the proof of formula (4), we have :

$$f(\lambda) = f\left(re^{i\arg(\lambda)}\right) = f\left(re^{i\theta}\right)$$
$$= \sum_{n=0}^{\infty} \widehat{f}(n)\, e^{in\theta} r^n = \frac{(1-r)\sum_{n=0}^{\infty} \widehat{f}(n)\, e^{in\theta} r^n}{1-r}$$
$$= \frac{\widetilde{D}_{\{\widehat{f}(n)e^{in\theta}\}}\left(\sqrt{r}\right)}{1-r}.$$

Thus

$$f\left(re^{i\theta}\right) = \frac{\widetilde{D}_{\{\widehat{f}(n)e^{in\theta}\}}\left(\sqrt{r}\right)}{1-r}, \tag{5}$$

where $\widetilde{D}_{\{\widehat{f}(n)e^{in\theta}\}}$ is the Berezin symbol of a bounded diagonal operator $D_{\{\widehat{f}(n)e^{in\theta}\}}$. Since $r \in [0,1)$ and $\theta \in [0, 2\pi)$ are arbitrary fixed numbers, formula (5) implies that

$$\lim_{r\to 1^-} f\left(re^{i\theta}\right) = \lim_{r\to 1^-} \frac{\widetilde{D}_{\{\widehat{f}(n)e^{in\theta}\}}\left(\sqrt{r}\right)}{1-r},$$

that is f has radial limits almost everywhere on the unit circle $\mathbb{T}$ if and only if

$$\widetilde{D}_{\{\widehat{f}(n)e^{in\theta}\}}\left(\sqrt{r}\right) = O(1-r) \text{ as } r \to 1^-$$

for almost all $\theta \in [0, 2\pi)$, which proves the theorem.

Remark 1. Note that Theorem 2 can be considered as one possible particular answer to the classical problem of Privalov in [18, Chapter 2, §10], where it is required to give in terms of Taylor coefficients $\left\{\widehat{f}(n)\right\}$ necessary and sufficient conditions ensuring existence of angular boundary values of the analytic function $f(z) = \sum_{n\geq 0} \widehat{f}(n)\, z^n$ almost everywhere on the unit circle $\mathbb{T}$. It is necessary, of course, to note also that there are many sufficient conditions in terms of Taylor coefficients ensuring existence of radial and angular limits of analytic functions on the unit disk $\mathbb{D}$ (see [18], [3], and also [20] and its references)

The following results are immediate from formula (5).

Corollary 3. Let $f \in l_A^\infty$. Then $f \in H^p$ $(0 < p < \infty)$ if and only if

$$\sup_{0<r<1} \frac{1}{(1-r)^p} \frac{1}{2\pi} \int_0^{2\pi} \left|\widetilde{D}_{\{\widehat{f}(n)e^{in\theta}\}}\left(\sqrt{r}\right)\right|^p d\theta < \infty.$$

For $p = \infty$, $f \in H^\infty$ if and only if

$$\sup_{\substack{0<r<1 \\ \theta\in[0,2\pi)}} \frac{\left|\widetilde{D}_{\{\widehat{f}(n)e^{in\theta}\}}\left(\sqrt{r}\right)\right|}{1-r} < \infty.$$

Corollary 4. Let $f \in l_A^{\infty}$. Then

$$\sup_{z\in\mathbb{D}}(1-|z|)\,|f(z)| \leq \sup_{z\in\mathbb{D}} ber\left(D_{\left\{\widehat{f}(n)e^{in\arg(z)}\right\}}\right), \tag{6}$$

where $ber\left(D_{\left\{\widehat{f}(n)e^{in\arg(z)}\right\}}\right)$ is the Berezin number of the operator $D_{\left\{\widehat{f}(n)e^{in\arg(z)}\right\}}$.

Note that, since

$$\begin{aligned} ber\left(D_{\left\{\widehat{f}(n)e^{in\arg(z)}\right\}}\right) &\leq w\left(D_{\left\{\widehat{f}(n)e^{in\arg(z)}\right\}}\right)\ (\text{ numerical radius}) \\ &\leq \left\|D_{\left\{\widehat{f}(n)e^{in\arg(z)}\right\}}\right\| \\ &= \sup_{n\geq 0}\left|\widehat{f}(n)\,e^{in\arg(z)}\right| = \sup_{n\geq 0}\left|\widehat{f}(n)\right| \end{aligned}$$

for all $z \in \mathbb{D}$, inequality (6) is better than the well-known and obvious inequality

$$(1-|z|)\,|f(z)| \leq \sup_{n\geq 0}\left|\widehat{f}(n)\right|. \tag{7}$$

L'Hospital's rule and a property of compact operators together with formula (5) give the proofs (which are omitted) of the following two results.

Corollary 5. Let $e^{i\theta} \in \mathbb{T}$ be a fixed point. Let f be an analytic function on $\mathbb{D}$ such that

(a) $\widehat{f}(n) \to 0\ (n \to 0)$;

(b) the sequence $\left\{n\widehat{f}(n)\,e^{in\theta}\right\}$ is Abel convergent to zero.

Then $\lim_{r\to 1^-} f\left(re^{i\theta}\right)$ exists and

$$\lim_{r\to 1^-} f\left(re^{i\theta}\right) = -\frac{1}{2}\lim_{r\to 1^-} \widetilde{D}'_{\left\{\widehat{f}(n)e^{in\theta}\right\}}\left(\sqrt{r}\right).$$

Corollary 6. Let $f \in l_A^p$ $(0 < p < \infty)$ be a function such that $\left\{n\widehat{f}(n)\,e^{in\theta}\right\}$ is Abel convergent to zero for almost all $\theta \in [0, 2\pi)$, and let $\varphi\left(e^{i\theta}\right)$ be a measurable function on the unit circle $\mathbb{T}$. Then $f\left(e^{i\theta}\right) = \varphi\left(e^{i\theta}\right)$ for almost all $\theta \in [0, 2\pi)$ if and only if

$$\varphi\left(e^{i\theta}\right) = -\frac{1}{2}\lim_{r\to 1^-} \widetilde{D}'_{\left\{\widehat{f}(n)e^{in\theta}\right\}}\left(\sqrt{r}\right)$$

for almost all $\theta \in [0, 2\pi)$.

4. Other applications of diagonal operators

For any $f \in l_A^p$ $(0 < p \leq \infty)$ and $\lambda \in \mathbb{D}$ we have

$$\begin{aligned}
\lambda f(\lambda) &= \lambda \sum_{n\geq 0} \widehat{f}(n)\lambda^n = \sum_{n\geq 0} \widehat{f}(n)\lambda^{n+1} \\
&= \sum_{n\geq 1} \widehat{f}(n-1)\lambda^n = \sum_{n\geq 1} \widehat{f}(n-1)e^{in\arg(\lambda)}|\lambda|^n \\
&= \frac{(1-|\lambda|)\sum_{n\geq 0}\widehat{f}(n-1)e^{in\arg(\lambda)}|\lambda|^n}{1-|\lambda|} \\
&= \frac{\widetilde{D}_{\left\{\widehat{f}(n-1)e^{in\arg(\lambda)}\right\}}\left(\sqrt{|\lambda|}\right)}{1-|\lambda|},
\end{aligned}$$

Thus

$$\lambda f(\lambda) = \frac{\widetilde{D}_{\left\{\widehat{f}(n-1)e^{in\arg(\lambda)}\right\}}\left(\sqrt{|\lambda|}\right)}{1-|\lambda|} \quad (\forall \lambda \in \mathbb{D}). \tag{8}$$

Our next result is immediate from formula (8).

Corollary 7. Let $E \subset l_A^p$ $(0 < p < \infty)$ be a closed subspace. Then $zE \subset E$ (i.e., E is invariant subspace for the shift operator) if and only if

$$\frac{\widetilde{D}_{\left\{\widehat{f}(n-1)e^{in\arg(z)}\right\}}\left(\sqrt{|z|}\right)}{1-|z|} \in E \text{ for all } f \in E.$$

If $g \in l_A^p$, $0 < p < \infty$, and $f \in l_A^\infty$, then it is standard to show that

$$|g(z)| \leq (1-|z|^q)^{-1/q}\|g\|_{l_A^p},$$

where $\frac{1}{q} + \frac{1}{p} = 1$, and(see formula (7))

$$(1-|z|)|f(z)| \leq \|f\|_{l_A^\infty} \tag{9}$$

for all $z \in \mathbb{D}$. Formula (9) implies that the values of the function $(1-|z|)f$ are contained in the closed disc $\overline{\mathbb{D}}_d$, where $d = \|f\|_{l_A^\infty}$. The following proposition represents the range of the function $(1-|z|)f$ in terms of Berezin sets of diagonal operators associated with the Taylor coefficients $\left\{\widehat{f}(n)\right\}$ of the function f.

Proposition 8. If $f \in l_A^\infty(\mathbb{D})$, then

$$Range\,(1-|z|)f = \bigcup_{\theta\in[0,2\pi)} Ber\left(D_{\left\{\widehat{f}(n)e^{in\theta}\right\}}\right). \tag{10}$$

Proof. For any $\theta \in [0, 2\pi)$, let $[0, e^{i\theta})$ denote the line segment with the ends 0 and $e^{i\theta}$. Since for any $z \in \mathbb{D}$, $z \in [0, e^{i\arg(z)})$, it is easy to see that

$$\mathbb{D} = \bigcup_{\theta\in[0,2\pi)} [0, e^{i\theta}). \tag{11}$$

Now, let $\theta \in [0, 2\pi)$ be any fixed number. Then for any $z \in [0, e^{i\theta})$ we have

$$(1-|z|)\, f(z) = \widetilde{D}_{\{\widehat{f}(n)e^{in\theta}\}}\left(\sqrt{|z|}\right),$$

which implies that $(1-|z|)\, f(z) \in Ber\left(D_{\{\widehat{f}(n)e^{in\theta}\}}\right)$, and therefore $(1-|z|)\, f(z) \in \underset{t\in[0,2\pi)}{\cup} Ber\left(D_{\{\widehat{f}(n)e^{int}\}}\right)$. This inclusion together with (5) and (11) shows that

$$\{(1-|z|)\, f(z) : z \in \mathbb{D}\} = \underset{t\in[0,2\pi)}{\cup} Ber\left(D_{\{\widehat{f}(n)e^{int}\}}\right),$$

which proves the proposition.

Remark 2. Since for any $t \in [0, 2\pi)$, $D_{\{\widehat{f}(n)e^{int}\}}$ is a normal operator on H^2, we have $\sigma\left(D_{\{\widehat{f}(n)e^{int}\}}\right) = clos\left\{\widehat{f}(n)\, e^{int} : n \geq 0\right\}$ and

$$\overline{W}\left(D_{\{\widehat{f}(n)e^{int}\}}\right) = conv\ \sigma\left(D_{\{\widehat{f}(n)e^{int}\}}\right).$$

Then,by considering the obvious inclusion

$$Ber\left(D_{\{\widehat{f}(n)e^{int}\}}\right) \subset W\left(D_{\{\widehat{f}(n)e^{int}\}}\right),$$

we have that

$$Range\,((1-|z|)\, f) \subseteq \underset{t\in[0,2\pi)}{\cup} conv\ clos\left\{\widehat{f}(n)\, e^{int} : n \geq 0\right\}.$$

Proposition 9. Let $f \in l^\infty_A$. For any $t \in [0, 2\pi)$, let $T_{\{\widehat{f}(n)e^{int}\}}$ be a weighted shift operator acting on the Hardy space H^2 by the formula

$$T_{\{\widehat{f}(n)e^{int}\}} z^n = \widehat{f}(n)\, e^{int} z^{n+1}, n \geq 0.$$

Then

$$Range\,((1-|z|)\, zf) \subset W\left(T_{\{\widehat{f}(n)\}}\right).$$

Proof. It is easy to verify that

$$\lambda f(\lambda) = \frac{\widetilde{T}_{\{\widehat{f}(n)e^{in\arg(\lambda)}\}}\left(\sqrt{|\lambda|}\right)}{1-|\lambda|}\ (\forall \lambda \in \mathbb{D}). \tag{12}$$

From this

$$(1-|\lambda|)\, \lambda f(\lambda) = \widetilde{T}_{\{\widehat{f}(n)e^{in\arg(\lambda)}\}}\left(\sqrt{|\lambda|}\right)\ (\forall \lambda \in \mathbb{D}). \tag{13}$$

Since $\left|\widehat{f}(n)\, e^{in\arg(\lambda)}\right| = \left|\widehat{f}(n)\right|$, for every fixed $\lambda \in \mathbb{D}$ there exists a unitary diagonal operator $D_{\{\delta_{n,\lambda}\}}$ on H^2 such that

$$D^{-1}_{\{\delta_{n,\lambda}\}} T_{\{\widehat{f}(n)\}} D_{\{\delta_{n,\lambda}\}} = T_{\{\widehat{f}(n)e^{in\arg(\lambda)}\}}.$$

Then, by considering that

$$Ber\left(T_{\left\{\widehat{f}(n)e^{in\arg(\lambda)}\right\}}\right) \subset W\left(T_{\left\{\widehat{f}(n)e^{in\arg(\lambda)}\right\}}\right)$$

and

$$W\left(T_{\left\{\widehat{f}(n)e^{in\arg(\lambda)}\right\}}\right) = W\left(T_{\left\{\widehat{f}(n)\right\}}\right),$$

we have from the equality (13) that

$$(1-|\lambda|)\,\lambda f(\lambda) = (D^{-1}_{\{\delta_{n,\lambda}\}} T_{\{\widehat{f}(n)\}} D_{\{\delta_{n,\lambda}\}})^{\widetilde{}}\left(\sqrt{|\lambda|}\right)$$
$$\in Ber\left(D^{-1}_{\{\delta_{n,\lambda}\}} T_{\{\widehat{f}(n)\}} D_{\{\delta_{n,\lambda}\}}\right) \subset W\left(D^{-1}_{\{\delta_{n,\lambda}\}} T_{\{\widehat{f}(n)\}} D_{\{\delta_{n,\lambda}\}}\right)$$
$$= W\left(T_{\{\widehat{f}(n)\}}\right)$$

for all $\lambda \in \mathbb{D}$. Thus, $Range\left((1-|z|)\,zf\right) \subset W\left(T_{\{\widehat{f}(n)\}}\right)$, which completes the proof.

Remark 3. By using formula (12), Corollary 7 can be reformulated as follows: if $E \subset l^p_A$ $(0 < p \leq \infty)$ is a closed subspace, then $zE \subset E$ if and only if

$$\frac{\widetilde{T}_{\left\{\widehat{f}(n)e^{in\arg(z)}\right\}}\left(\sqrt{|z|}\right)}{1-|z|} \in E$$

for all $f \in E$.

5. An analytic function without boundary values

There is a well known principle that very good behavior of a complex valued function of a complex variable on a disc does not necessarily imply even moderately good behavior at the boundary points of that disc. For example, Hadamard (see[19]) has constructed a function analytic on a disc that is not extendable to any neighborhood of any point of the boundary. Another example: K. G. Binmore has displayed a class of functions analytic in the unit disc $|z| < 1 = \{z \in \mathbb{C} : |z| < 1\}$ which cannot be extended continuously to any boundary point P, even if we interpret "extended continuously" in the very weak sense of only requiring that the function restricted to some curve in $|z| < 1$ terminating at P have a finite limiting value as P is approached.[1] We also note that apparently Lohwater and Piranian [15] gave a construction for analytic function which has radial limits at no point of the boundary $\partial\mathbb{D} = \mathbb{T}$. But their construction is not simple (see also Collingwood and Lohwater [2, Chapter 2]). The point of this section is to call the reader's attention to two examples of functions analytic on $\mathbb{D}$ that have radial limit at no point of $\mathbb{T}$. This result is quite a bit weaker than the result of Binmore, but both examples can be understood very quickly. The first example has the advantage of having an extremely short proof, but that proof depends on a powerful Tauberian theorem. The second example requires slightly more work, but the proof is completely self-contained. The very direct and simple second proof was discovered by a referee of an early version of this paper and appears with his permission.

There are many known results about functions analytic on $\mathbb{D}$ and also enjoying further good behavior, which nonetheless fail to have radial limits at almost every point of $\mathbb{T}$. See Theorems 2 and 3 of [16] for two results like this. Comparatively, the present results consider much less good functions since nothing except analyticity on $\mathbb{D}$ is assumed, but the resulting boundary behavior is a little worse since the set of nonexistence of radial limits is guaranteed to be empty rather than just of measure zero.

Theorem 10. The function $f(z) = \sum_{n=0}^{\infty} z^{2^n}$ is analytic on $\mathbb{D}$, but has radial limits at no point of the boundary $\mathbb{T}$. In other words, for each $\theta \in [0, 2\pi)$,

$$\lim_{r\to 1^-} \sum_{n=0}^{\infty} \left(e^{i\theta} r\right)^{2^n} = \lim_{r\to 1^-} \sum_{n=0}^{\infty} e^{i2^n\theta} r^{2^n}$$

does not exist.

Proof. For every positive $r < 1$, the function $f(z) = \sum_{n=0}^{\infty} z^{2^n}$ converges on $|z| \le r$ by the root test and thus f is analytic on $|z| < r$. Since r is arbitrary, f is analytic on all of $\mathbb{D}$.

Fix θ and set $a_n = e^{i2^n\theta}$. Our goal is to show that the limit

$$\lim_{r\to 1^-} \sum_{n=0}^{\infty} a_n r^{2^n} \tag{14}$$

does not exist. It is a fundamental theorem of Abel that the convergence of a series $\sum b_k$ implies the existence of the limit $\lim_{r\to 1^-} \sum b_k r^k$ (see Hardy [6], and also Pati[17])(see also Karaev [14] for a functional analysis proof of Abel's theorem based on the Berezin symbols). The easy proof of this is a simple summation by parts argument. But what we need here is the contrapositive of a limited converse theorem. That limited converse theorem, which is called the "high-indices" Tauberian theorem of Hardy and Littlewood, asserts that the existence of the limit $\lim_{r\to 1^-} \sum \alpha_n r^{2^n}$ implies the convergence of the series $\sum \alpha_n$. This is a much deeper result. The reason that this converse is limited is that $\alpha_n \longleftrightarrow b_{2^n}$ so that this converse only applies to numerical series $\sum b_n$ whose terms are zero when n is not a power of 2. It was proved by Hardy and Littlewood.[7] A simplified proof due to Ingham [10] appears as Theorem 115 on page 173 of [6]. (The statement is more general, the sequence $\{2^n\}$ may be replaced by any lacunary sequence). Such a limited converse theorem, i.e., one which imposes a prior condition on the coefficient sequence $\{b_n\}$, is called a Tauberian theorem.

Thus we need only show that the series $\sum a_n$ is divergent. But the modulus of every term of this series is 1 and a series of complex numbers whose moduli do not tend to zero diverges. Therefore, the limit (14) does not exist. This proves Theorem 10.

This proof is so short, basically it can be rephrased as "apply the high-indices Tauberian Theorem," that what is truly remarkable is that it seems to be missing from all of the standard texts discussing boundary values of functions harmonic on $\mathbb{D}$. It actually appears in the introduction of a multivariate paper of Ullrich.[21] We feel certain that it was solidly embedded in the folklore. Certainly any of the classical analysts of the first half of the twentieth century such as Hardy, Littlewood, Ingham, and Zygmund, could have instantly produced it upon request. We hope that it will not sink from sight again.

Theorem 11. The function $f(z) = \sum_{n=1}^{\infty} z^{n!}$ is analytic on $\mathbb{D}$, but has radial limits at no point of the boundary $\mathbb{T}$.

Proof. The only property of the sequence $\{2^k\}$ used in the proof of Theorem 10 was that $\lim_{k\to\infty} \frac{2^{k+1}}{2^k} > 1$, which is to say that $\{2^k\}$ is lacunary. Since the sequence $\{k!\}$ is also lacunary, the same proof works here also. The point of this example is that it allows us to give a complete and direct proof that works for sequences $\{\lambda_k\}$ satisfying $\lim_{k\to\infty} \frac{\lambda_{k+1}}{\lambda_k} = \infty$. Such sequences might be called super-lacunary.

Fix θ, $0 \le \theta < 2\pi$. If δ_k satisfies $|\delta_k| \le 1$ for all k, and $s_{k-1} = \sum_{j=1}^{k-1} e^{i\theta j!}$, then $\lim_{k\to\infty} \frac{\delta_k}{e} + s_{k-1}$ does not exist. One reason for this is that the difference between two successive terms does not tend to zero, since

$$\left| \frac{\delta_{k+1}}{e} + s_k - \frac{\delta_k}{e} - s_{k-1} \right| = \left| s_k - s_{k-1} + \frac{\delta_{k+1} - \delta_k}{e} \right|$$
$$\ge \left| e^{i\theta k!} \right| - \left| \frac{\delta_{k+1} - \delta_k}{e} \right| \ge 1 - \frac{2}{e} > 0.$$

We will now show that for each k, there is a δ_k satisfying $|\delta_k| \le 1$ so that

$$\lim_{k\to\infty} \left(f\left(\left(1 - 1/k!\right) e^{i\theta} \right) - \left(\frac{\delta_k}{e} + s_{k-1} \right) \right) = 0, \tag{5.1}$$

from which it is immediate that f does not have a radial limit at $e^{i\theta}$. Set $\delta_k = \left(1 - \frac{1}{k!}\right)^{k!} e^{i\theta k!} e$ and write

$$f\left(\left(1 - 1/k!\right) e^{i\theta} \right) = \sum_{j=1}^{k-1} \left(1 - \frac{1}{k!}\right)^{j!} e^{i\theta j!} + \left(1 - \frac{1}{k!}\right)^{k!} e^{i\theta k!} + \sum_{j=k+1}^{\infty} \left(1 - \frac{1}{k!}\right)^{j!} e^{i\theta j!}$$

$$= I + \frac{\delta_k}{e} + II, \tag{5.2}$$

Using the basic calculus identity that for each $M \ge 1$, there holds $1 - (1-x)^M \le Mx$ for all $x \in [0,1]$,

$$(5.3) \qquad |I - s_{k-1}| \leq \sum_{j=1}^{k-1} 1 - \left(1 - \frac{1}{k!}\right)^{j!}$$
$$\leq \sum_{j=1}^{k-1} \frac{j!}{k!} < \frac{(k-2)!}{k!} \sum_{j=1}^{k-2} 1 + \frac{(k-1)!}{k!}$$
$$= \frac{k-2}{(k-1)\,k} + \frac{1}{k} < \frac{2}{k} = o\,(1)\,.$$

Another basic calculus identity is that as n increases from 1 to ∞,

$$(5.4) \qquad \left(1 - \frac{1}{n}\right)^n \nearrow \frac{1}{e}.$$

By the definition of δ_k and (5.4),

$$(5.5) \qquad |\delta_k| \leq 1.$$

Finally we estimate II.

$$(5.6) \qquad |II| \leq \sum_{j=k+1}^{\infty} \left\{ \left(1 - \frac{1}{k!}\right)^{k!} \right\}^{\frac{j!}{k!}} \leq \sum_{j=k+1}^{\infty} \left\{ \frac{1}{e} \right\}^{\frac{j!}{k!}}$$
$$\leq \sum_{j=k+1}^{\infty} \left\{ \frac{1}{e} \right\}^{j} = \frac{1}{e^{k+1}} \frac{1}{1 - 1/e} = o\,(1)\,.$$

By Equation (5.2) and the definition of δ_k, it follows that

$$f\left((1 - 1/k!)\, e^{i\theta}\right) - \left(\frac{\delta_k}{e} + s_{k-1}\right) = (I - s_{k-1}) + II,$$

so that limit (5.1) follows from estimates (5.3) and (5.6).

References

[1] K. G. Binmore, *Analytic functions with Hadamard gaps,* Bull. London Math. Soc. **1**(1969), 211–217. MR0244473

[2] Collingwood E. F. and Lohwater A. J., The theory of cluster sets, Cambridge University Press, 1966.

[3] Duren P., Theory of H^p spaces, Dover, Mineola, 2000.

[4] Englis M. , *Toeplitz operators and the Berezin transform on* H^2, Linear algebra and its applications, **223/224**(1995),171–204.

[5] Halmos P.R., A Hilbert space Problem Book, Springer- Verlag, New York, 1982.

[6] Hardy G.H., Divergent series, Oxford Univ. Press, Oxford ,1973.

[7] Hardy G.H.and Littlewood J.E., *A further note on the converse of Abel's theorem,* Proc. London Math. Soc. **25**(1926), 219–236.

[8] Hedenmalm H., Korenblum B.and Zhu K., Theory of Bergman spaces, Graduate Texts in Mathematics, Springer, 2000.

[9] Hoffman K., Banach spaces of analytic functions, Prentice–Hall,1962.
[10] Ingham A.E., *On the "high-indices" theorem of Hardy and Littlewood,* Quarterly J. Math. **8**(1937), 1–7.
[11] Karaev M. T., Berezin set and Berezin number of operators and their applications, The 8th Workshop on Numerical Ranges and Numerical Radii (WONRA06), University of Bremen, July 15-17, 2006, Abstract, p.14.
[12] Karaev M. T., *Berezin symbol and invertibility of operators on the functional Hilbert spaces,* J. Funct. Anal. **238**(2006) ,181–192.
[13] Karaev M. T., *On some problems related to Berezin symbols,* C. R. Acad. Sci. Paris, Ser I, 340,10 (2005), 715–718.
[14] Karaev M.T., *Functional analysis proofs of Abel's theorems,* Proc. Amer. Math. Soc., **132**(2004), 2327–2329.
[15] Lohwater A.J. and Pranian G., *The boundary behavior of functions analytic in a disk,* Ann. Acad. Sci. Fenn., Ser. AI, No **239**(1957),1–17.
[16] Paley R.E.A.C. and Zygmund A., *A note on analytic functions in the unit circle,* Proc. Cambridge Philos. Soc. (2), **28**(1932),266–272.
[17] Pati T., *On a Tauberian theorem of Hardy and Littlewood,* Proc. Indian Acad. Sci. (Math. Sci.), **111**(2001), 221–227.
[18] Privalov I.I., Boundary Properties of Analytic Functions, (Moscow, 1941; 2nd ed., 1950) German transl: Deutscher- Verlag, Berlin, 1956(in Russian).
[19] Titchmarsh E., The Theory of Functions, 2nd ed. Oxford Univ. Press, London and New York, 1939.
[20] Tung J., *Taylor coefficients of functions in Fock spaces,* J. Math. Anal. Appl., **318**(2006), 397–409.
[21] Ullrich D.C., *A Bloch function on the ball with no radial limit,* Bull. Lon. Math. Soc., **20**(1988), 337–341.
[22] Zhu K., Operator Theory in Function Spaces, Marcel New York, 1990.
[23] Zorboska N., *The Berezin transform and radial operators,* Proc. Amer. Math. Soc., **131**(2003), 793–800.

Received: July, 2011