# Two Pointwise Characterizations of the Peano Derivative

J. Marshall Ash, Stefan Catoiu, and Hajrudin Fejzić

ABSTRACT. In 1936, J. Marcinkiewicz and A. Zygmund showed that the existence of the $n$th Peano derivative $f_{(n)}(x)$ of a function $f$ at $x$ is equivalent to the existence of both $f_{(n-1)}(x)$ and the $n$th generalized Riemann derivative $\widetilde{D}_n f(x)$, based at $x, x+h, x+2h, x+2^2h, \ldots, x+2^{n-1}h$.

Let $D_n^{\mathrm{sh}} f(x)$ be the *set* of the first $n-1$ forward shifts of the $n$th forward Riemann derivative $D_n f(x)$ of a function $f$ at $x$. We provide a second characterization of the $n$th Peano derivative $f_{(n)}(x)$ in terms of these sets: The existence of $f_{(n)}(x)$ is equivalent to the existence of $f_{(n-1)}(x)$ as well as the existence of each of the $n-1$ elements of $D_n^{\mathrm{sh}} f(x)$. The proof of this result involves an interesting combinatorial algorithm.

Compare three second derivatives:

1. The Schwarz derivative:

$$Sf(x) := \lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2},$$

2. The second Peano derivative, the number $f_{(2)}(x)$ making valid the relation:

$$f(x+h) = f_{(0)}(x) + f_{(1)}(x)\,h + \frac{f_{(2)}(x)}{2}h^2 + o\left(h^2\right),$$

3. The ordinary second derivative:

$$f''(x).$$

First, it is well known that if $f''(x)$ exists, then $f_{(2)}(x)$ exists and is equal to $f''(x)$; and if $f_{(2)}(x)$ exists, then $Sf(x)$ exists and is equal to $f_{(2)}(x)$. Both these generalizations are strict, since the function $g(x)$ which is $x^3\sin\left(x^{-1}\right)$ when $x \neq 0$ and $0$ when $x=0$ has $g_{(2)}(0)=0$, but $g''(0)$ nonexistent, while the function $\operatorname{sgn}(x)$ has Schwarz derivative $0$ at the origin but not a second (nor first nor zeroth) Peano derivative at the origin.

Second, the process of computing $Sf(x)$ at a fixed point $x$ is direct, just take a single limit. The other two second derivatives cannot be computed so quickly. To compute $f_{(2)}(x)$ one must first compute $f_{(0)}(x)$ and $f_{(1)}(x)$. To compute $f''(x)$ one must first compute $f'(x+h)$ at every $h$ in some neighborhood of 0 and finally compute the limit $\lim_{h\to 0}\left\{f'(x+h)-f'(x)\right\}/h$.

Suppose we know that a function $f$ has a first derivative at $x$, and we would like to fulfill the fantasy of finding its 2nd Peano derivative at $x$ by computing a single limit. Based on the above discussion, an obvious theorem that does this could be stated as follows: if

2010 *Mathematics Subject Classification.* Primary 26A24; Secondary 03F60, 15A06, 26A27.

*Key words and phrases.* Generalized Riemann derivative, $\mathcal{A}$-derivative, Peano derivative.

November, 2019. This paper is in final form and no version of it will be submitted for publication elsewhere.

both $f_{(1)}(x)$ and $Sf(x)$ exist, then so does $f_{(2)}(x)$. This would be lovely, but unfortunately is not true, because any odd function, such as $f(x)=x\sqrt{|x|}$, which is differentiable at 0, but not twice Peano differentiable at 0, would have $Sf(0)$ existing (and equal to zero).

The goal of this paper is to find a substitute for $S$ that would produce the fulfillment of the fantasy, here and for general $n$.

*Generalized Riemann derivatives*, or $\mathcal{A}$-derivatives, of order $n$ are given by limits of the form

$$D_{\mathcal{A}}f(x)=\lim_{h\to 0}h^{-n}\sum_{i=0}^{m}A_i f\left(x+a_i h\right),$$

where the data vector $\mathcal{A}=\{A_1,\dots,A_m;a_1,\dots,a_m\}$ satisfies the $n$th Vandermonde conditions $\sum_i A_i a_i^j=n!\delta_{jn}$, for $j=0,1,\dots,n$. For example, the generalized Riemann derivative of order $n=1$ and with data vector $\mathcal{A}=\{-1,1;0,1\}$ is just ordinary differentiation, while the Schwarz derivative $S$ given above is the $\mathcal{A}$-derivative of order $n=2$ with $\mathcal{A}=\{1,-2,1;-1,0,1\}$. The $n$th forward Riemann derivative $D_n f(x)$ is the $n$th generalized Riemann derivative with

$$\mathcal{A}=\left\{A_i=(-1)^i\binom{n}{i};a_i=n-i\,|\,i=0,\dots,n\right\}.$$

The *$n$th Peano derivative* of $f$ at $x$ is the number $f_{(n)}(x)$ that satisfies the relation

$$f(x+h)=f_{(0)}(x)+f_{(1)}(x)h+\frac{f_{(2)}(x)}{2!}h^2+\cdots+\frac{f_{(n)}(x)}{n!}h^n+o(h^n).$$

Notice that the existence of the $n$th Peano derivative of $f$ at $x$ assumes the existence of every lower order Peano derivative of $f$ at $x$. Moreover, it is well known that, by Taylor expansion about $x$, the existence of the $n$th Peano derivative $f_{(n)}(x)$ forces every generalized Riemann derivative of order $n$ to exist and to be equal to $f_{(n)}(x)$.

Riemann derivatives were introduced by Riemann in [**R**] (1892). Peano derivatives are due to Peano in [**P**] (1892), and were further developed by De la Vallée Poussin in [**VP**] (1908). Surveys on Peano derivatives are found in [**EW, O**]. Generalized Riemann derivatives were introduced by Denjoy in [**D**] (1935); they have many applications in the theory of trigonometric series [**Z**]. Generalized Riemann derivatives have been shown to satisfy basic properties of ordinary derivatives, such as monotonicity, convexity, or the mean value theorem. [**AJ, FFR, HL, HL1, T, W**] Multidimensional Riemann derivatives were studied in [**AC1**], and quantum Riemann derivatives appeared in [**AC, ACR**]. A survey on generalized Riemann derivatives is given in [**As2**].

Every generalized Riemann derivative of order $n\geq 1$ shares with $S$ the virtue of being immediately computable as a limit. But every generalized Riemann derivative of order $n\geq 2$ also shares with $S$ the defect of being a strict generalization of the $n$th Peano derivative. This means that no single generalized Riemann derivative of order greater than or equal to 2 can have it's existence at $x$ to be sufficient to force the existence of the Peano derivative of the same order.

We prove this by assuming to the contrary that a certain $n$th ($n\geq 2$) generalized Riemann derivative $\mathcal{A}$ has the property that, for each function $f$ and real number $x$, the existence of the derivative $D_{\mathcal{A}}f(x)$ implies the existence of $f_{(n)}(x)$, since the existence of the $n$th Peano derivative assumes the existence of every lower order Peano derivative, by transitivity, the existence of $D_{\mathcal{A}}f(x)$ implies the existence of $f_{(1)}(x)=f'(x)$.

On the other hand, there is a non-trivial analogue of the result for the ordinary derivative $f^{(n)}(x)$ instead of the Peano derivative $f_{(n)}(x)$, given in [**ACCs**, Theorem 1]. This result asserts that when $n \geq 2$, the existence of no single generalized Riemann derivative $D_{\mathcal{A}}f(x)$ can imply the existence of the ordinary derivative $f^{(n)}(x)$; and for $n = 1$, exactly the cases of

$$\lim_{h\to 0} \frac{Af(x+rh) + Af(x-rh) + f(x+h) - f(x-h) - 2Af(x)}{2h}, \text{ where } Ar \neq 0,$$

imply (hence are equivalent to) the first order ordinary derivative.

This proves that the original generalized Riemann derivative has order $n = 1$, a contradiction with the assumption that the same order is $n \geq 2$.

By taking $\mathcal{B} = \{-1, 1; 0, 1\}$ to be the data vector for ordinary first order differentiation, the above result in [**ACCs**] can be viewed as the classification of all pairs $(\mathcal{A}, \mathcal{B})$ of data vectors of generalized Riemann differentiations, subject to the condition that, for each function $f$ and point $x$, the existence of the derivative $D_{\mathcal{A}}f(x)$ either implies or is equivalent to the existence of the derivative $D_{\mathcal{B}}f(x)$. This classification for general $\mathcal{B}$ is given in [**ACCh**], and we will refer to it as *the classification of generalized Riemann derivatives.* The classification of complex generalized Riemann derivatives is given in [**ACCH**].

As we have just mentioned above, the problem of finding an $n$th generalized Riemann derivative whose existence at a point implies the existence of the $n$th Peano derivative at that point has no solution for $n \geq 2$. Thus the problem of characterizing the $n$th Peano derivative $f_{(n)}(x)$ by a single $n$th generalized Riemann derivative $D_{\mathcal{A}}f(x)$ is not well posed. The next question that one can ask is the one of characterizing the Peano derivative by a class $C$ of several generalized Riemann derivatives of orders $\leq n$, meaning that, for each function $f$ and real number $x$, the existence of $D_{\mathcal{A}}f(x)$, for all $\mathcal{A}$ in $C$, implies the existence of $f_{(n)}(x)$. Writing $C$ as the disjoint union $C = C_1 \cup \ldots \cup C_n$, where $C_k$ consists of all elements of $C$ of order $k$, and noticing that the existence of $f_{(n-1)}(x)$ implies the existence of the derivative of $f$ at $x$ in the sense of each element of $C_1 \cup \ldots \cup C_{n-1}$, the natural simpler question to ask is the one of characterizing the $n$th Peano derivative modulo the $n-1$st Peano derivative by a class $C_n$ of $n$th generalized Riemann derivatives. This is the problem of finding a class $C_n$ of $n$th generalized Riemann derivatives such that the existence of $f_{(n-1)}(x)$ as well as the existence of the derivatives of $f$ at $x$ in the sense of all elements of $C_n$ implies the existence of $f_{(n)}(x)$.

The first class $C_n$ that characterizes $f_{(n)}(x)$ modulo $f_{(n-1)}(x)$ was found in 1936 by Marcinkiewicz and Zygmund. It is a single element class $C_n = \{\widetilde{D}_n\}$, where $\widetilde{D}_n$ is a special $n$th generalized Riemann derivative which we will describe next. This is the first pointwise characterization of the Peano derivative by generalized Riemann derivatives. Our main result, Theorem 2, provides a second characterization of the $n$th Peano derivative modulo the $n-1$st Peano derivative by the class $C_n$ consisting of the first $n-1$ forward shifts of the $n$th forward Riemann derivative, which we will also describe in detail in the next part of the Introduction.

Marcinkiewicz and Zygmund in [**MZ**] introduced a special sequence of generalized Riemann derivatives, one for each $n$. They showed that the difference $\widetilde{\Delta}_n(x, h; f)$, defined recursively by

$$\begin{aligned} &\widetilde{\Delta}_1(x,h;f) = f(x+h) - f(x), \\ &\widetilde{\Delta}_n(x,h;f) = \widetilde{\Delta}_{n-1}(x,2h;f) - 2^{n-1}\widetilde{\Delta}_{n-1}(x,h;f) \quad (n = 2, 3, \ldots), \end{aligned} \tag{1}$$

when multiplied by a scalar $\lambda_n$, is an $n$th generalized Riemann difference. For simplicity, whenever possible, write $\Delta(h)$ for the difference $\Delta(x,h;f)$. In this way,

$$\begin{aligned}
\widetilde{\Delta}_2(h) &= f(x+2h) - 2f(x+h) + f(x)\\
\widetilde{\Delta}_3(h) &= f(x+4h) - 6f(x+2h) + 8f(x+h) - 3f(x),\\
\widetilde{\Delta}_4(h) &= f(x+8h) - 14f(x+4h) + 56f(x+2h) - 64f(x+h) + 21f(x),\\
\widetilde{\Delta}_5(h) &= f(x+16h) - 30f(x+8h) + 280f(x+4h) - 960f(x+2h)\\
&\quad + 1024f(x+h) - 315f(x),
\end{aligned}$$

with $\lambda_2 = 1$, $\lambda_3 = 1/4$, $\lambda_4 = 1/56$, and $\lambda_5 = 1/2688$.

By taking $\widetilde{D}_n f(x)$ to be the $n$th generalized Riemann derivative of $f$ at $x$,

$$\widetilde{D}_n f(x) = \lim_{h\to 0} \lambda_n \widetilde{\Delta}_n\,(x,h;f)/h^n,$$

corresponding to the generalized Riemann difference $\lambda_n\widetilde{\Delta}_n\,(x,h;f)$, they proved the strong implication in the following theorem, which characterizes the $n$th Peano derivative $f_{(n)}(x)$ in terms of the special $n$th generalized Riemann derivative $\widetilde{D}_n f(x)$:

THEOREM MZ. *For each function $f$ and real number $x$,*

$$f_{(n-1)}(x) \textit{ exists and } \widetilde{D}_n f(x) \textit{ exists } \iff f_{(n)}(x) \textit{ exists.}$$

This can also be read as $\widetilde{D}_n f(x)$ is equivalent to $f_{(n)}(x)$, for $n-1$ times Peano differentiable functions $f$ at $x$. The direct implication is explicitly proved in [**MZ**], Collected Papers, Page 134.

As we have mentioned earlier, by Taylor expansion, the $n$th Peano derivative $f_{(n)}(x)$ implies any $n$th generalized Riemann derivative. And since the definition of the $n$th Peano derivative $f_{(n)}(x)$ assumes the existence of any lower order Peano derivative of $f$ at $x$, the reverse implication in Theorem MZ is always true.

In 1936, Marcinkiewicz and Zygmund created $\widetilde{D}_n f(x)$ to establish one step of a very long and beautiful inductive proof. The question of finding a characterization of Peano differentiation was looked at explicitly in 1970 in [**As1**] and in 1998-2000 in [**GGR, GGR1, GR**]. While writing this paper, we realized that although $\widetilde{D}_n f(x)$ has provided a working tool in real analysis for a long time [**As, AC, ACR**], to the best of our knowledge, no one has ever referred to it as giving a characterization of $n$th order Peano differentiation.

With the exception of the Gaussian Riemann derivatives ${}_q\widetilde{D}_n f(x)$, for real $q$, with $q > 1$, which are introduced and studied in [**AC2**], there are no known $n$th generalized Riemann derivatives that are essentially different from $\widetilde{D}_n f(x) = {}_2\widetilde{D}_n f(x)$ and characterize the Peano derivative $f_{(n)}(x)$ in the same way as the derivative $\widetilde{D}_n f(x)$ does in Theorem MZ; see also Theorem 1 below. For this reason, the derivatives $\widetilde{D}_n f(x)$ are really special within the class of $n$th generalized Riemann derivatives.

We add that an iteration of the equivalence in Theorem MZ, for $n, n-1, \ldots, 1$, leads to another characterization of the $n$th Peano derivative $f_{(n)}(x)$ in terms of the special generalized Riemann derivatives $\widetilde{D}_n f(x)$:

COROLLARY MZ. *For each function $f$ and real number $x$,*

$$\widetilde{D}_1 f(x), \widetilde{D}_2 f(x), \ldots, \widetilde{D}_n f(x) \textit{ exist } \iff f_{(n)}(x) \textit{ exists.}$$

One can easily see that Corollary MZ is not only a consequence of Theorem MZ, but also an equivalent result. Based on this, both Theorem MZ and Corollary MZ provide the first pointwise characterization of the Peano derivative.

Also note that, by our earlier comments, the equivalence in Corollary MZ does not hold true if its left side is replaced by any single generalized Riemann derivative of any order.

**Results.** Our first result shows that, for $n \geq 3$, the two most widely known $n$th generalized Riemann derivatives, the $n$th symmetric Riemann derivative $D_n^s f(x)$ and the $n$th forward Riemann derivative $D_n f(x)$, fail to characterize the $n$th Peano derivative $f_{(n)}(x)$ at $x = 0$ in the same way as $\widetilde{D}_n f(x)$ does in Theorem MZ.

THEOREM 1. *Suppose $n \geq 3$. Then, for all functions $f$ and real numbers $x$,*
(i) *both $f_{(n-1)}(x)$ and $D_n^s f(x)$ exist $\not\Longrightarrow$ $f_{(n)}(x)$ exists.*
(ii) *both $f_{(2)}(x)$ and $D_3 f(x)$ exist $\not\Longrightarrow$ $f_{(3)}(x)$ exists.*

PROOF. (i) Consider the functions $g : [0, \infty) \to \mathbb{R}$, defined by $g(x) = x^{n-1/2}$, and $f : \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} g(x) & \text{, if } x \geq 0, \\ (-1)^{n-1} g(-x) & \text{, if } x < 0. \end{cases}$$

Clearly, $g(h) = o(h^{n-1})$ implies $f(h) = o(h^{n-1})$, so $f$ is $n-1$ times Peano differentiable at 0 and $f_{(0)}(0) = f_{(1)}(0) = \cdots = f_{(n-1)}(0) = 0$, while $\lim_{h \to 0^+} g(h)/h^n = \lim_{h \to 0^+} 1/\sqrt{h} = \infty$ implies that $f_{(n)}(0)$ does not exist. On the other hand, since $f$ is of opposite parity as $n$ is, the $n$th symmetric derivative $D_n^s f(0)$ exists and is equal to zero.

(ii) Let $G = \langle 2, 3 \rangle$ be the multiplicative subgroup of the rationals generated by the integers 2 and 3. Then $G = \{2^m 3^n | m, n \text{ integers}\}$, so that if $h \in G$, then $2h, 3h \in G$ and if $h \notin G$ then $2h, 3h \notin G$. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} (-1)^{m+n} x^3 & \text{, if } x = 2^m 3^n \in G, \\ 0 & \text{, if } x \notin G. \end{cases}$$

Since $0 \leq |f(x)| \leq x^3$, $f$ is continuous at 0, has two Peano derivatives at 0, and $f_{(0)}(0) = f_{(1)}(0) = f_{(2)}(0) = 0$. The third Peano derivative $f_{(3)}(0)$ does not exist, since the ratio $f(h)/h^3$ has three distinct limit points, 0, 1 and -1, as $h \to 0$.

Let $h = 2^m 3^n$. Then $\Delta_3(0, h; f) = f(3h) - 3f(2h) + 3f(h) - f(0)$

$$\begin{aligned} &= (-1)^{m+n+1} 3^3 h^3 - 3(-1)^{m+n+1} 2^3 h^3 + 3(-1)^{m+n} h^3 - 0 \\ &= (-1)^{m+n} h^3 (-27 + 24 + 3) \\ &= 0, \end{aligned}$$

so $f$ is three times forward Riemann differentiable at 0 and $D_3 f(0) = 0$. □

We suspect the result in Part (ii) of Theorem 1 to hold for a general $n$ in place of $n = 3$. Since the example we use here does not seem to extend to higher $n$, and such an extension does not seem to be of as much interest as the remainder of this paper, we leave this extension as an open question. However, Theorem 1 as stated fulfills its principal goal of motivating the main theorem of the paper, Theorem 2, as much as its most general version would; see the first paragraph of the preamble to Theorem 2.

By linear algebra, the $n$th Vandermonde conditions holding for a difference $\Delta_{\mathcal{A}}(x,h;f) = \sum_i A_i f(x+a_i h)$ imply that the same conditions hold for its *$r$-translate* (or *$r$-shift*)

$$\Delta_{\mathcal{A},r}(x,h;f) = \sum_i A_i f(x+(a_i+r)h).$$

In particular, the $n$th generalized Riemann differences

$$\Delta_{n,j}(x,h;f) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x+(n-i+j)h),$$

for $j = 0,1,\ldots,k-1$, are the first $k$ forward translates of the $n$th forward Riemann difference $\Delta_n(x,h;f) = \Delta_{n,0}(x,h;f)$, and the limits

$$D_{n,j}f(x) = \lim_{h\to 0} \Delta_{n,j}(x,h;f)/h^n,$$

for $j = 0,1,\ldots,k-1$, are the first $k$ forward translates (or shifts) of the $n$th forward Riemann derivative $D_n f(x) = D_{n,0}f(x)$.

Theorem 1 provides very good evidence that, once $n \geq 3$ and the Peano derivative $f_{(n-1)}(x)$ exists, neither the symmetric Riemann derivative $D_n^s f(x)$ nor the forward Riemann derivative $D_n f(x)$ are enough to force the existence of the Peano derivative $f_{(n)}(x)$. The next option available that might do the job would be to use sets of consecutive forward shifts of the $n$th forward Riemann derivative instead of just the $n$th forward Riemann derivative, and the smallest number of shifts needed when $n=3$ is $n-1$.

The second pointwise characterization of the $n$th Peano derivative is the main theorem of this paper. Let $D_n^{\mathrm{sh}} f(x)$ be the set of all shifts $D_{n,j}f(x)$ , for $j = 0,1,\ldots,n-2$, of the $n$th forward Riemann derivative $D_n f(x)$ of $f$ at $x$. The result says that the set $D_n^{\mathrm{sh}} f(x)$ is equivalent to the $n$th Peano derivative $f_{(n)}(x)$, for each $n-1$ times Peano differentiable function $f$ at $x$, thereby providing an analogue of the equivalence in Theorem MZ, where the special derivative $\widetilde{D}_n f(x)$ is replaced by a set of consecutive shifts of the Riemann derivative. The main result reads as follows:

THEOREM 2. *For each measurable function $f$ and real number $x$,*

$$\text{both } f_{(n-1)}(x) \text{ and } D_n^{\mathrm{sh}} f(x) \text{ exist} \iff f_{(n)}(x) \text{ exists.}$$

Notice that when $n=2$, $D_n^{\mathrm{sh}} f(x)$ is a set consisting of a single element $D_{2,0}f(x) = D_2 f(x) = \widetilde{D}_2 f(x)$, so Theorem 2 is the same as Theorem MZ. Thus the goal of replacing $S$ in order to fulfill the fantasy stated at the beginning has succeeded.

The motivation for Theorem 2 is due to an earlier result and conjecture of Ginchev, Guerragio and Rocca in [**GGR, GR**]. They made the following conjecture: if we replace $D_n^{\mathrm{sh}} f(x)$ by a set consisting of the first $n$ backward shifts instead of $n-1$ forward shifts of the $n$th forward Riemann derivative of $f$ at $x$, they conjectured the result of Theorem 2 with this new hypothesis and proved it by hand for $n = 1,2,3,4$ and with computer assistance for $n \leq 8$, leaving the cases $n > 8$ as a conjecture. Their approach is different from ours.

The next corollary is the analogue of the equivalence in Corollary MZ, where each special derivative $\widetilde{D}_n f(x)$ is replaced by the set $D_n^{\mathrm{sh}} f(x)$ of the first $n-1$ forward shifts of the $n$th forward Riemann derivative $D_n f(x)$. In essence, this result says that the $n$th Peano derivative $f_{(n)}(x)$ is equivalent to a triangular set of consecutive forward shifts of all forward Riemann derivatives of orders up to $n$.

COROLLARY 3. *Let $n \geq 2$. Then for each measurable function $f$ and real number $x$,*

$$D_1 f(x) \text{ and } D_k^{\mathrm{sh}} f(x), \text{ for } k = 2,\ldots,n, \text{ exist} \iff f_{(n)}(x) \text{ exists.}$$

Corollary 3 is deduced from Theorem 2 in the same way as the equivalence in Corollary MZ was deduced from the equivalence in Theorem MZ, so its formal proof is omitted. In this way, the entire article is devoted to the proof of Theorem 2. And as it was the case with Corollary MZ, the result of Corollary 3 is not only a consequence of Theorem 2, but an equivalent result. They both provide the second pointwise characterization of the Peano derivative.

Section 1 contains the statement of the characterization theorem of the Peano derivative, the combined result of Theorem MZ, Corollary MZ, Theorem 1 and Corollary 3, and an update of the status of its proof, based on some of the proofs being already addressed in the Introduction.

In Section 2, the proof of Theorem 2 is reduced to showing in Lemma 4 that the set $D_n^{\mathrm{sh}} f(x)$ implies the $n$th special generalized Riemann derivative $\widetilde{D}_n f(x)$ directly, without the hypothesis on $f_{(n-1)}(x)$. The most natural proof of this would require the use of linear algebra, namely a Gaussian elimination algorithm that inputs the first $n-1$ forward shifts of the $n$th Riemann difference, and outputs the unique $n$th generalized Riemann difference based at $x, x+h, x+2h, x+4h, \ldots, x+2^{n-1}h$. This linear algebra problem is reduced in Lemma 5 to the recursive set theory problem of finding a combinatorial (Gaussian) elimination algorithm that inputs the first $n-1$ forward shifts of the ordered set $\{0, 1, 2, \ldots, n\}$, and outputs the ordered set $\{0, 1, 2, 4, \ldots, 2^{n-1}\}$, using just two well-defined operations with ordered sets that correspond to two well-defined operations with differences, dilation by 2 and elimination. These will be explained in detail in Section 2.

Section 3 contains the proof of Lemma 5, or the above mentioned combinatorial elimination algorithm. The first half of the section will provide the proof of the result for $n = 3, 4, 7, 10$. In this way, all ideas behind the general algorithm are introduced in a gradual manner. The case $n = 10$ should suffice to understanding the general case. By now the reader is familiarized with the terminology, and especially with the more complicated operations with sets that are built as compounds of the two basic operations. These helped with the writing of the general algorithm in lowest terms. The general algorithm, given in the second half of the section, has four steps, and the repeat of the last step provides the result. A few comments are added to each step for a more convincing argument that the algorithm works.

In addition to simplifying the proof of Lemma 4, we found Lemma 5 and its proof to be interesting stand-alone results of combinatorics and recursive set theory.

## 1. The two characterizations of the Peano derivative

Recall that for each function $f$ at $x$, $f_{(n)}(x)$ is the $n$th Peano derivative, $\widetilde{D}_n f(x)$ is the unique $n$th generalized Riemann derivative based at $x, x+h, x+2h, x+2^2h, \ldots, x+2^{n-1}h$, and $D_n^{\mathrm{sh}} f(x)$ is the set $D_n^{\mathrm{sh}} f(x) = \{D_{n,j} f(x) \,|\, j = 0, 1, \ldots, n-2\}$ of the first $n-1$ forward shifts of the $n$th forward Riemann derivative $D_n f(x)$.

The following result provides the characterization of the Peano derivative $f_{(n)}(x)$, by combining in a single theorem the four results on two pointwise characterizations of the Peano derivative, that were outlined in the Introduction.

THEOREM (Pointwise characterization of the Peano derivative). *Suppose* $n \geq 2$. *Then for each function* $f$ *and real number* $x$,

(i) *both* $f_{(n-1)}(x)$ *and* $\widetilde{D}_n f(x)$ *exist* $\iff f_{(n)}(x)$ *exists;*
(ii) *all* $\widetilde{D}_1 f(x), \widetilde{D}_2 f(x), \dots, \widetilde{D}_n f(x)$ *exist* $\iff f_{(n)}(x)$ *exists;*
(iii) *both* $f_{(n-1)}(x)$ *and* $D_n^{\mathrm{sh}} f(x)$ *exist* $\iff f_{(n)}(x)$ *exists;*
(iv) $D_1 f(x)$ *and all* $D_2^{\mathrm{sh}} f(x), \dots, D_n^{\mathrm{sh}} f(x)$ *exist* $\iff f_{(n)}(x)$ *exists.*

Parts (i) and (ii) are easily equivalent to each other; they represent the first pointwise characterization of the Peano derivative. Part (i) is the result of Theorem MZ, due to Marcinkiewicz and Zygmund in [**MZ**]. Part (ii) is our result of Corollary MZ, whose easy proof appeared in the Introduction.

Similarly, Parts (iii) and (iv) are also equivalent to each other; they represent the second pointwise characterization of the Peano derivative. Part (iii) is the result of Theorem 2, the main result of the paper. Its proof is reduced to a result on recursive sets, Lemma 5, in Section 2, which is proved in Section 3 using an interesting combinatorial algorithm. Part (iv) is the result in Corollary 3, whose proof is discussed in the Introduction.

The two characterizations are different when viewed from a numerical analysis perspective. To illustrate this, suppose that the 10th Peano derivative exists at $x = 0$ and we want to check the existence of and find the value of the 11th Peano derivative, using only values of the original function. The first method computes a single limit involving the $12$ base points $0h, 1h, 2h, 4h, \dots, 512h, 1024h$. The other method involves the $21$ base points $0h, 1h, 2h, 3h, \dots, 18h, 19h, 20h$. Ten different limits, each using $12$ consecutive base points from the given $21$, are calculated. They must be equal (nearly equal in a numerical situation); the common value is the $11$th Peano derivative.

At first blush, the second method seems much more complicated than the first. However, it is easy to imagine situations where it is far easier and more accurate to evaluate a function on the second set of 21 regularly spaced points, than on the first set of 12 wildly unbalanced points. We will not pursue this line of thought further here.

## 2. Main theorem reduced to a combinatorial problem

The reverse implication in Theorem 2 is easily argued in the same way as we did for the reverse implication in Theorem MZ. For the direct implication, based on the direct implication in Theorem MZ, it suffices to show that

LEMMA 4. *For each measurable function $f$ and real number $x$,*

$$D_n^{\mathrm{sh}} f(x) \text{ exists} \implies \widetilde{D}_n f(x) \text{ exists.}$$

Note that, unlike Theorem 2, the above lemma relates only $n$th generalized Riemann derivatives, without involving any Peano derivative. When the left side of the implication in Lemma 4 is a set consisting of a single generalized Riemann derivative, then one available method of proof for the result would be by reference to the classification of generalized Riemann derivatives in [**ACCh**] that we mentioned in the Introduction. Unfortunately, this option is available only in the case $n = 2$, when both sides of the implication represent the same derivative $D_2 f(x) = \widetilde{D}_2 f(x)$, or when the implication is a triviality. All remaining cases ($n > 2$) fall outside the classification of generalized Riemann derivatives, so the proof will be based on new techniques.

An important property of all derivatives in Lemma 4, which turned out crucial to the combinatorial method of proof we chose from here on to the end of the paper, is that these are $n$th generalized Riemann derivatives based at $n + 1$ points. By the Vandermonde conditions, these derivatives are uniquely determined by their base points.

Before proceeding with the proof of Lemma 4, it is important to lay down a few ideas about generating new generalized Riemann derivatives from old, which we will refer to as *operations* with generalized Riemann derivatives. Specifically, we will concentrate on two basic operations with $n$th generalized Riemann derivatives based at $n+1$ points:

*1. Dilation.* The dilation by a real number $r$ of an $n$th generalized Riemann difference $\Delta_{\mathcal{A}}(x,h;f) = \sum_{i=0}^{n} A_i f(x+a_i h)$ corresponding to the data vector $\mathcal{A} = \{A_i; a_i \,|\, i = 0,\dots,n\}$ is the $n$th generalized Riemann difference

$$\Delta_{\mathcal{A}_r}(x,h;f) = \sum_{i=0}^{n} r^{-n} A_i f(x+a_i r h)$$

corresponding to vector $\mathcal{A}_r = \{r^{-n}A_i; ra_i \,|\, i = 0,\dots,n\}$. Moreover, a function $f$ is $\mathcal{A}$-differentiable at $x$ if and only if is $\mathcal{A}_r$-differentiable at $x$ and $D_{\mathcal{A}} f(x) = D_{\mathcal{A}_r} f(x)$.

To simplify the language, for fixed $f$ and $x$, we say that an ordered set $\mathcal{A}$ of $2n+2$ elements satisfying the $n$th Vandermonde conditions is *good* if $f$ is $\mathcal{A}$-differentiable at $x$. Then $\mathcal{A}$ is good if and only if $\mathcal{A}_r$ is good. And since each $n$th generalized Riemann derivative based at $n+1$ points is uniquely determined by its base points, $\mathcal{A}$ is uniquely determined by $\{a_0,\dots,a_n\}$, while $A_0,\dots,A_n$ are just place holders. In this way, a plain set $\{a_0,\dots,a_n\}$ is good if and only if, for each non-zero real number $r$, its *$r$-dilate* $\{ra_0,\dots,ra_n\}$ is good.

*2. Elimination.* By the $n$th Vandermonde conditions, only a unique non-zero scalar multiple of each non-zero linear combination of two $n$th generalized Riemann differences is a generalized Riemann difference. The set of base points of such a non-zero linear combination is the union of the base points of the terms, minus the base points corresponding to the terms that got eliminated by the linear combination. Focusing only on pairs of $n$th generalized Riemann differences based at $n+1$ points and having the additional property that they share $n$ base points, observe that by taking non-zero linear combinations of the differences corresponding to such pairs, one arrives at one of the following three possibilities:

- No common terms of the two differences were eliminated by the linear combination. Then the resulting $n$th difference has $n+2$ base points, a discarded case.
- At least two common terms were eliminated by the linear combination. Then the resulting $n$th difference has $\le n$ base points, an impossibility.
- Only one common term of the two differences got eliminated by the linear combination. Then the resulting $n$th difference has $n+1$ base points, hence it is of the desired kind. And since linear combinations of $n$th generalized differences are scalar multiples of $n$th generalized differences, if $S = \{a_0,\dots,a_n\}$ and $T = \{b_0,\dots,b_n\}$ are good sets such that $|S\cap T| = n$ then, for each $a\in S\cap T$, the set $S\cup T\setminus\{a\}$ is also a good set.

As an example highlighting the third bullet situation, consider the two shifts,

$$\Delta_{3,0}(h) = f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x) \text{ and}$$
$$\Delta_{3,1}(h) = f(x+4h) - 3f(x+3h) + 3f(x+2h) - f(x+h),$$

of the third Riemann difference $\Delta_3(h)$ of $f$ at $x$, so that the sets $\{0,1,2,3\}$ and $\{1,2,3,4\}$ representing their base points are good. The linear combination $3\Delta_{3,0}(h) + \Delta_{3,1}(h)$ that eliminates the term in $f(x+3h)$ is the difference $\widetilde{\Delta}_3(h)$ that we have seen earlier. This is a scalar multiple ($\lambda_3 = 1/4$) of a third generalized Riemann difference. The same third generalized Riemann difference can be obtained in a different way, from the Vandermonde system, as the unique third generalized Riemann difference whose base points set is the (new good) set $\{0,1,2,4\}$ obtained by eliminating 3 between the other two known good sets.

Back to Lemma 4, its statement in language of good sets goes as follows: If the sets $\{0,1,2,\dots,n\},\{1,2,3,\dots,n+1\},\dots,\{n-2,n-1,n,\dots,2n-2\}$ are good, then so is the set $\{0,1,2,4,\dots,2^{n-1}\}$.

As for its proof, since we already know two operations that produce new good sets from old, dilation and elimination, to prove the lemma, it suffices to provide an algorithm that inputs the given sets in the above hypothesis and outputs the set in the conclusion, by only using dilations of given or previously deduced sets, and elimination of a common element between of a pair of given or deduced sets that have $n$ common elements.

Summarizing, we have reduced the proof of Lemma 4, and implicitly the one of Theorem 2, to the following result of recursive set theory.

LEMMA 5. *Suppose a collection $\mathcal{S}$ of sets, each consisting of $n+1$ non-negative integers, is defined by the following properties:*

(i) $\{0,1,2,\dots,n\},\{1,2,3,\dots,n+1\},\dots,\{n-2,n-1,n,\dots,2n-2\}\in\mathcal{S}$;
(ii) *if* $S\in\mathcal{S}$, *then* $2S:=\{2s\,|\,s\in S\}\in\mathcal{S}$;
(iii) *if* $S,T\in\mathcal{S}$ *have* $|S\cap T|=n$, *then for each* $a\in S\cap T$, $S\cup T\setminus\{a\}\in\mathcal{S}$.

*Then* $\{0,1,2,4,\dots,2^{n-1}\}\in\mathcal{S}$.

## 3. Proof of the combinatorial problem

The general proof of Lemma 5 relies on a combinatorial elimination algorithm described in Section 3.2. All ideas behind the algorithm are brought up in Section 3.1, where several smaller cases are investigated.

As a simplifying terminology for the proof, we convene, in all displayed equations, to write each ordered set as a row-vector, that is, without braces and commas. The hypothesis (i) means *input* the given sets (vectors) of $\mathcal{S}$ into the algorithm, (ii) is *dilation* of a set by 2, and (iii) is *elimination* of a common element between two sets that share $n$ elements. And two elements $S,T$ of $\mathcal{S}$, with $|S\cap T|=n$, are said to be *set for elimination*.

**3.1. Smaller cases.** In this subsection we prove Lemma 5, for $n=3,4,7,10$. Each of these cases contains an extra idea for the general proof that was not shown in the previous cases, so that the case $n=10$ has all ideas needed for the general proof.

The proof of the $n=3$ case of Lemma 5 was accomplished at the end of the previous section by displaying $\widetilde{\Delta}_3(h)=3\Delta_{3,0}(h)+\Delta_{3,1}(h)$. As a first example, we rewrite the $n=3$ case here to begin illustrating the notation and language we'll use in giving the general proof of Lemma 5: Input the two given elements $\{0,1,2,3\},\{1,2,3,4\}\in\mathcal{S}$ as row vectors, with their common elements listed under each other, and highlight the number 3 as the only non-zero entry which is not a power of 2, which we will refer to as an *intruder*.

$$\begin{array}{ccccc} 0 & 1 & 2 & \mathbf{3} & \\ & 1 & 2 & \mathbf{3} & 4 \end{array}$$

Eliminate 3 between the two given sets to deduce that $\{0,1,2,4\}\in\mathcal{S}$, as needed.

**Case** $n = 4$**.** Write the three given sets as row-vectors in a parallelogram array, with the common elements written in the same column, and mark all intruders.

$$\begin{array}{ccccccc}
0 & 1 & 2 & \mathbf{3} & 4 & & \\
 & 1 & 2 & \mathbf{3} & 4 & \mathbf{5} & \\
 & & 2 & \mathbf{3} & 4 & \mathbf{5} & \mathbf{6}
\end{array}$$

Note that consecutive rows are set for elimination. This property will stay in place on all arrays till the end of the algorithm.

Consider the following chain of eliminations. Replace the third row with the result of eliminating 5 between itself and the second row, to deduce that $\{1, 2, 3, 4, 6\} \in \mathcal{S}$. Then replace this new third row with the result of eliminating 3 between itself and the first row, to deduce that $\{0, 1, 2, 4, 6\} \in \mathcal{S}$. The compound of these two eliminations is read as follows: on the row ending in 6 (even number), we eliminated as many odd numbers (two, namely 5 and 3, in this order) as there are rows above it, at the price of adding 0 and 1 at the beginning of the row.

Our first processing of the above parallelogram array is the set of all possible elimination compounds described above. Starting at the bottom and going up, on each row ending with an even entry (first and third), eliminate as many odd entries (high to low) as there are rows above it. Then delete the remaining (second) row(s), as shown on the left chart. Then double the first row and move it to the bottom,

$$\begin{array}{cccccc}
0 & 1 & 2 & \mathbf{3} & 4 & \\
0 & 1 & 2 & & 4 & \mathbf{6}
\end{array}
\qquad\qquad
\begin{array}{cccccc}
0 & 1 & 2 & 4 & \mathbf{6} & \\
0 & & 2 & 4 & \mathbf{6} & 8
\end{array}$$

as shown on the right chart. Finally, eliminate $6$ between the two rows to deduce that $\{0, 1, 2, 4, 8\} \in \mathcal{S}$, as needed.

As a shortcut for the compound of the last two operations, we say that the same conclusion was obtained directly from the earlier array by eliminating 6 between the second row and twice the first row. In this way, for a general $n$, from this stage on to the end of the algorithm, all arrays will have the addtional property that twice the top row and the bottom row are set for elimination.

**Case** $n = 7$**.** Input the given vectors in a, by now familiar, parallelogram array.

$$\begin{array}{ccccccccccccc}
0 & 1 & 2 & \mathbf{3} & 4 & \mathbf{5} & \mathbf{6} & \mathbf{7} & & & & & \\
 & 1 & 2 & \mathbf{3} & 4 & \mathbf{5} & \mathbf{6} & \mathbf{7} & 8 & & & & \\
 & & 2 & \mathbf{3} & 4 & \mathbf{5} & \mathbf{6} & \mathbf{7} & 8 & \mathbf{9} & & & \\
 & & & \mathbf{3} & 4 & \mathbf{5} & \mathbf{6} & \mathbf{7} & 8 & \mathbf{9} & \mathbf{10} & & \\
 & & & & 4 & \mathbf{5} & \mathbf{6} & \mathbf{7} & 8 & \mathbf{9} & \mathbf{10} & \mathbf{11} & \\
 & & & & & \mathbf{5} & \mathbf{6} & \mathbf{7} & 8 & \mathbf{9} & \mathbf{10} & \mathbf{11} & \mathbf{12}
\end{array}$$

Eliminate as many odd entries, high to low, from the rows ending in even entries as there are rows above them, and ignore the remaining rows.

$$\begin{array}{cccccccccc}
0 & 1 & 2 & \mathbf{3} & 4 & \mathbf{5} & \mathbf{6} & 8 & & \\
0 & 1 & 2 & \mathbf{3} & 4 & & \mathbf{6} & 8 & \mathbf{10} & \\
0 & 1 & 2 & & 4 & & \mathbf{6} & 8 & \mathbf{10} & \mathbf{12}
\end{array}$$

Note that all rows have the same number of intruders: three. Replace the third row with the result of the elimination of $12$ between itself and the double of the first row. Replace the second row with the result of the elimination of $10$ between itself and the newly established third row. Replace the first row with the result of the elimination of $6$ between itself and the new second row. The effect of this process is to replace the largest intruder in each row with 16, the next power of 2. We refer to the compound process as “cutting off the largest intruders” in each row.

$$\begin{array}{cccccccccc}
0 & 1 & 2 & \mathbf{3} & 4 & \mathbf{5} & & 8 & & 16 \\
0 & 1 & 2 & \mathbf{3} & 4 & & \mathbf{6} & 8 & & 16 \\
0 & 1 & 2 & & 4 & & \mathbf{6} & 8 & \mathbf{10} & 16
\end{array}$$

Repeat cutting off intruders from all rows, except for the first one, which is deleted. The difference between this cut off that deletes the first row and the preceding one that retained it is marked by the last intruder on the first row now being odd 5, instead of last time being even 6.

$$\begin{array}{ccccccccc}
0 & 1 & 2 & \mathbf{3} & 4 & & 8 & 16 & 32 \\
0 & 1 & 2 & & 4 & \mathbf{6} & 8 & 16 & 32
\end{array}$$

Finally, since the last intruder in the first row is the odd number 3, we delete the first row and cut off the last intruder in the second row to get $\{0, 1, 2, 4, 8, 16, 32, 64\} \in \mathcal{S}$.

**Case** $n = 10$**.** Start with the parallelogram made with the row-vectors corresponding to the given sets $\{0, 1, \ldots, 10\}, \{1, 2, \ldots, 11\}, \ldots, \{8, 9, \ldots, 18\}$ of $\mathcal{S}$, where each column has equal entries, and from the bottom to the top, in each row ending with an even entry, eliminate as many of the largest odd intruders as there are rows above it. Moreover, eliminate the rows ending in odd entries.

$$\begin{array}{ccccccccccccccc}
0 & 1 & 2 & \mathbf{3} & 4 & \mathbf{5} & \mathbf{6} & \mathbf{7} & 8 & \mathbf{9} & \mathbf{10} & & & & \\
0 & 1 & 2 & \mathbf{3} & 4 & \mathbf{5} & \mathbf{6} & \mathbf{7} & 8 & & \mathbf{10} & \mathbf{12} & & & \\
0 & 1 & 2 & \mathbf{3} & 4 & \mathbf{5} & \mathbf{6} & & 8 & & \mathbf{10} & \mathbf{12} & \mathbf{14} & & \\
0 & 1 & 2 & \mathbf{3} & 4 & & \mathbf{6} & & 8 & & \mathbf{10} & \mathbf{12} & \mathbf{14} & 16^* & \\
0 & 1 & 2 & & 4 & & \mathbf{6} & & 8 & & \mathbf{10} & \mathbf{12} & \mathbf{14} & 16 & \mathbf{18}
\end{array}$$

The largest numbers in each row are the even numbers between $n = 10$ and $2n - 2 = 18$. There is a single power of 2 among them, namely 16, which we mark by an asterisk. Note that the number of intruders in each of the rows above the asterisk is six, while the rows with asterisk or below it have only five intruders. Using the asterisk row as a base, cut off the largest intruders in all rows above it.

$$\begin{array}{ccccccccccccccc}
0 & 1 & 2 & \mathbf{3} & 4 & \mathbf{5} & \mathbf{6} & \mathbf{7} & 8 & \mathbf{9} & & & & 16 & \\
0 & 1 & 2 & \mathbf{3} & 4 & \mathbf{5} & \mathbf{6} & \mathbf{7} & 8 & & \mathbf{10} & & & 16 & \\
0 & 1 & 2 & \mathbf{3} & 4 & \mathbf{5} & \mathbf{6} & & 8 & & \mathbf{10} & \mathbf{12} & & 16 & \\
0 & 1 & 2 & \mathbf{3} & 4 & & \mathbf{6} & & 8 & & \mathbf{10} & \mathbf{12} & \mathbf{14} & 16^* & \\
0 & 1 & 2 & & 4 & & \mathbf{6} & & 8 & & \mathbf{10} & \mathbf{12} & \mathbf{14} & 16 & \mathbf{18}
\end{array}$$

Now there are five intruders in each row. As long as the last entry in the first row is odd, delete it and (using its double, which we do not write down, as a base) eliminate the highest intruders in the remaining rows. We deduce

$$\begin{array}{cccccccccccccc}
0 & 1 & 2 & \mathbf{3} & 4 & \mathbf{5} & \mathbf{6} & \mathbf{7} & 8 & & & & 16 & 32 \\
0 & 1 & 2 & \mathbf{3} & 4 & \mathbf{5} & \mathbf{6} & & 8 & \mathbf{10} & & & 16^* & 32 \\
0 & 1 & 2 & \mathbf{3} & 4 & & \mathbf{6} & & 8 & \mathbf{10} & \mathbf{12} & & 16 & 32 \\
0 & 1 & 2 & & 4 & & \mathbf{6} & & 8 & \mathbf{10} & \mathbf{12} & \mathbf{14} & 16 & 32
\end{array}$$

and

$$\begin{array}{ccccccccccccc}
0 & 1 & 2 & \mathbf{3} & 4 & \mathbf{5} & \mathbf{6} & 8 & & & 16^* & 32 & 64 \\
0 & 1 & 2 & \mathbf{3} & 4 & & \mathbf{6} & 8 & \mathbf{10} & & 16 & 32 & 64 \\
0 & 1 & 2 & & 4 & & \mathbf{6} & 8 & \mathbf{10} & \mathbf{12} & 16 & 32 & 64
\end{array}$$

When the last intruder in the first row is even, we keep the first row and cut off the largest intruders from all rows.

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | **3** | 4 | **5** | | 8 | | 16* | 32 | 64 | 128 |
| 0 | 1 | 2 | **3** | 4 | | **6** | 8 | | 16 | 32 | 64 | 128 |
| 0 | 1 | 2 | | 4 | | **6** | 8 | **10** | 16 | 32 | 64 | 128 |

Repeat the process until all intruders are eliminated. We get

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | **3** | 4 | | 8 | 16 | 32 | 64 | 128 | 256 |
| 0 | 1 | 2 | | 4 | **6** | 8 | 16 | 32 | 64 | 128 | 256 |

and, finally, deduce that $\{0, 1, 2, 4, 8, 16, 32, 64, 128, 256, 512\} \in \mathcal{S}$, as desired.

**3.2. The general case.** The proof of the general case in Lemma 5 follows ideas from the proofs of the smaller cases described in the first half of the section. The general proof is based on the following combinatorial (Gaussian) elimination algorithm:

*STEP 1.* Arrange (input) the given sets

$$\{0, 1, \ldots, n\}, \{1, 2, \ldots, n+1\}, \ldots, \{n-2, n-1, \ldots, 2n-2\}$$

in order, one under the other, as row vectors in a parallelogram array so that each next vector is shifted one position to the right of the previous vector, so that all equal entries are a part of the same column. Highlight all intruders.

*STEP 2.* Replace each row ending in an even entry with the one obtained from it by deleting as many of its largest odd entries as there are rows above it and adding all non-negative integers smaller than its smallest entry. Delete the remaining rows.

In this way, the new $k$th row from the bottom, for $k = 1, 2, \ldots, \lfloor (n-1)/2 \rfloor$, is coming from the old row $\{n-2k, n-2k+1, n-2k+2, \ldots, 2(n-k)\}$ by inserting all non-negative integers, $0, 1, \ldots, n-2k-1$, that can fit in front of it, and deleting its largest $n-2k$ odd entries $2(n-k)-1, 2(n-k)-3, \ldots, 2(n-k)-2(n-2k)+1 = 2k+1$. The new $k$th row (set) from the bottom has the expression:

$$\{0, 1, \ldots, 2k-1, 2k, 2(k+1), 2(k+2), \ldots, 2(n-k-1), 2(n-k)\}. \tag{2}$$

Then the top row is either $\{0, 1, \ldots, n\}$, for even $n$, or $\{0, 1, \ldots, n-2, n-1, n+1\}$, for odd $n$, and the bottom row is $\{0, 1, 2, 4, 6, 8, \ldots, 2(n-1)\}$. Note that all consecutive rows are set for elimination. This will continue to the end of the algorithm.

*STEP 3.* Mark with an asterisk the unique power of 2 among the last entries in all rows. Note that every row strictly above this unique row has the same number of intruders, and the rest of the rows all have one fewer than this common number. Cut off the largest intruders in the rows above the asterisk. The effect is that these will be replaced by the power of 2 marked with asterisk, and all rows will then have the same number of intruders.

Suppose the asterisk entry $2^\eta$ occurs in the $\ell$th row. Then after Step 3, the $k$th row from the bottom, for $k \leq \ell$, is as in (2), while the one for $k > \ell$ is

$$\{0, 1, \ldots, 2k-1, 2k, 2(k+1), 2(k+2), \ldots, 2(n-k-1), 2^\eta\}. \tag{3}$$

The top row is $\{0, 1, \ldots, n-1, 2^\eta\}$, regardless of the parity of $n$, and the bottom row is $\{0, 1, 2, 4, 6, 8, \ldots, 2(n-1)\}$. So the top and bottom rows are set for elimination, and the next step is granted.

*STEP 4.* If the last intruder in the top row is even, cut off the largest intruders in all rows. Otherwise, delete the first row and cut off the largest intruders in the remaining rows.

Repeat Step 4 until all intruders are eliminated and the array has shrunk to a single row, the desired $\{0, 1, 2, 4, 8, \ldots, 2^{n-1}\}$.

**Remarks.** We close the section with a few more details on how each application of either of the last two steps is granted by the conclusion of the previous step.

After Step 2, the odd intruders in the $k$th row from the bottom are $3, 5, \ldots, 2k-1$. Their number is $\alpha(k) = k-1$. The even intruders in the same row are all positive even entries, minus all powers of 2. Their count is $\beta(k) = n-k-\eta(k)$, where $\eta(k) = \lfloor \log_2 2(n-k) \rfloor$. Then $\eta(k) = \eta$, when $k \le \ell$, and $\eta(k) = \eta + 1$, when $k > \ell$. The total number of rows is $\nu = \lfloor n/2 \rfloor$, and the total number of intruders in row $k$, denoted as $\gamma(k)$, is $n-\eta$, when $k > \ell$, and $n-\eta-1$, when $k \le \ell$. Thus all actions in Step 3 are granted.

At the end of Step 3, or before each application of Step 4, if the last intruder in the top row is even, then Step 4 removes it together with its double in the bottom row, so that the new top and bottom rows are set for elimination, based on the same property for the old rows. If the last intruder in the first row is odd, say $2s+1$, then the top row is $0, 1, \ldots, 2s+1$ followed by 2-powers, the row below it is $0, 1, \ldots, 2s$, and one more even intruder followed by 2-powers, and the bottom row is $0, 1, 2, 4, 6, \ldots, 4s, 4s+2$ followed by 2-powers. After Step 4, the top row will be $0, 1, \ldots, 2s$ followed by 2-powers, and the bottom row will be $0, 1, 2, 4, 6, \ldots, 4s$ followed by 2-powers, making them set for elimination. Thus the next application of Step 4 is granted in all cases.

This completes the proof of Lemma 5 and implicitly the one of Theorem 2.

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DEPARTMENT OF MATHEMATICS, DEPAUL UNIVERSITY, CHICAGO, IL 60614
*Email address*: mash@depaul.edu

DEPARTMENT OF MATHEMATICS, DEPAUL UNIVERSITY, CHICAGO, IL 60614
*Email address*: scatoiu@depaul.edu

DEPARTMENT OF MATHEMATICS, CALIFORNIA STATE UNIVERSITY, SAN BERNARDINO, CA 92407
*Email address*: hfejzic@csusb.edu