$\sum_{n=2}^{\infty} \frac{1}{nH_{n-1}}$ diverges while $\sum_{n=2}^{\infty} \frac{1}{nH_n^{1+\varepsilon}}$ converges

The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is an example of a divergent series with positive terms where the general term tends to zero as n tends to infinty. For any integer number $n \geqslant 1$, the nth harmonic number H_n, is defined by $H_n = \sum_{k=1}^{n} \frac{1}{k}$. Then the terms of $\sum_{n=2}^{\infty} \frac{1}{nH_{n-1}}$ grow much more slowly than those of $\sum_{n=1}^{\infty} \frac{1}{n}$ since $\lim_{n \to \infty} \frac{\frac{1}{nH_{n-1}}}{\frac{1}{n}} = 0$. However,

Theorem 1: $\sum_{n=2}^{\infty} \frac{1}{nH_{n-1}}$ is divergent.

Proof: Here we present in Figure 1 a visual proof of this fact, following [1].

Raising the second factor of each summand to the $1 + \varepsilon$ power 'only slightly' shrinks the size of each summand, especially if ε is very small. But this modification is enough to transform divergence into convergence.

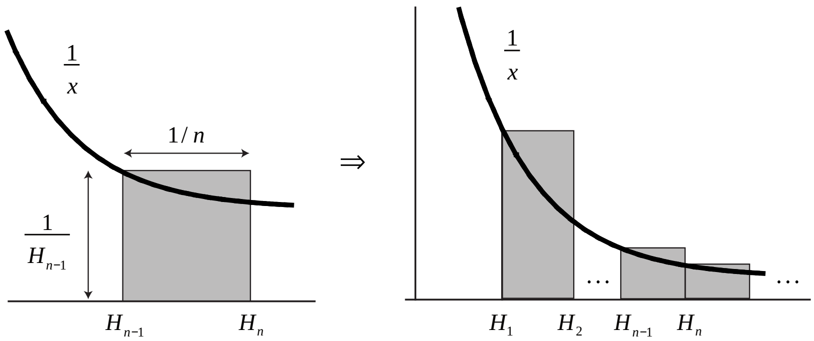

FIGURE 1

Theorem 2: Let $\varepsilon > 0$. Then $\sum_{n=2}^{\infty} \frac{1}{nH_n^{1+\varepsilon}}$ converges.

Proof: The proof follows from considering Figure 2 below.

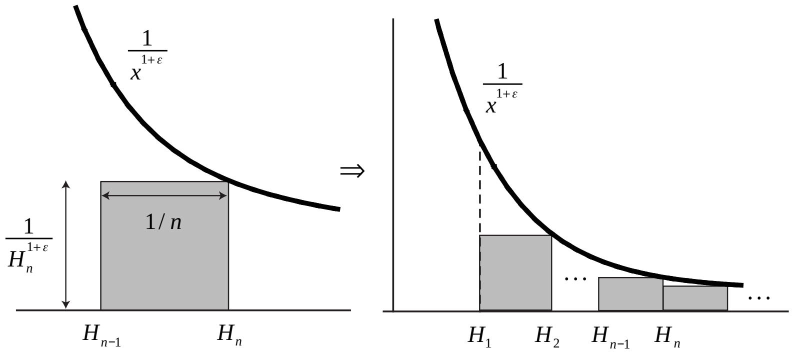

FIGURE 2

Remark: Each term of the series of Theorem 1 is a rational number. If ε is chosen to be a positive whole number, then this is also true for the series of Theorem 2.

Reference

1. J. Marshall Ash, Neither a Worst Convergent Series nor a Best Divergent Series Exists, *The College Mathematics J*. **28** (4) (1997) pp. 296-297.

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