A NEW PROOF OF THE GGR CONJECTURE

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Arguably, the most important application of higher order derivatives is Taylor's theorem, asserting that an n times differentiable function f at a point c is approximated near c by a polynomial p with error $f(c+h)-p(c+h)=o(h^n)$ as $h\to 0$. It is also well known that Taylor's theorem provides only a sufficient condition for this approximation to happen, and all functions f with this property are said to be n times Peano differentiable at c.

In 1998, Ginchev, Guerragio, and Rocca (GGR) conjectured the following result:

GGR Conjecture. *When $n\geq 2$, the following two conditions,*

(i) *f is $n-1$ times Peano differentiable at c and*

(ii) $\displaystyle\lim_{h\to 0}\frac{1}{h^n}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}f(c+(j-k)h)$ *exists for all k with $0\leq k\leq n-1$,*

are sufficient to make f an n times differentiable function at c.

They proved the theorem by hand for $n=2,3,4$ in [8], and with the use of a computer they proved it for $n=5,6,7,8$ in [9], leaving the rest as a conjecture. The GGR conjecture was recently proved in [2] and is now a theorem. The result proved in [2] is slightly stronger for n odd, by eliminating the second condition for $k=0$. A variant of the conjecture, where the bounds for k are replaced by $-(n-2)\leq k\leq 0$, is proved in [3].

The original statement of the GGR conjecture is actually an equivalent version of the above, by the principle of mathematical induction: If $D_{n,k}$ denote the above limits, they conjectured that the $\binom{n+1}{2}$ limits $D_{m,k}$ for $0\leq k<m\leq n$ will be enough for f to be n times Peano differentiable at c.

The purpose of this note is to provide a new, simple proof to both the GGR conjecture and its variant.

Let $R_n(h)$ be the difference defined recursively by $R_1(h)=f(c+h)-f(c)$, and $R_n(h)=R_{n-1}(2h)-2^{n-1}R_{n-1}(h)$ for $n\geq 2$.

We will use the following 1936 result of Marcinkiewicz and Zygmund in [10].

Theorem 1. *Suppose f is $n-1$ times Peano differentiable at c. If $\lim_{h\to 0}R_n(h)/h^n$ exists, then f is n times Peano differentiable at c.*

If we denote $\Delta_k(h)$ as the difference $\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}f(c+(j-k)h)$ in the GGR limit condition, then this condition can be concisely written as $\lim_{h\to 0}\Delta_k(h)/h^n$ exists. It is also obvious that if all GGR limit conditions are met then the following linear combination

$$\lim_{h\to 0}\frac{\sum_k c_k\Delta_k(s_kh)}{h^n}\text{ exists, where } c_k, s_k \text{ are arbitrary real constants.}$$

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Hence, if we can show that $R(h)$ is a linear combination $\sum_k c_k\Delta_k(s_k h)$, then the GGR conjecture follows from Theorem 1.

That this is indeed the case will follow from the analogous result for polynomials, via the linear isomorphism, $\Delta(h) := \sum c_j f(c+b_j h) \mapsto d(t) := \sum c_j t^{b_j}$, from the $\mathbb{R}$-space of all differences of f at c and h with integer nodes (the b_j), to the $\mathbb{R}$-space $\mathbb{R}[t,t^{-1}]$ of all Laurent polynomials in indeterminate t with real coefficients. In this way, (1) if $\Delta_k(h) = \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}f(c+(j-k)h)$, then $d_k(t) = \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}t^{j-k} = t^{-k}(t-1)^n$; (2) the polynomial corresponding to $\Delta_k(sh)$ is $d_k(t^s)$; and (3) if $r_n(t)$ is the polynomial that corresponds to $R_n(h)$ under this linear isomorphism, then its recursive definition is $r_1(t) = t-1$, and $r_n(t) = r_{n-1}(t^2) - 2^{n-1}r_{n-1}(t)$ for $n \geq 2$.

Based on these properties of the above linear isomorphism, we will be done by showing that the following result is true.

Theorem 2. *There are constants c_k and s_k such that $r_n(t) = \sum c_k d_k(t^{s_k})$.*

Before proceeding with the proof of Theorem 2, we need to make a clarification. Our solution to the theorem has the numbers s_k non-negative integers instead of real numbers, so we can think of s_k as s, with $s \geq 0$. Then the c_k are viewed as $c_{k,s}$, for a more precisely indexed sum $\sum c_k d_k(t^{s_k}) = \sum_{s=0}^{\infty}\sum_k c_{k,s}d_k(t^s) = \sum_{s=0}^{\infty}\sum_k c_{k,s}t^{-sk}(t^s-1)^n$, where the ranges for k will be clarified later, since these are different for different cases in the proof of the theorem. Summarizing, in order to prove Theorem 2, it suffices to show that

$$r_n(t) \in V_n := \text{span}\{t^{-sk}(t^s-1)^n \mid k = \ldots, s = 0, 1, \ldots\}.$$

The proof of Theorem 2 is much shorter for the variant of the GGR conjecture than it is for the conjecture itself. For this reason, we deal with the variant first.

Proof of Theorem 2 (Variant Case). In this case, by replacing the index of summation k with $-k$, the range $-(n-2) \leq k \leq 0$ becomes $0 \leq k \leq n-2$. In this way, V_n becomes

$$V_n = \text{span}\{t^{sk}(t^s-1)^n \mid k = 0, \ldots, n-2;\ s = 1, 2, \ldots\}.$$

The following lemma provides a new set of generators for the space V_n.

Lemma 3. $V_n = \text{span}\{(t^s-1)^{n+k} \mid k = 0, 1, \ldots, n-2,\ s = 1, 2, \ldots\}$.

Proof. It suffices to show the following equality of subspaces:

$$\text{span}\{t^k(t-1)^n \mid k = 0, 1, \ldots, n-2\} = \text{span}\{(t-1)^{n+k} \mid k = 0, 1, \ldots, n-2\}.$$

Indeed, this is the result of multiplying by $(t-1)^n$ both sides of the obvious equation $\text{span}\{1, t, t^2, \ldots, t^{n-2}\} = \text{span}\{1, t-1, (t-1)^2, \ldots, (t-1)^{n-2}\}$. □

We are now ready to proceed with the proof of Theorem 2 in its variant case.

Proof of Theorem 2 (Variant Case). Induct on n. When $n = 2$, $r_2 = (t-1)^2$ is clearly in V_2. Suppose $r_n \in V_n$, for some n, $n \geq 2$, and prove the same property for $n+1$. By Lemma 3, r_n is a linear combination of polynomials of the form $(t^s-1)^{n+k}$, where s is a positive integer and $k = 0, 1, \ldots, n-2$. By the recursion, $r_{n+1}(t) = r_n(t^2) - 2^n r_n(t)$ will be a linear combination of polynomials

$$(t^{2s}-1)^{n+k} - 2^n(t^s-1)^{n+k}, \text{ for various } k \text{ and } s.$$

By Lemma 3, these polynomials belong to V_{n+1} in all cases, except for $k = 0$, when

$$(t^{2s}-1)^n - 2^n(t^s-1)^n = (t^s-1)^n((t^s+1)^n - 2^n) = (t^s-1)^{n+1}p(t^s),$$

where p is a polynomial in t of degree $n-1$, so that $(t^s-1)^{n+1}p(t^s)$ belongs to the subspace $\text{span}\{(t^s-1)^{n+1}, (t^s-1)^{n+2}, \ldots, (t^s-1)^{2n}\}$ of V_{n+1}. □

Proof of Theorem 2 (GGR Case). The GGR case in the proof of Theorem 2 is similar to the variant case. The proof of the inductive step is now split further into two subcases, n odd and n even. In both cases, following the refined result of the GGR Theorem from [2],

$$V_n = \text{span}\{t^{-sk}(t^s - 1)^n \mid k = (0), 1, \ldots, n-1;\ s = 1, 2, \ldots\},$$

where (0) means that 0 is taken for n even, and not taken for n odd. More explicitly, this is

$$V_n = \begin{cases} \text{span}\{(t-1)^n, t^{-1}(t-1)^n, \ldots, t^{-(n-1)}(t-1)^n, \ldots\}, & n \text{ even}, \\ \text{span}\{t^{-1}(t-1)^n, t^{-2}(t-1)^n, \ldots, t^{-(n-1)}(t-1)^n, \ldots\}, & n \text{ odd}, \end{cases}$$

where the last dots in both cases mean that the generating set also includes the previously listed generators evaluated at t^s, for all s at least 2. Let W_n be the subspace of V_n spanned by all generators with $s = 1$. Then W_n has the expression

$$W_n = \begin{cases} \text{span}\{(t-1)^n, t^{-1}(t-1)^n, \ldots, t^{-(n-1)}(t-1)^n\}, & n \text{ even}, \\ \text{span}\{t^{-1}(t-1)^n, t^{-2}(t-1)^n, \ldots, t^{-(n-1)}(t-1)^n\}, & n \text{ odd}. \end{cases}$$

The following lemma provides new sets of generators for W_n in both parity cases.

Lemma 4. *With the above notation,*

$$W_n = \begin{cases} \text{span}\{t^{-n/2}(t-1)^n, t^{-1}(t-1)^{n+1}, \ldots, t^{-(n-1)}(t-1)^{n+1}\}, & n \text{ even}, \\ \text{span}\{t^{-(n-1)/2}(t-1)^n, t^{-2}(t-1)^{n+1}, \ldots, t^{-(n-1)}(t-1)^{n+1}\}, & n \text{ odd}. \end{cases}$$

Proof. When n is even, the result follows from $t^{-n/2}(t-1)^n$ being one of the generators in the definition of W_n, and $t^{-k}(t-1)^{n+1} = t^{-(k-1)}(t-1)^n - t^{-k}(t-1)^n$, for each $k = 1, \ldots, n-1$. The case when n is odd has a similar proof. □

We are now ready to prove Theorem 2 in the GGR case.

Proof of Theorem 2 (Case GGR). Induct on n. When $n = 1$, $r_1 = t - 1 \in V_1$. We assume that $r_n \in V_n$ and prove that $r_{n+1} \in V_{n+1}$ in two possible cases:

Case 1. When n is even,

$$V_{n+1} = \text{span}\{t^{-1}(t-1)^{n+1}, t^{-2}(t-1)^{n+1}, \ldots, t^{-n}(t-1)^{n+1}, \ldots\}.$$

The inductive hypothesis and Lemma 4 imply that $r_n(x)$ is a linear combination of

$$t^{-n/2}(t-1)^n \text{ and } t^{-k}(t-1)^{n+1}, \text{ for } k = 1, \ldots, n-1,$$

and their evaluations at t^s, for s at least 2. Then $r_{n+1}(t) = r_n(t^2) - 2^n r_n(t)$ is a linear combination of two kinds of polynomials and their evaluations at t^s, for $s \geq 2$. The first kind of polynomial has the form $t^{-n}(t^2-1)^n - 2^n t^{-n/2}(t-1)^n$

$$= t^{-n}(t-1)^n((t+1)^n - 2^n t^{n/2}) = t^{-n}(t-1)^{n+1}p(t),$$

where $p(t)$ is a polynomial degree $n-1$, hence the whole expression lives inside of

$$(t-1)^{n+1}\text{span}\{t^{-1}, t^{-2}, \ldots, t^{-n}\},$$

a subspace of V_{n+1}. The polynomials of the second kind are polynomials of the form

$$t^{-2k}(t^2-1)^{n+1} - 2^n t^{-k}(t-1)^{n+1}, \text{ for } k = 1, \ldots, n-1.$$

Their second term is a scalar multiple of a generator of V_{n+1}, while their first term is an $(s = 2)$-dilation of the same generator, so all polynomials of the second kind also belong to V_{n+1}. We conclude that $r_{n+1} \in V_{n+1}$, as needed.

Case 2. When n is odd,

$$V_{n+1} = \text{span}\{(t-1)^{n+1}, t^{-1}(t-1)^{n+1}, \ldots, t^{-n}(t-1)^{n+1}, \ldots\}.$$

The inductive hypothesis and Lemma 4 imply that $r_n(x)$ is a linear combination of

$$t^{-(n-1)/2}(t-1)^n \text{ and } t^{-k}(t-1)^{n+1}, \text{ for } k=2,\dots,n-1,$$

and their evaluations at t^s, for s at least 2. Then $r_{n+1}(t)=r_n(t^2)-2^n r_n(t)$ is a linear combination of two kinds of polynomials and their evaluations at t^s, for $s\ge 2$. The first kind of polynomial is of the form $t^{-(n-1)}(t^2-1)^n-2^n t^{-(n-1)/2}(t-1)^n$

$$=t^{-(n-1)}(t-1)^n((t+1)^n-2^n t^{(n-1)/2})=t^{-(n-1)}(t-1)^{n+1}p(t),$$

where $p(t)$ is a polynomial degree $n-1$, hence the above expression lives inside of

$$(t-1)^{n+1}\mathrm{span}\{1,t^{-1},t^{-2},\dots,t^{-(n-1)}\},$$

a subspace of V_{n+1}. The polynomials of the second kind are polynomials of the form

$$t^{-2k}(t^2-1)^{n+1}-2^n t^{-k}(t-1)^{n+1}, \text{ for } k=2,\dots,n-1.$$

Their second term is a scalar multiple of a generator of V_{n+1}, while their first term is an $(s=2)$-dilation of the same generator, so all polynomials of the second kind also belong to V_{n+1}. We conclude that $r_{n+1}\in V_{n+1}$ in this case as well. □

The limit $D_{n,0}=D_{n,0}f(c)$ we had at the beginning is called the n-th Riemann derivative of f at c, and $D_{n,n/2}f(c)$ is the n-th symmetric Riemann derivative. Both of these derivatives were invented by Riemann in the mid 1800s, see [12]. The Peano derivatives were invented by Peano in [11] in 1892, and then developed greatly by de la Vallée Poussin in [7]. For this reason, they are often referred to as de la Vallée Poussin derivatives. Closed form formulas for the differences $R_n(h)$ are deduced in [4]; they involve the Gaussian or q-binomial coefficients, so they are q-analogues of the Riemann differences. Other q-analogues of Riemann differences are found in [1] and [5]. Article [6] sheds some light towards extending the GGR conjecture for $n=1$.

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