Iterated harmonic numbers

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Summary The harmonic numbers are the sequence $1, 1 + 1/2, 1 + 1/2 + 1/3, \cdots$. Their asymptotic difference from the sequence of the natural logarithm of the positive integers is Euler's constant gamma. We define a family of natural generalizations of the harmonic numbers. The jth iterated harmonic numbers are a sequence of rational numbers that nests the previous sequences and relates in a similar way to the sequence of the jth iterate of the natural logarithm of positive integers. The analogues of several well-known properties of the harmonic numbers also hold for the iterated harmonic numbers, including a generalization of Euler's constant. We reproduce the pretty proof that only the first harmonic number is an integer and, providing some numeric evidence for the cases $j = 2$ and $j = 3$, conjecture that the same result holds for all iterated harmonic numbers.

Introduction: definitions and properties

The harmonic numbers are the sequence $\left\{1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{3}, \cdots\right\}$; we denote them as $h_1(n) := \sum_{k=1}^{n} \frac{1}{k}$. Some of their properties are:

1. They are positive rational numbers, starting at 1 and monotonically increasing,
2. They are the partial sums of the divergent infinite series $\sum_{k=1}^{\infty} \frac{1}{k}$,
3. They have a direct connection with the natural logarithm, in particular there is a constant γ so that as $n \to \infty, h_1(n) - \ln n \to \gamma$,
4. They are "close" to very similar, but convergent, sequences; $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, but, for any fixed small positive number ϵ, $\sum_{k=1}^{\infty} \frac{1}{k^{1+\epsilon}}$ converges,
5. The only harmonic number that is an integer is 1.

We define the iterated harmonic numbers of order j, $h_j(n)$, for $j = 2, 3, \cdots$. First define $h_2(n) := \sum_{k=1}^{n} \frac{1}{kh_1(k)}$, then $h_3(n) := \sum_{k=1}^{n} \frac{1}{kh_1(k)h_2(k)}$, and so on. Thus, for every integer $j \geq 2$,

$$h_j(n) := \sum_{k=1}^{n} \frac{1}{kh_1(k)\, h_2(k) \cdots h_{j-1}(k)}.$$

To motivate these definitions, start with $\ln x = \int_1^x \frac{dt}{t}$. Second, let $u = \ln t, du = \frac{dt}{t}$ to see that $\int_e^x \frac{1}{t \ln t}\, dt = \int_1^{\ln x} \frac{du}{u} = \ln \ln x - 0$, so that $\ln_2 x := \ln \ln x = \int_e^x \frac{dt}{t \ln t}$.* Third, let $u = \ln_2 t, du = \frac{dt}{t \ln t}$,so that $\int_{e^e}^x \frac{dt}{t \ln t \ln_2 t} = \int_1^{\ln_2 x} \frac{du}{u} = \ln(\ln_2 x) =: \ln_3 x$. By now it is clear that each h_j is like the corresponding iterated logarithm $\ln_j$ in the sense that given appropriate conditions on the function f, the sum $\sum_{k=1}^{n} f(k)$ is like the integral $\int_a^n f(x)\, dx$.

We define p-adic valuations. We use 2-adic valuations to prove Property 5.

Then we study empirically the conjecture that Property 5 extends to h_j for every j. At this point we will have seen from the previously presented proof of Property 5 that for all $n \geq 2$, $h_1(n)$ has an even denominator, so that it cannot be an integer. We only present numerical data for the cases of $j = 3$ and $j = 2$. Our computations indicate that this even denominator property essentially holds for h_3. The case of h_2

*This notation for the second iteration of the natural logarithm should not be confused with $\log_2 x$, the base 2 logarithm of x. Only natural logarithms will be iterated in this work.

is different, but our evidence, presented in Table 1, suggests a different argument can be found here also.

Now we define for each positive integer j two analogues of Euler's constant γ. We define γ'_j to be a constant satisfying

$$\sum_{a<k\leq n} \frac{1}{k \ln k \ln_2 k \cdots \ln_{j-1} k} - \ln_j n = \gamma'_j + o\left(1\right). \tag{1}$$

Here $\ln_j x$ denotes the jth iterated logarithm: $\ln_1 x = \ln x$, and $\ln_j x = \ln(\ln_{j-1} x)$ for $j = 2, 3, \ldots$. (Pick $a = a\left(j\right)$ so large that $\ln_{j-1}\left(a\right) \geq 0$; for example, if $j \geq 3$, one may choose a to be ${}^{j-2}e$, where the constants ${}^{j}e$ are defined recursively by ${}^{0}e := 1$ and ${}^{j+1}e := e^{({}^{j}e)}$ for $j = 1, 2, \ldots$. This choice is natural, since $\int_{{}^{j-2}e}^{x} \frac{dt}{t \ln t \cdots \ln_{j-1} t} = \ln_j x$.) The sum on the left side of equation (1) will be called $l_j\left(n\right)$ and is an analogue of $h_j\left(n\right)$. Note that $l_1\left(n\right) = h_1\left(n\right)$.

We also define γ_j to be a constant satisfying

$$h_j\left(n\right) - \ln_j n = \sum_{k=1}^{n} \frac{1}{k h_1\left(k\right) h_2\left(k\right) \cdots h_{j-1}\left(k\right)} - \ln_j n = \gamma_j + o\left(1\right). \tag{2}$$

Also observe that $\gamma'_1 = \gamma_1 = \gamma$. Estimates like equation (1) for iterated logarithms are already known, with much better estimates for the error term than $o\left(1\right)$. After Table 1 this fact will be used as a lemma assisting the proof of a sharper version of equation (2) which will prove the existence of γ_j for all $j \geq 2$.

The motivation for Property 4 and its generalization is explained in [**3**]. The infinite series $\sum_{a<k<\infty} \frac{1}{k \ln k \ln_2 k \cdots \ln_{j-1} k}$ is divergent since by relation (1) its partial sums increase boundlessly. However, for each $\epsilon > 0$, the series

$$\sum_{a<k<\infty} \frac{1}{k \ln k \ln_2 k \cdots \ln_{j-2} k \left(\ln_{j-1} k\right)^{1+\epsilon}}$$

converges. Similar results also hold for iterated harmonic numbers. The infinite series

$$\sum_{k=1}^{\infty} \frac{1}{k h_1\left(k\right) h_2\left(k\right) \cdots h_{j-1}\left(k\right)}$$

is divergent by relation (2). The integral test shows that for each $\epsilon > 0$, the series

$$\sum_{k=1}^{\infty} \frac{1}{k h_1\left(k\right) h_2\left(k\right) \cdots h_{j-2}\left(k\right) \left(h_{j-1}\left(k\right)\right)^{1+\epsilon}}$$

converges. These pairs of infinite series are "closer" to each other as j increases in the sense that the divergent one diverges more slowly while the other one converges more slowly. The logarithmic examples do not have rational partial sums, but the divergent harmonic sums do. If we set $\epsilon = 1$, then the convergent harmonic series also have rational partial sums.

The case $j = 2$, where $\sum_{k=1}^{\infty} \frac{1}{k h_1(k)}$ being divergent is contrasted with $\sum_{k=1}^{\infty} \frac{1}{k (h_1(k))^{1+\epsilon}}$ being convergent appears in [**4**]. (What we denote as $h_1\left(n\right)$ is written as h_n in [**4**].) Working with $\sum \frac{1}{k(h_1(k))}$ led us to the idea of defining jth iterated harmonic numbers.

As our last topic, we compare our iterated harmonic numbers to the hyperharmonic numbers of JH Conway and R Guy, a different set of sequence generalizing the harmonic numbers.

The p-adic valuation and a proof of Property 5

Fix a prime number p. We define the p-adic valuation ν_p on the field of rational numbers by $\nu_p(0) = -\infty$ and for all other rationals a/b we define ν_p to be r where $a/b = p^r \cdot a'/b'$ with a', b' coprime to p.* Examples:

$$\nu_2(8) = 3, \nu_2(1/8) = \nu_2(-5/8) = \nu_2(-5/56) = \nu_2(10/112) = -3.$$

Notice in particular that the p-valuation of a fraction remains the same whether or not that fraction is reduced. We will make use of an important equality satisfied by every ν_p. It asserts that

$$\nu_p(x+y) = \min\{\nu_p(x), \nu_p(y)\} \text{ when } \nu_p(x) \neq \nu_p(y). \tag{3}$$

We generalize this to

$$\nu_p(x_1 + x_2 + \cdots + x_n) = \min\{\nu_p(x_1), \cdots, \nu_p(x_n)\}$$
$$\text{when the minimum occurs exactly once.} \tag{4}$$

To see why this is true, we write out the $n = 3$ case. Without loss of generality, suppose $x_i = p^{r_i}\frac{a_i}{b_i}, i = 1, 2, 3$ with all a_i, b_i coprime with p, $r_2 > r_1$, and $r_3 > r_1$. Compute

$$x_1 + x_2 + x_3 = p^{r_1}\frac{a_1}{b_1} + p^{r_2}\frac{a_2}{b_2} + p^{r_3}\frac{a_3}{b_3} =$$

$$p^{r_1}\frac{a_1b_2b_3 + p^{r_2-r_1}a_2b_1b_3 + p^{r_3-r_1}a_3b_1b_2}{b_1b_2b_3}$$

The denominator is coprime to p. The numerator is congruent mod p to $a_1b_2b_3$ and hence is also coprime with p. So $\nu_p(x_1 + x_2 + x_3) = r_1$.

Property 5 was proved by Theisinger in 1915 [**11**]. Conway and Guy present his proof thus: Look at the term [of $h_1(n)$] with the highest power of 2 in it. It has nothing with which to pair. So $h_1(2), h_1(3), h_1(4), \cdots$ have odd numerator and even denominator. □ This proof is correct but insufficiently explicit. We now give our version using the language of valuations.

Note that a rational number with a negative 2-valuation cannot be an integer. More precisely, if a rational number a/b is such that $\nu_2(a/b) = -r < 0$ then b is divisible by the even integer 2^r. Thus the absolute value of the denominator remains at least 2, even after writing a/b in reduced form, and a/b is not an integer. So to prove Property 5, we need only show $\nu_2(h_1(n)) < 0$ for all $n \geq 2$.

Fix $n \geq 2$. Choose r maximal so that $2^r \leq n$. We note that $\nu_2(\frac{1}{2^r}) = -r$. Also by equation (4),

$$\nu_2(h_1(n)) = \nu_2(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^r} + \cdots + \frac{1}{n}) = -r.$$

This holds since for each positive integer $k = 2^\rho\sigma$ with ρ an integer and σ an odd integer such that $k \in [1, n]$ and $k \neq 2^r$, the definition of r forces ρ to be less than r. So $\nu_2(\frac{1}{k}) = -\rho > -r$; the unique term where the minimum $-r$ occurs is $\frac{1}{2^r}$. □

*The literature splits on defining *valuation* either: as the p-adic *norm*, $|p^r \cdot a'/b'|_p = p^{-r}$ except $|0|_p = 0$; or as the convention we follow, $\nu_p(p^r \cdot a'/b') = r$ except $\nu_p(0) = -\infty$. The norm definition satisfies the triangle inequality, $|x+y|_p \leq |x|_p + |y|_p$, and the strong triangle inequality, $|x+y|_p \leq \max(|x|_p, |y|_p)$. The equality case of the strong triangle inequality is equivalent to equation (3).

We also provide Kürschák's simple and elegant proof that invokes a number theory result.[**8**] Bertrand's Postulate, i.e., there is always a prime p in $(\lfloor n/2 \rfloor, n]$, implies that $h_1(n) = \sum_1^n \frac{1}{k} = 1 + \frac{1}{2} + \cdots + \frac{1}{\lfloor n/2 \rfloor} + \cdots + \frac{1}{p} + \cdots + \frac{1}{n} = \frac{1}{p} + \frac{a}{b}$ (with a, b both integers) is never an integer. Every denominator in the sum, except p itself, is relatively prime to p. (Other than p, the smallest number not relatively prime to p is $2p$, which is too large to be among the denominators ending at n.) The denominator b of the sum of fractions with denominators relatively prime to p is also relatively prime to p. Thus $h_1(n) = \frac{1}{p} + \frac{a}{b}$ is not an integer, since marching from $1/p$ in a steps of length $1/b$ cannot reach an integer. To formalize this, change scale by multiplying everything by b. We are now marching from b/p to $bh_1(n)$ in a steps of length 1. Since b and p are relatively prime, b/p is not an integer, so neither is its translation $bh_1(n)$, so neither is $h_1(n)$.

[**1**] provides a proof of Bertrand's Postulate. The Theisinger proof of property 5 requires only arithmetic and does not rely on Bertrand's Postulate.

A conjecture for *j*-times iterated harmonic numbers

From the definitions, it is immediate that for each $j \geq 1$, there holds the identity

$$h_j(1) = 1.$$

Conjecture 1. For each integer $j \geq 2$, the only jth iterated harmonic number that is an integer is $h_j(1)$.

Recall that when $j = 1$, the statement of the conjecture becomes Property 5 that was mentioned and proved in the introduction.

In computing numerical evidence, we restrict interest to the cases of $j = 2$ **and** $j = 3$**.** Write every rational number $h_j(n)$, $j = 2, 3$; $n = 2, 3, \ldots$ as a reduced ratio of positive integers, $\frac{n_j(n)}{d_j(n)}$. It suffices to prove that each such denominator $d = d_j(n)$ is at least 2. All the evidence we have accumulated points to the conjecture's truth. The rough argument is that the denominators of both $h_2(n)$ and $h_3(n)$ seem to grow steadily larger as n increases, whereas the failure of the conjecture would have a denominator of 1 occurring. The proof for the h_1 case involved showing that all denominators were even and hence not equal to 1. The evidence for h_3 does not rule out such a proof. The evidence for h_2 rules out an even denominator proof. Nevertheless it suggests many other possible proofs. If positive integer d is divisible by prime p then the p-valuation of d is $\max\{\nu : p^\nu | d\}$. The p-valuation of d is zero when p does not divide d. Notice that the proof of Property 5. given above shows that the 2-valuation of the denominator of $h_1(n)$ is $\lfloor \log_2 n \rfloor$.

Numerical evidence when $j = 3$: We computed the 2-valuation of the (reduced) denominator of $h_3(n)$ for $n = 1$ to $2,000$. For $n = 1$, the 2-valuation is, of course, 0. For $n = 2$ to $n = 11$ the 2-valuation is 2. For $n = 12$, the 2-valuation is, shockingly, 0. (Nonetheless, $h_3(12)$ is not an integer.) From $n = 13$ through $n = 31$ the 2-valuation is 3. From $n = 32$ to $n = 2,000$, the 2-valuation is always equal to 6.

Numerical evidence when $j = 2$: We attempted to calculate the p-valuation for the denominator of $h_2(n)$ for $n = 1$ to $n = 40,000$ for all 46 primes less than 200. We ceased computation when we hit computer system limits at $n = 27,477$. The 2-valuation is always 0. The 3-valuation is 1 from $n = 2$ to $n = 53$. The 3-valuation then alternates irregularly between 0 and 1. The left panel of Table shows the variation in the 3-valuation of $h_2(n)$ up to $n = 27,477$.

3-valuation of denom($h_2(n)$)		97-valuation of denom($h_2(n)$)	
n	3-valuation	n	97-valuation
1	0	1–10	0
2–53	1	11–95	1
54–62	0	96–9,322	2
63–65	1	9,323–9,407	1
66–161	0	9,408–27,477	0
162–188	1		
189–197	0		
198–1,457	1		
1,458–1,700	0		
1,701–1,781	1		
1,782–4,373	0		
4,372–5,102	1		
5,103–5,345	0		
5,346–27,477	1		

TABLE 1: Selected p-valuations of the denominator of $h_2(n)$. These data were calculated using PARI/GP [**10**].

The 5-valuation is 2 from $n = 4$ to $n = 2,499$. The 5-valuation is then 0 from $n = 2,500$ to $n = 2,999$. The 5-valuation is then 1 from $n = 3,000$ to $n = 12,499$, and then the 5-valuation remains at 2 through the end of the run (at $n = 27,477$).

Of the 43 remaining primes less than 200, all of them enter with non-zero p-valuations as n grows. For some primes the first non-zero valuation is 1 and in other cases the first non-zero p-valuation is 2. For example, up to $n = 6$, the 7-valuation is 0; beginning with $n = 6$, the 7-valuation is 2 through the end of the run. Only the 11-valuation exceeds 2 in the run; at $n = 848$, the 11-valuation of $h_2(n)$ becomes 3 and remains 3 through the end of the run.

With a single exception among the primes between 7 and 200, once the prime acquires a non-zero p-valuation, the p-valuation does not decline. At $n = 9,323$, the 97-valuation returns to 1 (from 2), and beginning at $n = 9,408$, the 97-valuation falls to 0 where it remains through the remainder of the run. Thus, from $n = 9,408$ through the end of the run ($n = 27,477$), 2 and 97 are the only primes less than 200 for which in the denominator of $h_2(n)$ has a p-valuation of zero. The behavior of the 97-valuation is tabulated in right panel of Table .

Our best guess is that all the odd h_j have similar behavior so that the conjecture will be true for the jth iterated harmonic numbers and provable by showing the 2-valuation of the denominators to be positive. We also guess that the conjecture will hold for all the even h_j, but that the proof will be quite difficult.

Asymptotic estimates for iterated harmonic numbers

Recall $l_j(n)$ was defined just after equation (1) to be equal to $\sum_{a<k\le n} \frac{1}{k \ln k \ln_2 k \cdots \ln_{j-1} k}$ for an appropriate constant a.

Theorem 2. *Notice that $l_1(n) = h_1(n)$ for $n \ge 1$. For each integer $j \ge 2$, there is a constant γ_j' such that*

$$l_j(n) - \ln_j(n) = \gamma_j' + \frac{1}{2n \ln n \ln_2 n \cdots \ln_{j-1} n} + O\left(\frac{1}{n^2 \ln n \ln_2 n \cdots \ln_{j-1} n}\right). \quad (5)$$

Proof. The function $f(x) = \frac{1}{x \ln x \ln_2 x \cdots \ln_{j-1} x}$ has antiderivative $\ln_j(x)$ and satisfies

$f''(n) = O\left(\frac{1}{n^3 \ln n \ln_2 n \cdots \ln_{j-1} n}\right)$. Apply the Euler summation formula to f. See exercises 2 and 6 of section 15.23 of [**2**]. □

The finding in Theorem 2 that the error in relation (1) is smaller than $O(\frac{1}{n^2})$ implies that the left side of relation (1) well approximates the γ_j'. We now show that there are much larger errors when approximating the γ_j from relation (2), so large that the γ_j cannot be effectively calculated nor can the approximation be improved. We write $f(n) \asymp g(n)$ to mean that there are two positive constants a and b so that $af(n) < g(n) < bf(n)$; in words, f is of the same order as g [**7**, page 7].

Theorem 3. *It is well known that*

$$h_1(n) - \ln n - \gamma \asymp \frac{1}{n} \tag{6}$$

where $\gamma = .577\ldots$ is Euler's constant. For each integer $j \geq 2$, $h_j(n)$ tends to ∞ at the same rate as the jth iterated logarithm. More quantitatively, for every $j = 2, 3, \ldots$, there are constant γ_j such that

$$h_j(n) - \ln_j n - \gamma_j \asymp \frac{1}{\ln_{j-1} n}. \tag{7}$$

On the positive side, this theorem establishes the existence of the γ_j for all $j = 2, 3, \cdots$. But on the negative side, this theorem proves that the natural extension of Property 4 converges so slowly that it can not provide an efficient way to compute the numerical value of γ_j when $j \geq 2$.

We defer its proof to an appendix to improve the flow of of the paper.

The hyperharmonic numbers of Conway and Guy

In *The Book of Numbers*, Conway and Guy generalize the harmonic numbers to the hyperharmonic numbers [**5**]. In their notation, the sequence of harmonic numbers are designated as $H_n^{(1)}$, the sequence of second harmonics is defined by $H_n^{(2)} = H_1^{(1)} + H_2^{(1)} + \cdots + H_n^{(1)}$, the sequence of third harmonics is defined by $H_n^{(3)} = H_1^{(2)} + H_2^{(2)} + \cdots + H_n^{(2)}$, and so on.

The Book of Numbers just displays the hyperharmonic numbers without motivating their existence. A natural motivation comes from summability of infinite series. We look at Ernest Cesàro's sequence of summation methods $(C, k), k = 0, 1, 2, \ldots$. (See section 5.4 of [**6**].) Let $\Sigma = \sum_{i=1}^{\infty} a_i$ be an infinite series. Let $A_n^{(1)}$ be the sequence of partial sums of the series, let $A_n^{(2)} = A_1^{(1)} + A_2^{(1)} + \cdots + A_n^{(1)}$, and $A_n^{(3)} = A_1^{(2)} + A_2^{(2)} + \cdots + A_n^{(2)}$, and so on.

Obviously the hyperharmonic number $H_n^{(k)}$ is exactly the number $A_n^{(k)}$ for the special case where the original series is specialized to $a_i = \frac{1}{i}$ for all i. However it seems like the motivation goes no deeper. The next paragraph sketches how the $A_n^{(k)}$ fit into summability theory.

When $\lim_{n\to\infty} A_n^{(1)} = A^{(1)}$, say that Σ is $(C, 0)$ summable to $A^{(1)}$, so $(C, 0)$ summability is ordinary convergence. When $\lim_{n\to\infty} \frac{A_n^{(2)}}{n} = A^{(2)}$, say that Σ is $(C, 1)$ summable to $A^{(2)}$. When $\lim_{n\to\infty} \frac{A_n^{(3)}}{\binom{n+1}{2}} = A^{(3)}$, say that Σ is $(C, 2)$ summable to $A^{(3)}$. For $i < j, A^{(i)}$ exists implies $A^{(j)}$ exists and $A^{(j)} = A^{(i)}$. The series $1 -$

$1+1-1+\cdots$ is not $(C,0)$ summable, but is $(C,1)$ summable to $\frac{1}{2}$. The series $1-2+3-4+\cdots$ is not $(C,1)$ summable, but is $(C,2)$ summable to $\frac{1}{4}$.

During the twenty year period after the creation of the hyperharmonic numbers, a lot of evidence, both numerical and theoretical, was piling up in support of the conjecture that 1 was the only hyperharmonic integer. However, in 2020, some very large hyperharmonic integers were revealed, the smallest of which is $H_{33}^{(64(2^{2659}-1)+32)}$ [**9**].

Musings. For each $j=2,3,\cdots$, our iterated harmonic numbers $h_j(\cdot)$ and the Conway–Guy hyperharmonic numbers $H_\cdot^{(j)}$ are sequences of rational numbers tending to infinity. When $j=1$, both generalizations coincide with the harmonic numbers and hence contain no integers larger than 1. The most naive perspective suggests that the very slowly increasing sequences h_j are more likely to intersect the integers than are the much more rapidly increasing $H^{(j)}$; and that this effect increases with increasing j. Both our computations and Sertbas' example indicate that this perspective is far too simplistic.

It is easy to see that for each j, the sequence $\{h_j(n)\}$ is increasing. It might be fun to prove that each sequences is also convex, i.e., $h_j(n)-2h_j(n+1)+h_j(n+2)>0$ for $n=1,2,\cdots$. This is easy for the harmonic numbers, and we also confirmed the $j=2$ case.

Comparing Theorems 2 and 3 shows that the γ_j' can easily be estimated to several decimal places, while the γ_j cannot. That is, although $h_j-\gamma_j$ is indeed an estimator for the jth iterated logarithm, it is not nearly as good as $l_j-\gamma_j'$. The h_j are rational, but they do not give practical rational approximations for iterated logarithms. Even for just γ_2 and γ_3, it would probably require a new idea to find numerical values to two significant figures.

With respect to the main conjecture, we have a feeling that the odd iteration cases all involve the 2-valuations and may be more like the $j=1$ classical case and the even iteration cases may all be similar to the $j=2$ case. In particular, $j=3$ may be the easiest of all the open cases.

Here is an infinite list of very difficult open questions: are all the γ_j and γ_j' transcendental, or at least irrational? This is a very well known open question when $j=1$. In this case $\gamma_1=\gamma_1'=\gamma$, where $\gamma=.577\ldots$ is Euler's constant.

We have not found any direct generalizations of the harmonic numbers other than the Conway-Guy hyperharmonic numbers and our own iterated harmonic numbers. It would be interesting to see if there have been other generalizations. One place to look could be among various generalizations of Euler's constant γ associated with Stieltjes.

Appendix. The proof of Theorem 3.

Proof. We illustrate an inductive proof by working out the $j=4$ step. It might be a nice exercise for the reader to do the next step, $j=5$, or even to write out the general inductive step. This proof contains all the ideas necessary for the general inductive step. We assume that relation (5) holds for $j=2$ and $j=3$. We estimate

$$h_4(n)-l_4(n)=\sum_{k=a}^{n}\frac{1}{kh_1(k)\,h_2(k)\,h_3(k)}-\frac{1}{k\ln k\ln_2 k\ln_3 k},$$

where $a=\lceil e^e\rceil=16$ is the smallest positive integer for which $\ln_3$ is positive. We subtract and add successively $\frac{1}{kh_1(k)h_2(k)\ln_3 k}$ and $\frac{1}{kh_1(k)\ln_2 k\ln_3 k}$ to the kth summand, getting

$$\sum_{k=a}^{n}\left\{\underbrace{\frac{1}{kh_1(k)\,h_2(k)\,h_3(k)}-\frac{1}{kh_1(k)\,h_2(k)\ln_3 k}}_{I}\right.$$
$$+\underbrace{\frac{1}{kh_1(k)\,h_2(k)\ln_3 k}-\frac{1}{kh_1(k)\ln_2 k\ln_3 k}}_{II}$$
$$\left.+\underbrace{\frac{1}{kh_1(k)\ln_2 k\ln_3 k}-\frac{1}{k\ln k\ln_2 k\ln_3 k}}_{III}\right\}$$

Use relation (7) with $j=3$ to decompose I into I_A and I_B,

$$I=\sum_{k=a}^{n}\frac{\ln_3 k-h_3(k)}{kh(k)\,h_2(k)\,h_3(k)\ln_3 k}$$
$$=\underbrace{\sum_{k=a}^{n}\frac{\gamma_3}{kh(k)\,h_2(k)\,h_3(k)\ln_3 k}}_{I_A}+\underbrace{\sum_{k=a}^{n}O\left(\frac{1}{kh(k)\,h_2(k)\,h_3(k)\ln_3 k\ln_2 k}\right)}_{I_B}.$$

It follows from relation (6) that $h_1(k)\asymp\ln k$. It also holds that $h_2(k)\asymp\ln_2 k$, and $h_3(k)\asymp\ln_3 k$, so that from $\int_x^\infty\frac{dt}{t\ln t\ln_2 t\ln_3^2 t}=\frac{1}{\ln_3 x}$ and the integral test we may write I_A as C_A-R_A,where C_A is the value of the entire infinite sum from a to ∞ and $R_A=\sum_{k=n+1}^{\infty}\frac{1}{k\ln k\ln_2 k\ln_3^2 k}\asymp\frac{1}{\ln_3(n+1)}\asymp\frac{1}{\ln_3 n}$. The argument for I_B is the same, but now the relevant integral is $\int_x^\infty\frac{dt}{(\ln_2 t)\left(t\ln t\ln_2 t\ln_3^2 t\right)}<\frac{1}{\ln_2 x\ln_3 x}$, so $I_B=C_B-R_B$ where $R_B=O\left(\frac{1}{\ln_2 n\ln_3 n}\right)$ is of smaller order than R_A. Thus $I-(C_A+C_B)\asymp\frac{1}{\ln_3 n}$.

Similar calculations show that there are constants C_{II} and C_{III} and so that $II=C_{II}-R_{II}$ and $III=C_{III}-R_{III}$ with both remainders of smaller order than $\frac{1}{\ln_3 n}$. We have shown that $h_4(n)-l_4(n)-(C_A+C_B+C_{II}+C_{III})\asymp\frac{1}{\ln_3 n}$. By Theorem 2, $l_4(n)-\ln_4(n)=\gamma_4'+O\left(\frac{1}{n\ln n\ln_2 n\ln_3 n}\right)$. So defining $\gamma_4=C_A+C_B+C_{II}+C_{III}$, we have shown that $h_4(n)-\ln_4 n-\gamma_4\asymp\frac{1}{\ln_3 n}$. □

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