

A survey of multidimensional generalizations of Cantor's uniqueness theorem for trigonometric series

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ABSTRACT. Georg Cantor's pointwise uniqueness theorem for one dimensional trigonometric series says that if, for each x in $[0, 2\pi)$, $\sum c_n e^{inx} = 0$, then all $c_n = 0$. In dimension d , $d \geq 2$, we begin by assuming that for each x in $[0, 2\pi)^d$, $\sum c_n e^{inx} = 0$ where $n = (n_1, \dots, n_d)$ and $nx = n_1x_1 + \dots + n_dx_d$. It is quite natural to group together all terms whose indices differ only by signs. But here there are still several different natural interpretations of the infinite multiple sum, and, correspondingly, several different potential generalizations of Cantor's Theorem. For example, in two dimensions, two natural methods of convergence are circular convergence and square convergence. In the former case, the generalization is true, and this has been known since 1971. In the latter case, this is still an open question. In this historical survey, I will discuss these two cases as well as the cases of iterated convergence, unrestricted rectangular convergence, restricted rectangular convergence, and simplex convergence.

1. Introduction

The idea of this paper is to provide an overview and an organization of other surveys I have authored or coauthored on uniqueness for multiple trigonometric series.

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Let $\{d_n\}_{-\infty < n < \infty}$ be a sequence of complex numbers and let $x \in \mathbb{T}^1 = [0, 2\pi)$. Suppose a function has a representation of the form

$$\sum d_n e^{inx} = \lim_{N \rightarrow \infty} d_0 + \sum_{n=1}^N (d_{-n} e^{-inx} + d_n e^{inx}).$$

To see why it is natural to combine the n th and $-n$ th terms, suppose that a_n and b_n are real, and that $d_n = (a_n + ib_n)/2$ and d_{-n} are complex conjugates. Since $e^{i\theta} = \cos \theta + i \sin \theta$, $d_n e^{inx} + d_{-n} e^{-inx}$ is immediately computed to be $a_n \cos nx + b_n \sin nx$, the “natural” n th term of a real valued trigonometric series. Is this representation unique? In other words, if $\sum d_n e^{inx} = \sum d'_n e^{inx}$ for every x , does it necessarily follow that $d_n = d'_n$ for every n ? Subtract and set $c_n = d_n - d'_n$ to get a cleaner formulation.

(U) Let $\sum c_n e^{inx} = 0$ for every $x \in \mathbb{T}^1$. Does this imply that $c_n = 0$ for every n ?

In 1870, Georg Cantor showed that the answer to question (U) is “yes.”

THEOREM C. *Let $\sum c_n e^{inx} = 0$ for every $x \in \mathbb{T}^1$. Then $c_n = 0$ for every n .*

In all dimensions we will always combine terms whose indices differ only by signs. This reduction in dimension 1 leads to the definite meaning of $\sum d_n e^{inx}$ given above. When $d \geq 2$, the meaning of $\sum d_n e^{inx}$ is not yet definite, so there are many variants of question (U).

First, for each $n \in Z^{+d} = \{0, 1, 2, \dots\}^d$, we write $\sum_{n \in Z^d} C_n = \sum_{n \in Z^{+d}} T_n$, where $T_n = \sum_{\{\nu: \text{each } \nu_i = n_i \text{ or } -n_i\}} C_\nu$. For example, when $d = 2$, $T_{3,4} = C_{3,4} + C_{-3,4} + C_{3,-4} + C_{-3,-4}$ and $T_{5,0} = C_{5,0} + C_{-5,0}$. This reduction still leaves many possible ways of interpreting the multiple sum. Here are six very natural ones. For simplicity, each will only be described in dimension 2 and we will write (n_1, n_2) as (ℓ, m) to avoid indices.

Square convergence:

$$Sq \sum_{n \in Z^{+2}} T_n = \lim_{N \rightarrow \infty} \left(\sum_{\ell=0}^N \sum_{m=0}^N T_{(\ell, m)} \right).$$

The N th partial sum contains all terms with indices in the square with lower left corner $(0, 0)$ and upper right corner (N, N) .

Spherical convergence:

$$Sp \sum_{n \in Z^{+2}} T_n = \lim_{N \rightarrow \infty} \sum_{k=0}^N \left(\sum_{\{(\ell, m): \ell^2 + m^2 \leq k\}} T_{(\ell, m)} \right).$$

The N th partial sum contains all terms with indices in the intersection of the disk of radius \sqrt{N} and the first quadrant.

One way iterated convergence:

$$It \sum_{n \in Z^{+2}} T_n = \lim_{N \rightarrow \infty} \sum_{k=0}^N \left(\lim_{J \rightarrow \infty} \sum_{j=0}^J T_{(j, k)} \right).$$

The terms with indices of height 0 are summed yielding a first intermediate number, then the terms with indices of height 1 are summed yielding a second intermediate number, and so on, producing a one dimensional one way sequence of intermediate numbers. Finally all the numbers of that sequence are added together. In dimension d , there are $d!$ distinct versions of one way iterated convergence, but they are all very similar and it will be enough for us to pick any one of them.

Unrestricted rectangular convergence:

$$UR \sum_{n \in Z^{+2}} T_n = \lim_{\min \{M, N\} \rightarrow \infty} \sum_{j=0}^M \sum_{k=0}^N T_{(j, k)}.$$

Restricted rectangular convergence:

$$RR \sum_{n \in Z^{+2}} T_n = t \text{ if for every } E \geq 1, \text{ no matter how large,}$$

$$\lim_{\min \{M, N\} \rightarrow \infty} \sum_{j=0}^M \sum_{k=0}^N T_{(j, k)} = t.$$

$$1/E \leq M/N \leq E$$

Simplex convergence:

$$Sm \sum_{n \in \mathbb{Z}^{+2}} T_n = \lim_{N \rightarrow \infty} \sum_{k=0}^N \left(\sum_{m=0}^k T_{(m, k-m)} \right).$$

We discuss 6 generalizations of Cantor's Theorem (U). They are

THEOREM 1 (Iterated). *Fix any $d \geq 2$. Let $It \sum c_n e^{inx} = 0$ for every $x \in \mathbb{T}^d$. Then $c_n = 0$ for every $n \in \mathbb{Z}^d$.*

THEOREM 2 (Unrestricted Rectangular). *Fix any $d \geq 2$. Let $UR \sum c_n e^{inx} = 0$ for every $x \in \mathbb{T}^d$. Then $c_n = 0$ for every $n \in \mathbb{Z}^d$.*

THEOREM 3 (Spherical). *Fix any $d \geq 2$. Let $Sp \sum c_n e^{inx} = 0$ for every $x \in \mathbb{T}^d$. Then $c_n = 0$ for every $n \in \mathbb{Z}^d$.*

QUESTION 4 (Restricted Rectangular). Let $RR \sum c_n e^{inx} = 0$ for every $x \in \mathbb{T}^2$. Does this imply that $c_n = 0$ for every $n \in \mathbb{Z}^2$?

QUESTION 5 (Square). Let $Sq \sum c_n e^{inx} = 0$ for every $x \in \mathbb{T}^2$. Does this imply that $c_n = 0$ for every $n \in \mathbb{Z}^2$?

QUESTION 6 (Simplex). Let $Si \sum c_n e^{inx} = 0$ for every $x \in \mathbb{T}^2$. Does this imply that $c_n = 0$ for every $n \in \mathbb{Z}^2$?

Many additional questions can be asked. For a lot of one dimensional generalizations of Theorem 1 see chapter 9 of [Z]. We will later need to mention one higher dimensional extension of Theorem 3 which involves replacing the condition of spherical convergence: $Sp \sum c_n e^{inx}$ exists, by the weaker condition of spherical Abel summability: $Sp \sum c_n e^{inx} r^{\|n\|}$ exists for all positive $r < 1$, where $\|n\| = \sqrt{n_1^2 + \cdots + n_d^2}$, and $\lim_{r \rightarrow 1^-} Sp \sum c_n e^{inx} r^{\|n\|}$ exists.

2. History of the three theorems

Our discussion here will be informed by drawing comparisons with the steps of Cantor's original proof. Here are the four major steps of his proof.

- (1) Establish the Cantor-Lebesgue Theorem, that $|c_n| + |c_{-n}| \rightarrow 0$,
- (2) show that the Riemann function $F(x) = c_0 \frac{x^2}{2} + \sum_{n \neq 0} \frac{c_n}{(in)^2} e^{inx}$ is continuous,
- (3) establish the consistency of Riemann summability, that the Schwarz second derivative D^2 defined by

$$(2.1) \quad D^2 F(x) = \lim_{h \rightarrow 0} \frac{F(x+h) - 2F(x) + F(x-h)}{h^2}$$

satisfies

$$D^2 F(x) = \lim_{h \rightarrow 0} c_0 + \sum_{n \neq 0} c_n e^{inx} \left(\frac{\sin n \frac{h}{2}}{n \frac{h}{2}} \right)^2 = 0, \text{ and}$$

- (4) prove Schwarz's Theorem, that continuous functions with identically zero Schwarz second derivative are of the form $ax + b$.

The theorem about iterated convergence has a direct simple inductive proof, the starting point being Cantor's Theorem.

The unrestricted rectangular theorem was given an erroneous proof in 1919. The false proof was given for $d = 2$. The idea was to copy the steps of Cantor's proof very directly. There was defined the natural analogue of the Riemann function, namely the following termwise fourth integral of the original series,

$$F(x, y) := c_{00} \frac{x^2 y^2}{4} + \sum_{n \in \mathbb{Z}, n \neq 0} c_{0n} \frac{x^2}{2} \frac{e^{iny}}{(in)^2} + \sum_{m \in \mathbb{Z}, m \neq 0} c_{m0} \frac{e^{imx}}{(im)^2} \frac{y^2}{2} + \sum_{m \in \mathbb{Z}, n \in \mathbb{Z}, mn \neq 0} c_{mn} \frac{e^{imx}}{(im)^2} \frac{e^{iny}}{(in)^2}.$$

Differentiating formally (but without any justification) shows that

$$\frac{\partial^4 F}{\partial^2 x \partial^2 y} = \sum c_{mn} e^{imx} e^{iny} = 0.$$

Next the author showed that F is continuous and does have generalized fourth derivative $D_{xxyy}F(x, y) = 0$ for every point (x, y) , where

$$D_{xxyy} := \lim_{\substack{h, k \rightarrow 0 \\ hk \neq 0}} \frac{\begin{array}{rrr} +1F(x-h, y+k) & -2F(x, y+k) & +1F(x+h, y+k) \\ -2F(x-h, y) & +4F(x, y) & -2F(x+h, y) \\ +1F(x-h, y-k) & -2F(x, y-k) & +1F(x+h, y-k) \end{array}}{h^2k^2}.$$

Finally the truth of the following “analog” of Schwarz’s Theorem was assumed without proof.

CONJECTURE 1. *If $F(x, y)$ is continuous and if for all (x, y)*

$$D_{xxyy}F(x, y) = 0,$$

then F behaves as if F were C^4 and satisfied $\frac{\partial^4 F}{\partial^2 x \partial^2 y} = 0$.

This was assumed to be correct and easily extendable to all dimensions.

Since it appeared that the unrestricted rectangular case had been resolved, nothing happened in that area for over 50 years. But when we looked at this around 1970, we could find no proof for the conjecture and felt strongly that one was needed.

The next area to receive attention was that of circular uniqueness. In 1957 Victor Shapiro proved a powerful d dimensional theorem. Shapiro worked in a somewhat more general context, also considering questions of summability. He did not prove Theorem 3 because his proof required an extra assumption on the coefficient size.[S] For $m \in \mathbb{Z}^d$, let $|m|$ denote $\sqrt{m_1^2 + \cdots + m_d^2}$. A corollary of one of Shapiro’s results was this.

COROLLARY 4. *If $\sum c_n e^{inx} = 0$ for all $x \in \mathbb{T}^d$, and if*

$$(2.2) \quad \lim_{r \rightarrow \infty} \frac{1}{r} \sum_{r-1 < |m| \leq r} |c_m| = 0,$$

then all $c_n = 0$.

The coefficient size assumption 2.2 is natural in dimension 2: since there are $O(r)$ lattice points being summed over in condition 2.2, this assumption asserts that the c_m tend to zero “on the average” as $|m| \rightarrow \infty$. But the assumption becomes much stronger as the dimension increases; specifically in dimension d there are $O(r^{d-1})$ terms in the sum, so that the coefficients are required to be decaying like $o(r^{2-d})$ on the average. Fourteen years later, in 1971, Roger Cooke found this generalization to the Cantor-Lebesgue Theorem for dimension $d = 2$. [Coo]

THEOREM 5 (Cooke). *Let $d = 2$. If $\{c_m\}$ is a doubly indexed set of complex numbers such that*

$$\sum_{|m|=r} c_m e^{imx}$$

tends to zero for almost all x , then

$$(2.3) \quad \sqrt{\sum_{|m|=r} |c_m|^2} \text{ tends to 0 as } r \rightarrow \infty.$$

From the definition of spherical convergence it is clear that spherical convergence at x to 0 (or to any other finite value for that matter) implies that the hypothesis of Cooke’s theorem holds at x . Now it is a very easy calculation that when $d = 2$, the conclusion of Cooke’s Theorem implies the validity of condition (2.2) and thus the unconditional spherical uniqueness theorem in dimension $d = 2$. [AWa1], page 42 But an unconditional proof of Theorem 3 seemed well out of reach.

In the early 1970s, the pendulum swung back to the unrestricted rectangular convergence uniqueness question. Just at the time of Cooke’s work, Grant Welland and I looked at the 1919 paper with the gap mentioned above. [AWe] We were unable to fill the gap, but we did discover that when a series converges UR almost everywhere, “most” coefficients tend to zero, while all coefficients are bounded. The first fact is easy, but the second required a clever idea that we found in the unpublished thesis of Paul Cohen. [Coh] From this control of the coefficient size, it follows that Shapiro’s condition 2.2 holds in dimension 2. So the UR uniqueness

theorem for two dimensions would follow immediately from corollary 4 if everywhere UR convergence implies everywhere Sp convergence. It does not. However, by a lucky stroke of fate, everywhere UR convergence *does* imply everywhere spherical Abel summability and it turns out that the hypotheses of the quite general theorem of Victor Shapiro which yielded Corollary 4 above are satisfied. Thus Theorem 2 was proved in two dimensions.

The first precursor to a higher dimensional theory came in 1976, when Connes extended the Cantor Lebesgue result of Cooke, whose proof was exceedingly two dimensional, to all dimensions.**[Con]** At this point, we knew that if we wanted to prove Theorem 3, we could use the fact that

$$\lim_{k \rightarrow \infty} \sum_{\{n: n_1^2 + \dots + n_d^2 = k\}} |c_n|^2 = 0$$

without having to add any further hypothesis.

From the middle seventies until the early nineties was a period of hibernation.

In the early 1990s, attention turned to Theorem 2. First of all, Cris Freiling and Dan Rinne showed me the function

$$F(x, y) := (x + y) |x + y|$$

which satisfies the hypothesis of Conjecture 1 above. This analogue of Schwarz's Theorem had been stated as fact in 1919. But F does not have the expected form of

$$a(y)x + b(y) + c(x)y + d(x),$$

with $a(y)$ and $c(x)$ being C^4 . So the proposed analogue of Schwarz's Theorem is false! We went on to give a proof for all dimensions of Theorem 2.**[AFR]** The proof involved replacing the simple but false Conjecture 1 by adding more hypotheses.

Rename D_{xxyy} as $S_{(1,1)}$. Further define

$$S_{(1,0)} := \lim_{\substack{h, k \rightarrow 0 \\ hk \neq 0}} \frac{\begin{array}{ccc} +1F(x-h, y+2k) & -2F(x, y+2k) & +1F(x+h, y+2k) \\ -2F(x-h, y+k) & +4F(x, y+k) & -2F(x+h, y+k) \\ +1F(x-h, y) & -2F(x, y) & +1F(x+h, y) \end{array}}{h^2},$$

$$S_{(0,1)} := \lim_{\substack{h, k \rightarrow 0 \\ hk \neq 0}} \frac{\begin{array}{ccc} +1F(x, y+k) & -2F(x+h, y+k) & +1F(x+2h, y+k) \\ -2F(x, y) & +4F(x+h, y) & -2F(x+2h, y) \\ +1F(x, y-k) & -2F(x+h, y-k) & +1F(x+2h, y-k) \end{array}}{k^2},$$

$$S_{(0,0)} := \lim_{\substack{h, k \rightarrow 0 \\ hk \neq 0}} \frac{\begin{array}{ccc} +1F(x, y+2k) & -2F(x+h, y+2k) & +1F(x+2h, y+2k) \\ -2F(x, y+k) & +4F(x+h, y+k) & -2F(x+2h, y+k) \\ +1F(x, y) & -2F(x+h, y) & +1F(x+2h, y) \end{array}}{1}$$

What is true is that all four of these are zero everywhere. Notice that the square of the step size of the second difference appears in the denominator exactly when the difference is symmetric in that direction. To motivate these definitions, note that the 1-dimensional function $A(x) = ce^{inx} + de^{-inx}$ has a second symmetric difference $-4A(x) \sin^2 \frac{mh}{2}$, whereas its second forward difference is $-4A(x+h) \sin^2 \frac{mh}{2}$. The symmetric differences are so nice that they can overcome the damage done to the quotient by the step size squared term in the denominator; the forward differences are not as nice, but they do not have corresponding denominator terms fighting against their movement toward zero. However, the proof of the corollary still remained quite difficult. We developed a complicated covering technique to get the job done. Probably the best way to understand the technique of the proof, is

to see it applied to the much, much simpler one dimensional case. There it gives a proof by means of covering of Schwarz's original theorem. This proof is much longer and more involved than Schwarz's original, short, and beautiful proof. It has the virtue of extending to our higher dimensional situation and also it avoids using the maximum principle.[**As1**] I will only give a small one dimensional analogue of how a covering might come into play here. Suppose you want to prove that if a function is uniformly differentiable to zero on the interval $[a, b]$, then $f(a) = f(b)$. One way would be to let $\{x_i\}$ be a very fine partition of $[a, b]$ and to begin by writing

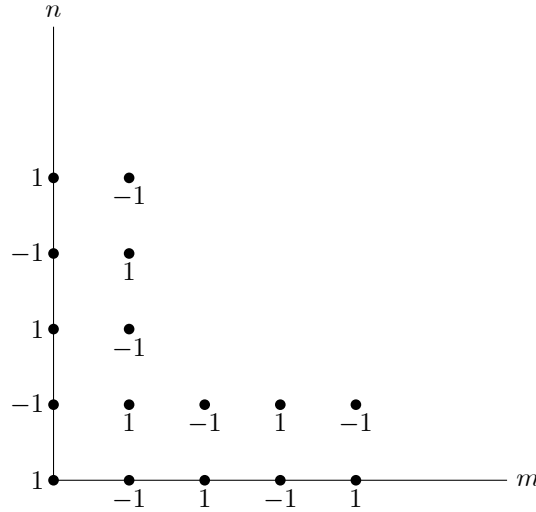
$$F(b) - F(a) = \sum_i \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} (x_{i+1} - x_i).$$

This proof can even be fiddled with to prove that having a zero derivative implies being constant. See [**As1**] for how to deal with second differences in a similar way.

By one of those historical coincidences that happen in mathematics from time to time, even though there had been a 15 year period of complete inactivity, just as we were proving Theorem 2, Shakro Tetunashvili was also proving it in Tbilisi, Georgia.[**Tet**] He saw our article and sent me a copy of his proof. It is important to note that his proof was published first. His proof is completely different. It is very easy to give an example of a function which converges UR while diverging iteratively. Let

$$a_{mn} = \begin{cases} (-1)^{m+n} & \text{if } m \in \{0, 1\} \text{ or } n \in \{0, 1\} \\ 0 & \text{otherwise} \end{cases}$$

Here is a representation of this series where the value of a_{mn} is attached to the point (m, n) .



The UR limit is 0 since the rectangular partial sums S_{mn} are all 0 as soon as $\min\{m, n\} \geq 1$. Iterated convergence fails since neither $\lim_{m \rightarrow \infty} S_{m0}$ nor $\lim_{n \rightarrow \infty} S_{0n}$ exist.

Nevertheless, Tetunashvili was able to prove a lemma: UR convergence to zero everywhere implies one way iterated convergence everywhere. Applying Theorem 2 completes the proof of Theorem 2, UR uniqueness is true. Tetunashvili's Lemma uses a result from [AWe] that everywhere UR convergence implies that *all* partial sums are bounded which requires the aforementioned lemma from Paul Cohen's thesis. His proof also uses that Paul Cohen's lemma directly. It is much shorter than the proof of Ash, Freiling, and Rinne. A simple description of his proof can be found in [As2]. An irony in this tale of two proofs is that the first, "wrong" covering proof probably would not have happened if the second "right" proof had come to our attention earlier. Only time will tell if the covering techniques we developed will eventually have useful applications elsewhere.

The last positive result is Theorem 2. This was proved in 1996 by Jean Bourgain.[B] The basic approach was already outlined in Shapiro's 1957 paper. The idea there is to follow Cantor's original proof. The analogue of the Riemann

function is the formal antiLaplacian of $\sum c_n e^{inx}$, namely

$$F(x) = - \sum \frac{c_n}{\|n\|^2} e^{inx}.$$

Note that $\Delta e^{inx} = \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2} e^{inx} = - \sum_{j=1}^d (in_j)^2 e^{inx} = -\|n\|^2 e^{inx}$. The analogue of the one dimensional Schwarz derivative here is this generalized Laplacian of F :

$$\lim_{\rho \rightarrow \infty} \frac{c_d}{\rho^n} \left(\frac{1}{m(B(x, \rho))} \int_{B(x, \rho)} F(t) dt - F(x) \right),$$

where $B(x, \rho)$ is an x -centered solid d -dimensional ball of radius ρ , and m denotes Lebesgue measure. A calculation involving Bessel functions shows that this limit is $\Delta F(x)$ when F is C^2 ; and a theorem of Rado asserts that if F is continuous and this generalized Laplacian is identically 0, then F is harmonic. But harmonic functions are C^∞ and hence the coefficients of F decay very rapidly. In particular, the c_n decay rapidly enough so that Shapiro's condition (2.2) is satisfied. Since Shapiro had shown that the assumption was enough to guarantee that the generalized Laplacian of F is 0, the proof would be complete if F could be shown to be continuous. In the one dimensional case the continuity of the Riemann function followed very quickly from the Weierstrass M -test. Because of Conne's 1976 result we do know that $\sum_{\{\|n\|=k\}} |c_n|^2$ tends to zero, but this is not enough to allow applying the M -test to F . Showing F to be continuous is very difficult. What Bourgain did was show F to be continuous.[B]

Bourgain's proof requires numerous ideas as well as strong technique. The flow of his argument is given in [As2]. The proof itself appears in at least three places. There is Bourgain's precisely but concisely written original 15 page article[B], there is a somewhat expanded 22 page version in [AWa1], and there is a 42 page version of the proof specialized down to dimension 2 in [As3].

3. The questions

Since all three questions are completely unsolved, we will restrict our discussion to the two dimensional situation. A square can be rotated into a 2-simplex and a

simple rotation argument shows the equivalence of the square uniqueness question and the simplex uniqueness question in two dimensions, but this equivalence is not clear in higher dimensions, since, in particular, a cube has six edges while a 3-simplex has eight edges. Since we are going to stay in two dimensions, we will naturally consider only the RR and square uniqueness question.

There are two obvious inclusions. If a multiple numerical series converges Unrestricted Rectangularly, then it converges Restricted Rectangularly. If a multiple numerical series converges Restricted Rectangularly, then it converges Square. So

$$\begin{aligned} \text{Hypothesis of UR theorem} &\implies \text{Hypothesis of RR question} \\ &\text{and} \\ \text{Hypothesis of RR question} &\implies \text{Hypothesis of Square question.} \end{aligned}$$

This explains how it can be that UR uniqueness can be known while RR uniqueness remains an open question. This also suggests that in attacking the questions hoping to find affirming proofs one should try to prove the validity of RR uniqueness, while if one is thinking about a counterexample, one should try to find a counterexample to square uniqueness.

There is not much evidence either way for these questions. One reason to lean in the negative direction is the spectacular failure of the Cantor Lebesgue Theorem in this setting.

The weakest version of the one dimensional Cantor Lebesgue theorem says that if the sequence $\{c_{-n}e^{-inx} + c_ne^{inx}\}$ converges to zero for every x as $n \rightarrow \infty$, then

$$(3.1) \quad \text{the sequence } \{|c_{-n}| + |c_n|\} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

The natural analogue of this for two dimensional square convergence is to start with the assumption that the sequence with N th term

$$(3.2) \quad \sum_{\{(m,n):\max\{|m|,|n|\}=N\}} c_{mn}e^{i(mx+ny)}$$

converges to zero for every (x, y) as $N \rightarrow \infty$. The conclusion should be something about the decay of the moduli of the coefficients. For $N \geq 2$, consider the sequence with N th term

$$A_N(x, y) = e^{\frac{N}{\log \log N}} \cos^2 \frac{x}{2} \sin^{2N-2} \frac{x}{2} \cos Ny.$$

At each point (x, y) this sequence tends to zero rapidly as $N \rightarrow \infty$, for if $x = \pm\pi$, every term is identically zero, while if $|x| < \pi$ and $a = \sin^2 \frac{x}{2}$, $|A_N(x, y)| \leq e^{\frac{N}{\log \log N}} a^{N-1}$, which tends very rapidly to zero since $a < 1$. Using the Euler identities to write $\sin \theta$ and $\cos \theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$ and then expanding by the binomial theorem we see that A has the form (3.2) and a calculation shows that

$$c_{0N} = \frac{1}{4^N} \binom{2N}{N} \frac{1}{2N-1} e^{\frac{N}{\log \log N}}$$

which, by Stirling's Formula, is on the order of $N^{-3/2} e^{\frac{N}{\log \log N}}$ and hence enormously divergent. This is an equally strong counterexample for restricted rectangular convergence, but the details are slightly messy. [AWa2] (Because of the example I have just shown you, a Cantor-Lebesgue analogue for Square and RR convergence would have to be very, very weak. Actually, the Cantor-Lebesgue analogue here is this: if (3.2) tends to zero for every (x, y) , the growth of the $\{c_{mn}\}$ must be less than exponential. The above example can be slightly modified to show that this is sharp.)

The example that is mentioned here also occurs in the study of spherical harmonics. The question of uniqueness is essentially open for spherical harmonics also. There is a uniqueness result that appeared 60 years ago in the PhD thesis of Walter Rudin. [R] Rudin constructs an analogue of the Riemann function, and the crux of the matter then comes down to showing that his Riemann function is continuous. He does not prove continuity, but rather restricts his result to the set of all series for which his associated Riemann function is continuous. This is a very strong auxiliary condition, but Rudin's result has never been improved. I feel that the fact that this counterexample fits both situations makes it very likely that progress on the open trigonometric questions will be strongly correlated with progress on the uniqueness question for spherical harmonics.

The first step of the one dimensional Cantor program, namely the application of a Cantor Lebesgue theorem to gain knowledge of some decay of the coefficients, seems out of reach because of the counterexample. However this is not the end of the difficulties. Even the following two questions are beyond what I know how to do.

QUESTION 4' (Restricted Rectangular). Let $d = 2$, $RR \sum c_n e^{inx} = 0$ for every $x \in \mathbb{T}^2$, and further assume that $|c_{mn}| = o\left(\frac{1}{\sqrt{m^2+n^2}}\right)$. Does this imply that $c_n = 0$ for every $n \in \mathbb{Z}^2$?

QUESTION 5' (Square). Let $d = 2$, $Sq \sum c_n e^{inx} = 0$ for every $x \in \mathbb{T}^2$, and further assume that $\sum_{\{(m,n): \max\{|m|, |n|\}=k\}} |c_{mn}|^2 = o\left(\frac{1}{k \ln k}\right)$. Does this imply that $c_n = 0$ for every $n \in \mathbb{Z}^2$?

I picked the auxilliary hypotheses here pretty much at random. The idea is to keep the set of series for which the proof is valid broader than L^2 . In other words, is anything at all is true about uniqueness for either of these two methods as soon as one moves out into the realm of trigonometric series that may not be Fourier series?

Here are two approaches that I have recently thought of, but haven't yet tried out.

1. Try to reverse a normally irreversible hypothesis using the facts that the assumed convergence is both everywhere and to zero. For example, try to move from the RR hypothesis to the UR hypothesis, thereby using the UR uniqueness theorem as a lemma for proving RR uniqueness; or try to move from the Sq hypothesis to the RR hypothesis to prove equivalence of Questions 4 and 5. Examples of this sort of procedure can be found directly in Tetunashvili's proof of Theorem 2 and indirectly in the result of Ash and Welland that while UR convergence at a point trivially implies the finiteness of $\limsup_{\min\{m,n\} \rightarrow \infty} |S_{mn}|$ at the same point, UR

convergence everywhere unexpectedly implies the boundedness of *all* partial sums at each point.

2. The Laplacian seems to be a better derivative than $\frac{\partial^4}{\partial x^2 \partial y^2}$. But the spherical generalized Laplacian used by Shapiro and Bourgain doesn't seem to fit the rectangular methods very well. Perhaps a generalized Laplacian formed by integral averaging a function over the boundary of a small square and subtracting the function value at the center might work better. Another possibility is a generalized Laplacian formed by averaging a sum of function values spaced around the edge of a small square and subtracting the function value at the center.

The first seven references below can be found using links from <http://condor.depaul.edu/mash/realvita.html>.

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