Gaussian Riemann Derivatives

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ABSTRACT. For a real parameter q, $q \neq 0, \pm 1$, we introduce: two q-analogues of the nth Riemann derivative $D_n f(x)$ of f at x, the nth Gaussian Riemann derivatives ${}_qD_n f(x)$ and ${}_q\bar{D}_n f(x)$ are the nth generalized Riemann derivatives based at $x, x+h, x+qh, x+q^2h, \ldots, x+q^{n-1}h$ and $x+h, x+qh, x+q^2h, \ldots, x+q^n h$; and one analog of the nth symmetric Riemann derivative $D_n^s f(x)$, the nth Gaussian symmetric Riemann derivative ${}_qD_n^s f(x)$ is the nth generalized Riemann derivative based at $(x), x \pm h, x \pm qh, x \pm q^2h, \ldots, x \pm q^{m-1}h$, where $m = \lfloor (n+1)/2 \rfloor$ and (x) means that x is taken only for n even. We provide the exact expressions for their associated differences in terms of Gaussian binomial coefficients; we show that f has n Peano derivatives at x if and only if it has n Gaussian Riemann derivatives at x, and f has n symmetric Peano derivatives at x if and only if it has n Gaussian symmetric Riemann derivatives at x; and we conjecture that these two results are false for every larger classes of generalized Riemann derivatives, thereby extending two recent conjectures by Ash and Catoiu, the second of which we update by answering it in a few cases.

The first family of generalized derivatives was invented by Riemann in [**R**] (1892). For a positive integer n, the nth *Riemann derivative* of a function f at x is defined by the limit

$$D_n f(x) = \lim_{h \to 0} \frac{1}{h^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(x + (n-k)h).$$

The above sum is denoted by $\Delta_n(x, h; f)$ and called the nth *Riemann difference* of f at x and h. The sequence of Riemann differences satisfies the recursive relations:

$$\Delta_1(x, h; f) = f(x+h) - f(x),$$
$$\Delta_n(x, h; f) = \Delta_{n-1}(x+h, h; f) - \Delta_{n-1}(x, h; f), \qquad (n \geq 2).$$

The Riemann derivatives were generalized by Denjoy in [**D**] (1935). An nth *generalized Riemann derivative* of a function f at x is defined by the following limit:

$$D_{\mathcal{A}} f(x) = \lim_{h \to 0} \frac{1}{h^n} \sum_{k=0}^{\ell} A_k f(x + a_k h).$$

The above sum is denoted by $\Delta_{\mathcal{A}}(x, h; f)$ and called an nth *generalized Riemann difference*. Its data vector $\mathcal{A} = \{A_0, \ldots, A_\ell; a_0, \ldots, a_\ell\}$ for which the A_k are non-zero and the a_k are distinct is required to satisfy the nth *Vandermonde relations* $\sum_{k=0}^{\ell} A_k a_k^j = \delta_{j,n} \cdot n!$, for $j = 0, 1, \ldots, n$. The points $x + a_0 h, \ldots, x + a_\ell h$ are called *base points* of either the derivative or the difference. The Vandermonde linear system is consistent precisely when

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$\ell \geq n$, and has a unique solution $(A_0, \dots, A_\ell)$ precisely when $\ell = n$. In this case, the generalized Riemann derivative is considered to be *without excess*. Unless otherwise stated, all generalized Riemann derivatives in this paper will be without excess.

More examples of generalized Riemann derivatives include the nth *symmetric Riemann derivative* $D_n^s f(x)$ whose associated nth *symmetric Riemann difference*,

$$\Delta_n^s(x,h;f) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f\left(x + \left(\frac{n}{2} - k\right) h\right),$$

satisfies the recursive relations:

$$\Delta_1^s(x,h;f) = f(x+h/2) - f(x-h/2),$$
$$\Delta_n^s(x,h;f) = \Delta_{n-1}^s(x+h/2,h;f) - \Delta_{n-1}^s(x-h/2,h;f), \qquad (n \geq 2).$$

In general, an nth generalized Riemann difference $\Delta_{\mathcal{A}}(x,h;f)$ is *symmetric* if it satisfies $\Delta_{\mathcal{A}}(x,-h;f) = (-1)^n \Delta_{\mathcal{A}}(x,h;f)$, and is *even* or *odd* if it is symmetric of even or odd order n. The symmetry of a nth generalized Riemann derivative without excess is equivalent to the symmetry of the set $\{a_0, \dots, a_n\}$ relative to the origin.

More generalized Riemann differences are obtained by scaling. A *scale* by r of an nth generalized Riemann difference $\Delta_{\mathcal{A}}(x,h;f)$, where $\mathcal{A} = \{A_k; a_k\}$, is the nth generalized Riemann difference $\Delta_{\mathcal{B}}(x,h;f)$, where $\mathcal{B} = \{B_k = r^{-n} A_k; b_k = r a_k\}$. Moreover, f is $\mathcal{A}$-differentiable at x if and only if f is $\mathcal{B}$-differentiable at x and $D_{\mathcal{A}} f(x) = D_{\mathcal{B}} f(x)$.

The results in this paper come in two different flavors. One flavor comes from results that are more algebraic and combinatorial:

We introduce three new kinds of nth generalized Riemann derivatives, all depending on a real parameter q, with $q \neq 0, \pm 1$. The first two kinds are q-analogues of the nth Riemann derivative $D_n f(x)$ and we call them together nth *Gaussian Riemann derivatives*. These are the nth derivatives ${}_qD_n f(x)$ based at $x, x+h, x+qh, x+q^2h, \dots, x+q^{n-1}h$ and ${}_q\bar{D}_n f(x)$ based at $x+h, x+qh, \dots, x+q^n h$. The third, or the nth *Gaussian symmetric Riemann derivative* ${}_qD_n^s f(x)$, is a q-analogue of the nth symmetric Riemann derivative $D_n^s f(x)$ and is based at $(x), x \pm h, x \pm qh, x \pm q^2 h, \dots, x \pm q^{m-1} h$, where $m = \lfloor (n+1)/2 \rfloor$ and (x) means that x is taken only for n even. In section 1 we provide the explicit expressions for their associated differences in terms of the Gaussian binomial coefficients and determine the recursive relations for each of these three classes of differences.

The other flavor comes from results, which are included in sections 2 and 3, and are more like classical analysis. These are about equivalences and implications between generalized Riemann derivatives and Peano derivatives. We describe them next.

The second family of generalized derivatives was introduced by Peano in [**P**] (1892) and further developed by de la Vallée Poussin in [**dlVP**] (1908). A function f has n *Peano derivatives* at x if there exist numbers $f_{(0)}(x), f_{(1)}(x), \dots f_{(n)}(x)$ such that

$$f(x+h) = f_{(0)}(x) + f_{(1)}(x)h + f_{(2)}(x)\frac{h^2}{2!} + \cdots + f_{(n)}(x)\frac{h^n}{n!} + o(h^n),$$

as h approaches zero. The number $f_{(n)}(x)$ is the nth *Peano derivative* of f at x. Its existence assumes the existence of every lower order Peano derivative of f at x.

A function f is said to have n *symmetric Peano derivatives* at x if there exist numbers $f_{(0)}^s(x), f_{(1)}^s(x), \dots, f_{(n)}^s(x)$ such that

$$\frac{1}{2}\{f(x+h) + (-1)^n f(x-h)\} = f_{(0)}^s(x) + f_{(1)}^s(x)h + \cdots + f_{(n)}^s(x)\frac{h^n}{n!} + o(h^n),$$

as h approaches zero. The number $f_{(n)}^s(x)$ is the nth *symmetric Peano derivative* of f at x. Replacing h with $-h$ in the above displayed equation yields $f_{(n-1)}^s(x) = f_{(n-3)}^s(x) =$

$f^s_{(n-5)}(x) = \cdots = 0$. In this way, if f has n symmetric Peano derivatives at x, then f has symmetric Peano derivatives at x of orders $n-2, n-4$, and so on.

The following three additional simple facts about Peano and generalized Riemann derivatives are a motivation for the definitions below:

- The existence of the nth Peano derivative $f_{(n)}(x)$ implies the existence of every nth generalized Riemann derivative $D_{\mathcal{A}}f(x)$ and $f_{(n)}(x) = D_{\mathcal{A}}f(x)$.
- The existence of an nth generalized Riemann derivative does not enjoy the nice property of forcing the existence of lower order derivatives.
- A chain of generalized Riemann derivatives, one for each order i, $i \le n$, existing is not enough to force the nth Peano derivative to exist.

We illustrate the last bullet property by looking at the simple example of the function $g(x) = x^n \operatorname{sgn} x$ at $x = 0$. Every chain $D_{\mathcal{A}_0}, D_{\mathcal{A}_1}, D_{\mathcal{A}_2}, \ldots, D_{\mathcal{A}_{n-1}}, D^s_n$ where $D_{\mathcal{A}_0}f(x) = f_{(0)}(x)$, each $D_{\mathcal{A}_i}$ is an arbitrary ith generalized Riemann derivative, for $1 \le i \le n-1$, and D^s_n is the nth symmetric Riemann derivative, exists for g at 0, but g does not have an nth Peano derivative at 0. Proof: Since $g(h) = o(h^{n-1})$, $D_{\mathcal{A}_0}g(0) = 0$, and for $1 \le i \le n-1$, $g_{(i)}(0) = 0$ and hence $D_{\mathcal{A}_i}g(0) = 0$ also. Since the parity of g is opposite to the parity of n, $D^s_n g(0) = 0$. Finally, $g_{(n)}(0) = \lim_{h\to 0} R(h)$, where $R(h) = n!g(h)/h^n$, does not exist since $\lim_{h\to 0^+} R(h) = n!$ but $\lim_{h\to 0^-} R(h) = -n!$.

To make the statement of the last bullet item more precise, we start with a definition.

DEFINITION. (i) A chain of $D_{\mathcal{A}_0}, \ldots, D_{\mathcal{A}_n}$ of generalized Riemann differentiations, with each $D_{\mathcal{A}_i}$ of order i, is a *Peano-chain* for f at x, if

$$\text{all } D_{\mathcal{A}_0}f(x), \ldots, D_{\mathcal{A}_n}f(x) \text{ exist} \iff f_{(n)}(x) \text{ exists.}$$

Fix n and let Γ be a class of generalized Riemann differentiations of orders up to n.

(ii) A function f has n *derivatives in* Γ *at* x, if there exist $D_{\mathcal{A}_0}, \ldots, D_{\mathcal{A}_n} \in \Gamma$, with each $D_{\mathcal{A}_i}$ of order i, such that $D_{\mathcal{A}_i}f(x)$ exists for each i.

(iii) Γ is a *Peano-class* at x if, for all f, f has n derivatives in Γ at $x \iff f_{(n)}(x)$ exists. This means that each chain of n derivatives f has in Γ at x is a Peano-chain.

The above example implies that the class of *all* generalized Riemann derivatives of orders up to n is not a Peano-class. Is there any class of generalized Riemann derivatives that is a Peano-class? The answer is: Yes, up till now there has been only one known Peano-class, namely the single Peano-chain $\Gamma_2 = \{{}_2D_k \mid k = 0, \ldots, n\}$ highlighted in [**ACF**, Corollary MZ]. If Γ' and Γ'' are two Peano-classes, then $\Gamma' \cup \Gamma''$ is such a class. As a consequence, the union $\boldsymbol{\Gamma}$ of all Peano-classes of generalized Riemann derivatives of orders up to n is the largest such class. This class is maximal in the sense that every other Peano-class is contained in it or, equivalently, any strictly larger class of generalized Riemann derivatives is not a Peano-class.

The same implications, discussion, definitions and results apply to the symmetric case. In particular, there is only one known symmetric Peano-class of symmetric generalized Riemann derivatives, namely the class $\Gamma^s_2 = \{{}_2D^s_k \mid k = n, n-2, \ldots\}$ obtained in [**AC2**, Corollary 2.3], and there is the largest symmetric Peano-class $\boldsymbol{\Gamma}^s$ of symmetric generalized Riemann derivatives of orders up to n.

Let $\Gamma_G = \{{}_qD_k, {}_q\bar{D}_k \mid q \ne 0, \pm 1, k = 0, \ldots, n\}$ be the class of all Gaussian Riemann derivatives of orders up to n, and let $\Gamma^s_G = \{{}_qD^s_k, \mid q \ne 0, \pm 1, k = 0, \ldots, n\}$ be the class of all Gaussian symmetric Riemann derivatives of orders up to n, where ${}_qD_0 = {}_q\bar{D}_0 = f_{(0)}$, ${}_qD_1 = D_1$, ${}_qD^s_0 = f^s_{(0)}$, ${}_qD^s_1 = D^s_1$, ${}_qD^s_2 = D^s_2$, for all q, $q \ne 0, \pm 1$.

The (second) analysis-flavored goal of the paper is to prove that $\Gamma_G = \mathbf{\Gamma}$ and $\Gamma_G^s = \mathbf{\Gamma}^s$. We prove the direct inclusion in both of this equalities, and leave the reverse inclusions as a conjecture. We prove and conjecture the following:

THEOREM A. *Fix a non-negative integer n. Then:*

(i) *The class Γ_G of all Gaussian Riemann derivatives of orders up to n is a Peano-class;*
(ii) *The class Γ_G^s of all Gaussian symmetric Riemann derivatives of orders up to n is a symmetric Peano-class.*

CONJECTURE A. Fix a positive integer n. Let Γ, with $\Gamma \supsetneq \Gamma_G$, be a class of generalized Riemann derivatives and let Γ^s, with $\Gamma^s \supsetneq \Gamma_G^s$, be a class of symmetric generalized Riemann derivatives. Then:

(i) There is a function f such that f has n derivatives in Γ at x, and f has no n Peano derivatives at x.
(ii) There is a function f such that f has n derivatives in Γ^s at x, and f has no n symmetric Peano derivatives at x.

When Conjecture A will be proved, its result combined with the one of Theorem A will replace the old vague perception that "generalized Rieman derivatives are just more general than Peano derivatives" with a more exact perception, that "there is a concrete class of generalized Riemann derivatives whose differentiation is equivalent to the Peano differentiation, and the differentiation in any larger class is strictly more general than the Peano differentiation."

Details. The remaining part of the introduction gives more details on the specific results in each section.

Section 1. The first section computes explicitly the nth forward and the nth symmetric Gaussian Riemann differences in terms of the Gaussian or q-binomial coefficients. Moreover, we provide recursive algorithms for generating these two kinds of differences.

Let q be a real number, with $q \neq 0, \pm 1$, and let n be a positive integer. The quantum integer n, the quantum n factorial, and the quantum n-choose-k are

$$[n]_q = 1 + q + \cdots + q^{n-1},\ [n]_q! = [1]_q[2]_q\cdots[n]_q, \text{ and } \left[\begin{array}{c} n \\ k \end{array}\right]_q = \frac{[n]_q!}{[n-k]_q!\cdot[k]_q!},$$

for $k = 0, 1, \ldots, n$, where $[0]_q! = 1$ and $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q = 0$ for $k > n$. Taking the limit as $q \to 1$ in $[n]_q$, $[n]_q!$ and $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$, respectively leads to n, $n!$ and $\binom{n}{k}$. The Gaussian or q-binomial coefficients $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$ are related to the Gaussian or q-binomial formula,

$$(a-b)(a-bq)(a-bq^2)\ldots(a-bq^{n-1}) = \sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}} \left[\begin{array}{c} n \\ k \end{array}\right]_q a^{n-k}b^k, \tag{1}$$

whose limit as $q \to 1$ is the classical (Newton's) binomial formula.

n	${}_q\widetilde{\Delta}_n(x,h;f)$	λ_n
1.	$f(x+h) - f(x)$	1
2.	$f(x+qh) - qf(x+h) + (q-1)f(x)$	$\frac{2}{q^2-q}$
3.	$f(x+q^2h) - q(q+1)f(x+qh) + q^3f(x+h) - (q^2-1)(q-1)f(x)$	$\frac{6}{(q^3-q)(q^3-q^2)}$
n	${}_q\widetilde{\bar{\Delta}}_n(x,h;f)$	$\bar{\lambda}_n$
1.	$f(x+qh) - f(x+h)$	$\frac{1}{q-1}$
2.	$f(x+q^2h) - (q+1)f(x+qh) + qf(x+h)$	$\frac{2}{(q^2-1)(q^2-q)}$
3.	$f(x+q^3h) - (q^2+q+1)f(x+q^2h) + (q^3+q^2+q)f(x+qh) - q^3f(x+h)$	$\frac{6}{(q^3-1)(q^3-q)(q^3-q^2)}$

n	${}_q\widetilde{\Delta}^s_n(x,h;f)$	λ^s_n
1.	$f(x+h)-f(x-h)$	$1/2$
3.	$f(x+qh)-qf(x+h)+qf(x-h)-f(x-qh)$	$\frac{3}{q^3-q}$
5.	$f(x+q^2h)-q(q^2+1)f(x+qh)+q^4f(x+h)-q^4f(x-h)$ $+q(q^2+1)f(x-qh)-f(x-q^2h)$	$\frac{60}{(q^5-q)(q^5-q^3)}$
2.	$f(x+h)-2f(x)+f(x-h)$	1
4.	$f(x+qh)-q^2f(x+h)+(q^2-1)f(x)-q^2f(x-h)+f(x-qh)$	$\frac{12}{q^4-q^2}$
6.	$f(x+q^2h)-q^2(q^2+1)f(x+qh)+q^6f(x+h)-2(q^2-1)(q^4-1)f(x)$ $+q^6f(x-h)-q^2(q^2+1)f(x-qh)+f(x-q^2h)$	$\frac{360}{(q^6-q^2)(q^6-q^4)}$

To get an idea on how quantum integers and Gaussian binomial coefficients are involved in the expression of the nth forward Gaussian Riemann differences ${}_q\Delta_n(x,h;f)$, based at $x, x+h, x+qh, a+q^2h, \dots, x+q^{n-1}h$, we list the first couple of these differences. For simplicity, we write ${}_q\Delta_n(x,h;f) = \lambda_n \cdot {}_q\widetilde{\Delta}_n(x,h;f)$, where λ_n is the dominant coefficient and ${}_q\widetilde{\Delta}_n(x,h;f)$ is a monic difference, or the coefficient of the highest term $f(x+q^{n-1}h)$ is 1, and list only ${}_q\widetilde{\Delta}_n(x,h;f)$ and λ_n. A similar list is compiled for the other (forward) Gaussian Riemann differences ${}_q\bar{\Delta}_n(x,h;f) = \bar{\lambda}_n \cdot {}_q\widetilde{\bar{\Delta}}_n(x,h;f)$ with base points $x+h, x+qh, a+q^2h, \dots, x+q^nh$. A new list consists of the symmetric Gaussian Riemann differences ${}_q\Delta^s_n(x,h;f) = \lambda^s_n \cdot {}_q\widetilde{\Delta}^s_n(x,h;f)$, based at $x, x\pm h, x\pm qh, x\pm q^2h, \dots, x\pm q^{m-1}h$, $m = \lfloor (n+1)/2 \rfloor$, which is divided into two halves: one for the odd differences, and one for the even differences. All these formulas were obtained by solving the Vandermonde system of linear equations for each difference. The exact expressions for ${}_q\Delta_n(x,h;f)$, ${}_q\bar{\Delta}_n(x,h;f)$, and ${}_q\Delta^s_n(x,h;f)$, which involve q-binomial coefficients, are given in Lemmas 1.1-1.4. Recursive formulas for these differences are given in (7), (9), and (13). All these and more make the subject of section 1.

Before describing the next section's results, note that the formulas in the first half of Table 1 resemble the formulas in the first half of Table 2, and the formulas in the second half of Table 1 resemble the formulas in the second half of Table 2. In this way, it makes sense to have two q-analogues of the nth forward Riemann difference for each n, so that the Gaussian symmetric case is a symmetric analogue of the (forward) Gaussian case.

Section 2. We prove the following Theorem B, an equivalent simplified version of Theorem A. In the same way, Conjecture A simplifies as the following Conjecture B:

THEOREM B. *Let q be a real number with $q \neq 0, \pm 1$, and let n be a positive integer. Then for each function f and point x,*

(i) *both $f_{(n-1)}(x)$ and one of ${}_qD_nf(x)$ or ${}_q\bar{D}_nf(x)$ exist $\iff$ $f_{(n)}(x)$ exists.*
(ii) *both $f^s_{(n-2)}(x)$ and ${}_qD^s_nf(x)$ exists $\iff$ $f^s_{(n)}(x)$ exists.*

CONJECTURE B. Fix a positive integer n and let $D_{\mathcal{A}}$ be an nth generalized Riemann differentiation without excess. Then:

(i) If $D_{\mathcal{A}}$ is not Gaussian, then for all f and x,

$$\text{both } f_{(n-1)}(x) \text{ and } D_{\mathcal{A}}f(x) \text{ exist } \;\not\Longrightarrow\; f_{(n)}(x) \text{ exists.}$$

(ii) If $n \geq 3$ and $D_{\mathcal{A}}$ is symmetric but not Gaussian symmetric, then for all f and x,

$$\text{both } f^s_{(n-2)}(x) \text{ and } D_{\mathcal{A}}f(x) \text{ exists } \;\not\Longrightarrow\; f^s_{(n)}(x) \text{ exists.}$$

When $n = 1$, part (ii) of Theorem B is still valid by ignoring the term $f^s_{(n-2)}(x)$ that does not make sense. Part (i) of Conjecture B is shown to be true in Proposition 2.5 for $n = 1$, and in Proposition 2.6 for $n = 2$, leaving it open for $n \geq 3$. Part (ii) is easily proved false for $n = 1$ and 2, and is proved true for $n = 3$ and 4 in Proposition 2.7, leaving it open for $n \geq 5$.

Section 3. Conjecture B is more general than the following conjecture on Peano and Riemann derivatives, the two oldest and most important generalized derivatives:

CONJECTURE C ([**AC2**]). For all functions f and points x,
(i) When $n \geq 3$, $f_{(n-1)}(x)$ and $D_n f(x)$ exist $\not\Longrightarrow f_{(n)}(x)$ exists.
(ii) When $n \geq 5$, $f^s_{(n-2)}(x)$ and $D^s_n f(x)$ exist $\not\Longrightarrow f^s_{(n)}(x)$ exists.

Both parts of this conjecture were first formally stated for $n \geq 3$ in [**AC2**] as Conjectures 4.2 and 4.1. Part (i) of Conjecture C was proved for $n = 3$ in [**ACF**, Theorem 1(ii)] via a clever example that does not extend to higher n; we will not be able to add anything more to it here. In section 3 we are updating part (ii) of Conjecture C to $n \geq 5$, based on us proving the asserted result false for $n = 3$ and 4 in Theorem 3.1(i). In addition, Theorem 3.1(ii) proves Conjecture C(ii) for $n = 5, 6, 7, 8$, leaving it open for $n \geq 9$.

The equivalence between generalized derivatives is an almost a century old problem. It was initiated by Kintchine in [**Ki**] (1927), who proved that the first symmetric derivative is a.e. equivalent to the first Peano derivative. This was greatly extended by Marcinkiewicz and Zygmund in [**MZ**] (1936) to the a.e. equivalence between the nth Peano and the nth symmetric Riemann derivatives, and further by Ash in [**As**] (1967) who showed that any nth generalized Riemann derivative of a function f is a.e. equivalent to the nth Peano derivative on a measurable set. In particular, any two generalized Riemann derivatives of f of the same order are a.e. equivalent on a measurable set. The pointwise equivalence between Peano and generalized Riemann differentiation is studied in [**ACCs**]; an application of this to continuity is found in [**AAC**]. Pointwise equivalences and pointwise implications between any two generalized Riemann derivatives of a real or complex function f are investigated in [**ACCh**] and [**ACCh1**]. In particular, the above mentioned single equivalent class breaks up into numerous smaller equivalence classes, and these are described explicitly. Quantum Riemann derivatives were introduced in [**AC, ACR**]. Multidimensional Riemann derivatives are explored in [**AC1**]. These recent articles have shown numerous connections between generalized Riemann derivatives and linear and abstract algebra, recursive set theory, symmetric functions, complex and numerical analysis. For more on Peano and generalized Riemann differentiation, see [**As1, F, F1, FR, GR, LPW, RAA**].

Generalized Riemann derivatives have many applications in the theory of trigonometric series [**SZ, Z**]. They were shown to satisfy properties similar to those for ordinary derivatives, such as convexity, monotonicity, and the mean value theorem [**AJ, FFR, GGR, HL, HL1, MM, T, W**]. Surveys on generalized derivatives are found in [**As2**] and [**EW**].

1. Explicit formulas for Gaussian Riemann differences

In this section we both introduce and prove explicit formulas for the nth forward and the nth symmetric Gaussian Riemann differences. These are respectively the nth generalized Riemann differences ${}_q\Delta_n(x, h; f)$ based at $x, x+h, x+qh, x+q^2h, \ldots, x+q^{n-1}h$ and ${}_q\bar{\Delta}_n(x, h; f)$ based at $x+h, x+qh, x+q^2h, \ldots, x+q^n h$, and the nth symmetric generalized Riemann difference ${}_q\Delta^s_n(x, h; f)$ based at $(x), x\pm h, x\pm qh, x\pm q^2h, \ldots, x\pm q^{m-1}h$, where $m = \lfloor (n+1)/2 \rfloor$ and (x) means that x is taken only for n even. These formulas are proved by reference to different versions of the Gaussian q-binomial formula.

1.1. Forward Gaussian Riemann differences. Taking $n-1$ instead of n in (1) and setting $b = q$ leads to the Gaussian binomial formula

$$(a-q)(a-q^2)\ldots(a-q^{n-1}) = \sum_{k=0}^{n-1}(-1)^k q^{\binom{k+1}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q a^{n-1-k}. \tag{2}$$

When $a=1$, equation (2) is equivalent to

$$\sum_{k=0}^{n-1}(-1)^k q^{\binom{k+1}{2}}\left[\begin{array}{c} n-1 \\ k \end{array}\right]_q-(1-q)(1-q^2)\dots(1-q^{n-1})=0; \tag{3}$$

when $a=q^j$, for $j=1,\dots,n-1$, the same equation is

$$\sum_{k=0}^{n-1}(-1)^k q^{\binom{k+1}{2}}\left[\begin{array}{c} n-1 \\ k \end{array}\right]_q\left(q^{n-1-k}\right)^j=0; \tag{4}$$

and when $a=q^n$, the same equation becomes

$$\sum_{k=0}^{n-1}(-1)^k q^{\binom{k+1}{2}}\left[\begin{array}{c} n-1 \\ k \end{array}\right]_q\left(q^{n-1-k}\right)^n=(q^n-q)(q^n-q^2)\dots(q^n-q^{n-1}). \tag{5}$$

The following lemma provides the expression of the nth Gaussian Riemann difference.

LEMMA 1.1. *The nth forward Gaussian Riemann difference has the expression*

$${}_q\Delta_n(x,h;f)=\lambda_n\cdot{}_q\widetilde{\Delta}_n(x,h;f),$$

where

$${}_q\widetilde{\Delta}_n(x,h;f)=\sum_{k=0}^{n-1}(-1)^k q^{\binom{k+1}{2}}\left[\begin{array}{c} n-1 \\ k \end{array}\right]_q f(x+q^{n-1-k}h)-(1-q)(1-q^2)\dots(1-q^{n-1})f(x)$$

and $\lambda_n=n!/\left((q^n-q)(q^n-q^2)\dots(q^n-q^{n-1})\right)$.

PROOF. By (3), (4), (5) the difference $\lambda_n\cdot{}_q\widetilde{\Delta}_n(x,h;f)$ based at $x,x+h,x+qh,\dots,x+q^{n-1}h$ satisfies the nth Vandermonde relations, so it must equal ${}_q\Delta_n(x,h;f)$. □

Using the quantum Pascal triangle identity

$$\left[\begin{array}{c} n-1 \\ k \end{array}\right]_q=\left[\begin{array}{c} n-2 \\ k \end{array}\right]_q+q^{n-1-k}\left[\begin{array}{c} n-2 \\ k-1 \end{array}\right]_q, \tag{6}$$

and the expressions for the differences $\widetilde{\Delta}_n(x,h;f)$ given in Lemma 1.1, one can easily show that the same differences can also be computed recursively as

$$\begin{aligned} &{}_q\widetilde{\Delta}_1(x,h;f)=f(x+h)-f(x),\\ &{}_q\widetilde{\Delta}_n(x,h;f)={}_q\widetilde{\Delta}_{n-1}(x,qh;f)-q^{n-1}\cdot{}_q\widetilde{\Delta}_{n-1}(x,h;f), \qquad (n\geq 2). \end{aligned} \tag{7}$$

Another recursive way to define the nth Gaussian Riemann difference is by using the sequence of generalized Riemann difference quotients given by

$${}_q\widetilde{D}_n(x,h;f)={}_q\Delta_n(x,h;f)/h^n.$$

By (7) and the expression of λ_n in Lemma 1.1, one can prove by induction on n that this sequence satisfies the following recursive relation:

$$\begin{aligned} &{}_q\widetilde{D}_1(x,h;f)=\frac{f(x+h)-f(x)}{h},\\ &{}_q\widetilde{D}_n(x,h;f)=n\frac{{}_q\widetilde{D}_{n-1}(x,qh;f)-{}_q\widetilde{D}_{n-1}(x,h;f)}{(q^{n-1}-1)h}, \qquad (n\geq 2). \end{aligned} \tag{8}$$

As an nth generalized Riemann difference quotient, the same sequence enjoys the property that for each n, ${}_q\widetilde{D}_n(x,h;f)=f^{(n)}(x)$, for all polynomials f of degree $\leq n$.

The *nth Gaussian Riemann derivative* of a function f at x is the nth generalized Riemann derivative

$${}_qD_nf(x):=\lim_{h\to 0}{}_q\widetilde{D}_n(x,h;f)=\lim_{h\to 0}\frac{{}_q\Delta_n(x,h;f)}{h^n}=\lim_{h\to 0}\frac{\lambda_n\cdot{}_q\widetilde{\Delta}_n(x,h;f)}{h^n}.$$

It is a q-analogue of the nth (forward) Riemann derivative, but not the only one. For example, it is different from the nth quantum Riemann derivative defined in [**ACR**], which satisfies q-Vandermonde relations instead of ordinary Vandermonde relations, hence is not an nth generalized Riemann derivative.

Another q-analogue of the nth Riemann derivative which *is* an nth generalized Riemann derivative is the unique nth generalized Riemann derivative based at $x+h, x+qh, \dots, x+q^n h$ whose expression is given explicitly in the following lemma:

LEMMA 1.2. *The nth generalized Riemann difference based at $x+h, x+qh, \dots, x+q^n h$ has the expression*

$$ {}_q\bar{\Delta}_n(x,h;f) = \bar{\lambda}_n \sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q f(x+q^{n-k}h), $$

where $\bar{\lambda}_n = n!/\left((q^n-1)(q^n-q)\dots(q^n-q^{n-1})\right)$.

PROOF. As in Lemma 1.1, the expression is implied by the q-binomial formula

$$ (a-1)(a-q)(a-q^2)\dots(a-q^{n-1}) = \sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q a^{n-k} $$

obtained from (1) by taking $b=1$, since its Vandermonde relations are deduced from this formula, by taking $a=q^j$ for $j=0,1,\dots,n$. ☐

The identity (6) for n instead of $n-1$ and Lemma 1.2 can be employed to deduce the recursive relations between the Gaussian differences ${}_q\bar{\Delta}_n(x,h;f)$. These are

$$ \begin{aligned} &{}_q\tilde{\bar{\Delta}}_1(x,h;f) = f(x+qh) - f(x+h), \\ &{}_q\tilde{\bar{\Delta}}_n(x,h;f) = {}_q\tilde{\bar{\Delta}}_{n-1}(x,qh;f) - q^{n-1}\cdot {}_q\tilde{\bar{\Delta}}_{n-1}(x,h;f), \qquad (n \ge 2), \end{aligned} \tag{9} $$

which in turn can be involved in deducing the following recursive relations satisfied by the defining difference quotients ${}_q\tilde{\bar{D}}_n(x,h;f) := {}_q\bar{\Delta}_n(x,h;f)/h^n$:

$$ \begin{aligned} &{}_q\tilde{\bar{D}}_1(x,h;f) = \frac{f(x+qh)-f(x+h)}{(q-1)h}, \\ &{}_q\tilde{\bar{D}}_n(x,h;f) = n\frac{{}_q\tilde{\bar{D}}_{n-1}(x,qh;f) - {}_q\tilde{\bar{D}}_{n-1}(x,h;f)}{(q^n-1)h}, \qquad (n \ge 2). \end{aligned} \tag{10} $$

The nth (forward) Gaussian Riemann derivative ${}_q\bar{D}_n f(x)$ is defined by the limit:

$$ {}_q\bar{D}_n f(x) = \lim_{h\to 0} {}_q\tilde{\bar{D}}_n(x,h;f) = \lim_{h\to 0} {}_q\bar{\Delta}_n(x,h;f)/h^n. $$

1.2. Symmetric Gaussian Riemann differences. Throughout this section, n will be a fixed positive integer and $m = \lfloor (n+1)/2 \rfloor$. Recall from the introduction that the expressions of the symmetric Gaussian Riemann differences depend on the parity of n. Their exact formulas for general n will be deduced from two new q-binomial formulas in a similar way as we did for the forward Gaussian Riemann derivatives. The first of these two new q-binomial formulas,

$$ (a-q^2)(a-q^4)\dots(a-q^{2(m-1)}) = \sum_{k=0}^{m-1} (-1)^k q^{k(k+1)} \begin{bmatrix} m-1 \\ k \end{bmatrix}_{q^2} a^{m-1-k}, \tag{11} $$

is easily obtained from (2) by replacing n with m and q with q^2.

The following lemma provides the expression for the nth Gaussian symmetric Riemann derivative in the case when n is even.

LEMMA 1.3. *When $n = 2m$, the nth symmetric Gaussian Riemann difference is the nth generalized Riemann difference based at $x, x \pm h, x \pm qh, x \pm q^2h, \ldots, x \pm q^{m-1}h$. This has the expression ${}_q\Delta_n^s(x, h; f) = \lambda_n^s \cdot {}_q\widetilde{\Delta}_n^s(x, h; f)$, where*

$$\begin{aligned}{}_q\widetilde{\Delta}_n^s(x, h; f) = \sum_{k=0}^{m-1} (-1)^k q^{k(k+1)} \left[\begin{array}{c} m-1 \\ k \end{array} \right]_{q^2} \left\{ f(x + q^{m-1-k}h) + f(x - q^{m-1-k}) \right\} \\ - 2(1-q^2)(1-q^4)\ldots(1-q^{2(m-1)})f(x)\end{aligned}$$

and $\lambda_n^s = n!/\left(2(q^n - q^2)(q^n - q^4)\ldots(q^n - q^{n-2})\right)$.

PROOF. The Vandermonde conditions for j even are verified in the same way as in the proof of Lemma 1.1, this time using (3), (4) and (5) with n replaced by m and q replaced by q^2. For odd j and even difference, the Vandermonde conditions are trivially satisfied. □

The second q-binomial formula that will be needed in dealing with symmetric Gaussian Riemann differences is

$$(a-q)(a-q^3)\ldots(a-q^{2m-3}) = \sum_{k=0}^{m-1} (-1)^k q^{k^2} \left[\begin{array}{c} m-1 \\ k \end{array} \right]_{q^2} a^{m-1-k}. \tag{12}$$

The following lemma provides the expression for the nth Gaussian symmetric Riemann derivative in the case when n is odd.

LEMMA 1.4. *When $n = 2m + 1$, the nth symmetric Gaussian Riemann difference is the nth generalized Riemann difference based at $x \pm h, x \pm qh, x \pm q^2h, \ldots, x \pm q^{m-1}h$. This has the expression ${}_q\Delta_n^s(x, h; f) = \lambda_n^s \cdot {}_q\widetilde{\Delta}_n^s(x, h; f)$, where*

$$\widetilde{\Delta}_n^s(x, h; f) = \sum_{k=0}^{m-1} (-1)^k q^{k^2} \left[\begin{array}{c} m-1 \\ k \end{array} \right]_{q^2} \left\{ f(x + q^{m-1-k}h) - f(x - q^{m-1-k}) \right\}$$

and $\lambda_n^s = n!/\left(2(q^n - q)(q^n - q^3)\ldots(q^n - q^{n-2})\right)$.

PROOF. The Vandermonde conditions for j odd are verified in the same way as in Lemma 1.1, this time using (3), (4), (5) with n replaced by m and q replaced by q^2. The Vandermonde conditions are trivially satisfied for even j and odd difference. □

Since both expressions in Lemmas 1.3 and 1.4 involve the same q-binomial coefficients, the Pascal triangle identity (6) with m instead of n and q^2 instead of q can be used to inductively deduce the following combined recursive relation for all symmetric Gaussian Riemann differences:

$$\begin{aligned}&{}_q\widetilde{\Delta}_1^s(x, h; f) = f(x+h) - f(x-h), \\ &{}_q\widetilde{\Delta}_2^s(x, h; f) = f(x+h) - 2f(x) + f(x-h), \\ &{}_q\widetilde{\Delta}_n^s(x, h; f) = {}_q\widetilde{\Delta}_{n-1}^s(x, qh; f) - q^{n-2} \cdot {}_q\widetilde{\Delta}_{n-2}^s(x, h; f), \qquad (n \geq 3).\end{aligned} \tag{13}$$

Finally, we can use the recursive relations (13) and the expressions for λ_n^s provided by Lemmas 1.3 and 1.4 to inductively prove that the nth symmetric Gaussian Riemann quotients

$${}_q\widetilde{D}_n^s(x, h; f) := {}_q\Delta_n^s(x, h; f)/h^n = \lambda_n^s \cdot {}_q\widetilde{\Delta}_n^s(x, h; f)/h^n$$

satisfy the following recursive relations:

$$\begin{aligned}&{}_q\widetilde{D}_1^s(x, h; f) = \frac{f(x+h) - f(x)}{h}, \ {}_q\widetilde{D}_2^s(x, h; f) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}, \\ &{}_q\widetilde{D}_n^s(x, h; f) = n(n-1)\frac{{}_q\widetilde{D}_{n-2}^s(x, qh; f) - {}_q\widetilde{D}_{n-2}^s(x, h; f)}{(q^{n-2+(n \mod 2)} - 1)h^2}, \qquad (n \geq 3).\end{aligned} \tag{14}$$

The *nth symmetric Gaussian Riemann derivative* of a function f at x is the nth generalized Riemann derivative

$$_qD^s_nf(x) := \lim_{h\to 0} {}_q\widetilde{D}^s_n(x,h;f).$$

It is a q-analogue of the nth symmetric Riemann derivative, and is different from the nth quantum symmetric Riemann derivative defined in [**AC**].

2. Proof of Theorem B and Conjecture B

This section has three parts: the translation of Theorem/Conjecture A into Theorem/Conjecture B; the proof of Theorem B; and the evidence for Conjecture B.

2.1. Translating A into B. The explicit statement of Theorem A reads as follows:

THEOREM 2.1. *Fix a non-negative integer n. Then:*

(i) *For each function f and point x,*

there exist real numbers $q_0, q_1, \dots, q_n \neq 0, \pm 1$ such that, for each $i = 0, \dots, n$, either ${}_{q_i}D_if(x)$ or ${}_{q_i}\bar{D}_if(x)$ exists $\iff$ *$f_{(n)}(x)$ exists.*

(ii) *For each function f and point x,*

there exist real numbers $q_0, q_1, \dots, q_n \neq 0, \pm 1$ such that, for each $i = 0, \dots, n$, the derivative ${}_{q_i}D^s_if(x)$ exists $\iff$ *$f^s_{(n)}(x)$ exists.*

And the following is the explicit working statement of Conjecture A:

CONJECTURE 2.2. *Fix an integer n at least 2, and let $D_{\mathcal{A}_0}, \dots, D_{\mathcal{A}_n}$ be a sequence of generalized Riemann differentiations, with each $D_{\mathcal{A}_i}$ of order i. Then:*

(i) *If not all of these differentiations are Gaussian, then for all f and x,*

$$\text{all } D_{\mathcal{A}_0}f(x), \dots, D_{\mathcal{A}_n}f(x) \text{ exist } \;\not\Longrightarrow\; f_{(n)}(x) \text{ exists.}$$

(ii) *If all of these differentiations are symmetric, but not all are Gaussian symmetric, then for all f and x, all $D_{\mathcal{A}_n}f(x), D_{\mathcal{A}_{n-2}}f(x), \dots$ exist $\not\Longrightarrow$ $f^s_{(n)}(x)$ exists.*

The next theorem translates Theorem A into Theorem B and Conjecture A into Conjecture B. For simplicity, during its proof, all references to Theorem A are meant to refer to its equivalent version, Theorem 2.1.

THEOREM 2.3. *The following equivalences of results are satisfied.*

(i) *Theorem A $\iff$ Theorem B;*
(ii) *Conjecture A $\iff$ Conjecture B.*

PROOF. We only prove the two equivalences in the Gaussian cases; the Gaussian symmetric cases are similar.

(i) When $n = 0$, Theorem A is a tautology. We prove the equivalent statement "Theorem A, for $k = 0, \dots n \iff$ Theorem B, for $k = 1, \dots, n$" by induction upon n, for $n \geq 1$. When $n = 1$, Theorems A and B have the same statements. Assume the result for $n-1$ and prove it for n.

"$\Longrightarrow$" It suffices to show that Theorem A is true for $k = 1, \dots, n$ implies that Theorem B is true for n. Indeed, by Theorem A for $n-1$, we can substitute "there exist real numbers $q_0, q_1, \dots, q_{n-1} \neq 0, \pm 1$ such that, for each $i = 0, \dots, n-1$, the derivative ${}_{q_i}D^s_if(x)$ exists" with "$f_{(n-1)}(x)$ exists" in the left side of the equivalence in Theorem A for n, leading to Theorem B for n.

"$\Longleftarrow$" It suffices to show that Theorem B is true for $k = 1, \dots, n$ implies that Theorem A is true for n. Since the inductive hypothesis implies that Theorem A is true for $n-1$, we can substitute "$f_{(n-1)}(x)$ exists" with "there exist real numbers $q_0, q_1, \ldots, q_{n-1} \neq 0, \pm 1$ such that, for each $i = 0, \ldots, n-1$, the derivative ${}_{q_i}D_i^s f(x)$ exists" with "$f_{(n-1)}(x)$ exists" in the left side of the equivalence in Theorem B for n, leading to Theorem A for n.

(ii) Conjecture A says that the class Γ_G is maximal with respect to making the result of Theorem A true, and Conjecture B says that Γ_G is maximal with respect to making the result of Theorem B true. Part (ii) then follows from part (i). □

2.2. Proof of Theorem B. We are now ready to proceed with the proof of this theorem. For this we will need the following lemma:

LEMMA 2.4. *Let q be a real number with $q \neq 0, \pm 1$, and let n be a positive integer. Then for each function f and point x,*

(i) ${}_qD_n f(x)$ *exists* $\iff$ ${}_{q^{-1}}D_n f(x)$ *exists.*
(ii) ${}_qD_n^s f(x)$ *exists* $\iff$ ${}_{q^{-1}}D_n^s f(x)$ *exists.*

PROOF. Part (i) follows from the Gaussian Riemann differences ${}_{q^{-1}}\Delta_n(x, h; f)$ and ${}_q\Delta_n(x, h; f)$ being scales of each other by $q^{\pm(n-1)}$, since they are respectively based at $x, x+h, x+q^{-1}h, \ldots, x+q^{-n+1}h$ and $x, x+h, x+qh, \ldots, x+q^{n-1}h$. Part (ii) follows from the similar property between ${}_{q^{-1}}\Delta_n^s(x, h; f)$ and ${}_q\Delta_n^s(x, h; f)$. □

For simplicity, in the proof of Theorem B we denote a difference $\Delta(0, h; f)$ as $\Delta(h)$.

PROOF OF THEOREM B. (i) As the existence of the nth Peano derivative $f_{(n)}(x)$ both assumes the existence of every lower order Peano derivatives of f at x and implies every nth generalized Riemann derivative of f at x, the reverse implication is clear.

Conversely, suppose that both $f_{(n-1)}(x)$ and ${}_qD_n f(x)$ exist. By Lemma 2.4, we may assume that $|q| > 1$. And eventually by translating the graph of f to the left by x we may assume that $x = 0$, and by subtracting from f a degree n polynomial we may further assume that $f_{(n-1)}(0) = 0$ and ${}_qD_n f(0) = 0$, or ${}_q\Delta_n(h) = o(h^n)$. This is equivalent to ${}_q\widetilde{\Delta}_n(h) = o(h^n)$ since ${}_q\lambda_n$ is independent of h. The last equality means that for each $\varepsilon > 0$, there is a $\delta > 0$ such that $|h| < \delta \Rightarrow |{}_q\widetilde{\Delta}_n(h)| < \varepsilon |h|^n$. Then by (7),

$$\left|{}_q\widetilde{\Delta}_{n-1}(qh) - q^{n-1}{}_q\widetilde{\Delta}_{n-1}(h)\right| < \varepsilon|h|^n, \quad \left|{}_q\widetilde{\Delta}_{n-1}(h) - q^{n-1}{}_q\widetilde{\Delta}_{n-1}\left(\frac{h}{q}\right)\right| < \varepsilon\left|\frac{h}{q}\right|^n, \ldots$$

$$\ldots, \left|{}_q\widetilde{\Delta}_{n-1}\left(\frac{h}{q^{k-1}}\right) - q^{n-1}{}_q\widetilde{\Delta}_{n-1}\left(\frac{h}{q^k}\right)\right| < \varepsilon\left|\frac{h}{q^k}\right|^n.$$

We multiply these inequalities resp. by $1, q^{n-1}, q^{2(n-1)}, \ldots, q^{k(n-1)}$ and add. The triangle inequality makes the left side telescope, while the right side is a geometric series. Then

$$\left|{}_q\widetilde{\Delta}_{n-1}(qh) - q^{(k+1)(n-1)} \cdot {}_q\widetilde{\Delta}_{n-1}\left(\frac{h}{q^k}\right)\right| < \frac{q}{q-1} \cdot \varepsilon\,|h|^n.$$

The second term on the left can be neglected, since it is $(qh)^{n-1}\Big({}_q\widetilde{\Delta}_{n-1}(h/q^k)\Big)/\big(h/q^k\big)^{n-1}$ and this approaches 0 as $k \to \infty$, by the hypothesis $f_{(n-1)}(0) = {}_qD_n f(0) = 0$. Therefore,

$$\left|{}_q\widetilde{\Delta}_{n-1}(qh)\right| < \frac{q}{q-1} \cdot \varepsilon|h|^n, \text{ that is, } \left|{}_q\widetilde{\Delta}_{n-1}(h)\right| = o(h^n).$$

By the independence on h of ${}_q\lambda_{n-1}$, this is equivalent to $|{}_q\Delta_{n-1}(h)| = o(h^n)$. Similarly, $|{}_q\Delta_{n-2}(h)| = o(h^n)$, and so on. At the end, $|{}_q\Delta_1(h)| = o(h^n)$ means that $f(h) - f(0) = f(h) = o(h^n)$, and hence $f_{(n)}(0) = 0$, as needed. The direct implication under the hypothesis that both $f_{(n-1)}(x)$ and ${}_q\bar{D}_n f(x)$ exist is proved along the same lines. The proof of part (ii) is similar to the proof of part (i). □

2.3. Evidence for Conjecture B. The remaining part of the section analyzes the evidence towards this conjecture by proving its asserted result in a number of cases.

The following proposition shows that part (i) of Conjecture B is true for $n = 1$.

PROPOSITION 2.5. *Let $D_{\mathcal{A}}f(x)$ be a first generalized Riemann derivative without excess which is not a Gaussian Riemann derivative. Then:*

(i) $D_{\mathcal{A}}f(x) = f^s_{(1)}(x)$;

(ii) *both $f_{(0)}(x)$ and $D_{\mathcal{A}}f(x)$ exist $\not\Longrightarrow$ $f_{(1)}(x)$ exists, for all f and x.*

PROOF. (i) The hypothesis that $D_{\mathcal{A}}f(x)$ is a first generalized Riemann derivative without excess makes its difference $\Delta_{\mathcal{A}}f(x) = A_1f(x + a_1h) + A_2f(x + a_2h)$, for some A_1, A_2, a_1, a_2, with $a_1 \neq a_2$. If $a_1a_2 = 0$, say $a_1 = 0$, then $\Delta_{\mathcal{A}}f(x)$ is the scale by a_2 of the Riemann difference $\Delta_1 f(x) = f(x+h) - f(x) = {}_q\Delta_1 f(x)$ for any q, hence $D_{\mathcal{A}}f(x)$ is Gaussian, a contradiction. If $a_1a_2 \neq 0$ and $|a_1| \neq |a_2|$, then $\Delta_{\mathcal{A}}f(x)$ is the scale by a_1 of the first difference based at $x + h, x + qh$ for $q = a_2/a_1$, which is ${}_q\bar{\Delta}_1 f(x)$, hence $D_{\mathcal{A}}f(x)$ is Gaussian, a contradiction. In the remaining case $a_1 = -a_2 \neq 0$, $\Delta_{\mathcal{A}}(x, h; f)$ is the scale by a_1 of the symmetric difference $\Delta^s_1(x, h; f)$, hence $D_{\mathcal{A}}f(x) = f^s_{(1)}(x)$.

(ii) As an example, take the function $f(t) = |t - x|$, which is continuous at $t = x$ hence $f_{(0)}(x) = f(x) = 0$, it has $f(x+h) - f(x-h) = 0$ hence $D_{\mathcal{A}}f(x) = f^s_{(1)}(x) = 0$, while $f_{(1)}(x) = \lim_{h\to 0}\{f(x + h) - f(x)\}/h = \lim_{h\to 0} |h|/h$ does not exist. □

The next proposition shows that part (i) of Conjecture B is also true for $n = 2$.

PROPOSITION 2.6. *Let $D_{\mathcal{A}}f(x)$ be a second generalized Riemann derivative without excess which is not a Gaussian Riemann derivative. Then:*

(i) *up to a scale, $\Delta_{\mathcal{A}}(x, h; f)$ is either based at $x \pm h, x + qh$, for $q \neq 0, \pm 1$, or based at $x + h, x + ph, x + qh$, where $p, q \neq 0, \pm 1$, $p \neq \pm q$ and none of p and q is the square of the other;*

(ii) *both $f_{(1)}(x)$ and $D_{\mathcal{A}}f(x)$ exist $\not\Longrightarrow$ $f_{(2)}(x)$ exists, for all f and x.*

PROOF. (i) The hypothesis that $D_{\mathcal{A}}f(x)$ is second order and without excess makes it have base points $x + a_1h, x + a_2h, x + a_3h$ for distinct a_1, a_2, a_3. If one of a_1, a_2, a_3 is zero, say $a_1 = 0$, then up to a scale by a_2^{-1}, $D_{\mathcal{A}}f(x)$ is based at $x, x + h, x + qh$, for $q = a_2/a_1$, that is $D_{\mathcal{A}}f(x)$ is Gaussian, a contradiction. Thus a_1, a_2, a_3 are all non-zero. If two of them add up to zero, say $a_1 + a_2 = 0$, then up to a scale by a_1^{-1}, the difference has the first outlined form. Otherwise, a scale by a_1^{-1} and discounting the Gaussian case makes the difference have the second outlined form.

(ii) Our example has $D_{\mathcal{A}}f(x)$ based at $x + h, x + 2h, x + 3h$, that is, $\Delta_{\mathcal{A}}(x, h; f) = f(x + 3h) - 2f(x + 2h) + f(x + h)$. Eventually by shifting the graph of f to the left by x, we may assume that $x = 0$. Take $G = \langle 2, 3\rangle = \{2^r3^s \mid r, s \text{ integers}\}$ and define f as

$$f(x) = (-1)^{r+s}x^2, \quad \text{if } x = 2^r3^s \in G,$$

and $f(x) = 0$, otherwise. Then $f(h) = o(h)$ as $h \to 0$, hence $f_{(0)}(0) = f_{(1)}(0) = 0$, while $f_{(2)}(0)$ does not exist, due to $\lim_{h\to 0} f(h)/h^2 = 0, \pm 1$. Moreover, when $h = 2^r3^s \in G$, $\Delta_{\mathcal{A}}(0, h; f) = f(3h)-2f(2h)+f(h) = (-1)^{r+s}(-3^2+2\cdot 2^2+1\cdot 1^2)h^2 = 0$, and when $h \notin G$, $\Delta_{\mathcal{A}}(0, h; f) = 0 - 2\cdot 0 + 0 = 0$, and so $D_{\mathcal{A}}f(0) = 0$. □

We now turn to the symmetric case addressed in part (ii) of Conjecture B. The first symmetric Riemann derivative $D^s_1 f(x)$ is up to a scale the only first symmetric generalized Riemann derivative of f at x without excess, and since its definition is the same as the definition of $f^s_{(1)}(x)$, the conjecture is false in the case $n = 1$. Same story for $n = 2$.

The following proposition shows that Conjecture B(ii) is true for $n = 3$ and 4.

PROPOSITION 2.7. *Each order 3 or 4 symmetric generalized Riemann derivative without excess is a symmetric Gaussian Riemann derivative.*

PROOF. Let $D_{\mathcal{A}}f(x)$ be a symmetric generalized Riemann difference of order $n = 3$ or 4. Then it is based at $(x), x \pm ph, x \pm qh$, for $0 < p < q$, which then scaled by p^{-1} becomes a symmetric Gaussian Riemann difference. □

3. Updating Conjecture C

In this section we are updating part (ii) of Conjecture C to $n \geq 5$ by disproving the asserted result for $n = 3$ and 4 in Theorem 3.1(i). In addition, we positively answer the conjecture for $n = 3, 4, \ldots, 8$ in Theorem 3.1(ii) and leave it open for $n \geq 9$.

The following theorem gives answers to Conjecture C(ii), for $n = 3, 4, \ldots, 8$.

THEOREM 3.1. *The following are answers to Conjecture C(ii) for small n.*

(i) *When $n = 3$ or 4, the conjecture is false.*
(ii) *When $n = 5, 6, 7, 8$, the conjecture is true.*

PROOF. (i) When $n = 3$ or 4, the symmetric Riemann derivative $D_n^s f(x)$ is symmetric Gaussian, either by Proposition 2.7 or directly by observing that

$$2\Delta_3^s(x, h; f) = f(x + 3h) - 3f(x + h) + 3f(x - h) - f(x - 3h) = 2 \cdot {}_3\Delta_3^s(x, h; f),$$
$$\Delta_4^s(x, h; f) = f(x + 2h) - 4f(x + h) + 6f(x) - 4f(x - h) + f(x - 2h) = {}_2\Delta_4^s(x, h; f).$$

The result then follows from Theorem B(ii).

(ii) When $n = 5$, let $G = \langle 3, 5\rangle = \{3^m 5^n \mid m, n \in \mathbb{Z}\}$ be the multiplicative subgroup of the rationals generated by 3 and 5, and let $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = (-1)^{m+n}x^k, \text{ for } x = 3^m 5^n \in G,$$

and $f(x) = 0$ for $x \notin G$, where k, $3 < k < 4$, is to be determined. Compute the expression

$$\frac{1}{2}\{f(h) - f(-h)\} = \frac{1}{2} \cdot (-1)^{m+n} \cdot (\pm|h|^k), \quad \text{for } h \in G,$$

and $\frac{1}{2}\{f(h) - f(-h)\} = 0$ for $h \notin G$, to deduce $f_{(3)}(0) = 0$, since $\frac{1}{2}\{f(h) - f(-h)\} = o(h^3)$, and $f_{(5)}(0)$ does not exist since $\lim_{h\to 0} \frac{1}{2}\{f(h) - f(-h)\}/h^5$ does not exist. A scale by 2 of the difference $\Delta_5^s(0, h; f)$ is the difference $2^{-5} \cdot \Delta_5^s(0, 2h; f)$, where the difference $\Delta_5^s(0, 2h; f) = f(5h) - 5f(3h) + 10f(h) - 10f(-h) + 5f(-3h) - 6f(-5h) = (-1)^{m+n+1}5^k|h|^k - 5\cdot(-1)^{m+n+1}3^k|h|^k + 10\cdot(-1)^{m+n}|h|^k = (-1)^{m+n+1}(5^k - 5\cdot 3^k - 10)|h|^k$. Denote $\varphi(k) = 5^k - 5\cdot 3^k - 10$ and observe that $\varphi(3)\varphi(4) < 0$, and so by continuity, $\varphi(k) = 0$ for some k in the open interval $(3, 4)$. Then the f for this k has $D_5^s f(0) = 0$.

When $n = 6$, let $G = \langle 2, 3\rangle = \{2^m 3^n \mid m, n \in \mathbb{Z}\}$ and take f to be the function

$$f(x) = (-1)^{m+n}x^k, \quad \text{for } x = 2^m 3^n \in G,$$

and $f(x) = 0$, for $x \notin G$, where the real number k, $4 < k < 5$, is to be determined. Then

$$\frac{1}{2}\{f(h) + f(-h)\} = \frac{1}{2} \cdot (-1)^{m+n} \cdot (\pm|h|^k) \quad \text{, for } h \in G,$$

and $\frac{1}{2}\{f(h) + f(-h)\} = 0$ for $h \notin G$. Then $f_{(4)}(0) = 0$, since $\frac{1}{2}\{f(h) + f(-h)\} = o(h^4)$, and $f_{(6)}(0)$ does not exist, since $\lim_{h\to 0} \frac{1}{2}\{f(h) + f(-h)\}$ is either 0 or $\pm\infty$. We compute $\Delta_6^s(0, h; f) = f(3h) - 6f(2h) + 15f(h) - 20f(0) + 15f(-h) - 6f(-2h) + f(-3h) = (-1)^{m+n+1}3^k|h|^k - 6\cdot(-1)^{m+n+1}2^k|h|^k + 15\cdot(-1)^{m+n}|h|^k = (-1)^{m+n+1}(3^k - 6\cdot 2^k - 15)|h|^k$ and denote $\varphi(k) = 3^k - 6 \cdot 2^k - 15$. Since $\varphi(4)\varphi(5) < 0$, by continuity, $\varphi(k) = 0$ for some k in the open interval $(4, 5)$. For that particular k, $D_6^s f(0) = 0$, as we needed.

When $n = 7$, let $G = \langle 3, 5, 7\rangle$ and let f be the function

$$f(x) = (-1)^{n+p}x^k, \quad \text{for } x = 3^m 5^n 7^p \in G,$$

and $f(x) = 0$, otherwise, where $5 < k < 7$. Then $\Delta_7^s(0, 2h; f) = f(7h)-7f(5h)+21f(3h)-35f(h) + \cdots = (-1)^{n+p+1}(7^k - 7\cdot 5^k - 21\cdot 3^k + 35)|h|^k$, if $h = 3^m5^n7^p \in G$. As usual, denote $\varphi(k) = 7^k - 7\cdot 5^k - 21\cdot 3^k + 35$ and check that $\varphi(5)\varphi(7) < 0$. The rest is the same as in the other two cases.

When $n = 8$, let $G = \langle 2, 3\rangle$ and let f be the function

$$f(x) = (-1)^m x^k, \quad \text{for } x = 2^m 3^n \in G,$$

and $f(x) = 0$, for $x \notin G$, where $7 < k < 8$. Then $\Delta_8^s(0, h; f) = f(4h) - 8f(3h) + 28f(2h) - 56f(h) + 70f(0) - \cdots = (-1)^m \left(4^k - 8\cdot 3^k - 28\cdot 2^k - 56\right)|h|^k$, for $h = 2^m3^n \in G$. The function $\varphi(k) = 4^k - 8\cdot 3^k - 28\cdot 2^k - 56$ has $\varphi(7)\varphi(8) < 0$, and the rest is folklore. □

Following the same method as in the proof of Theorem 3.1 for $n = 5, 6, 7, 8$, when $n = 9$ we would start by letting $G = \langle 3, 5, 7\rangle$ and then look for an expression of $f(x)$ of the form

$$f(x) = (-1)^{am+bn+cp}x^k, \quad \text{if } x = 3^m5^n7^p \in G,$$

and $f(x) = 0$, otherwise, where $7 < k < 9$ and $a, b, c \in \{0, 1\}$. Then $\Delta_9^s(0, 2h; f) = f(9h) - 9f(7h) + 36f(5h) - 84f(3h) + 126f(h) - \cdots$ will be of the form

$$(-1)^{am+bn+cp}\left(\pm 9^k \pm 9\cdot 7^k \pm 36\cdot 5^k \pm 84\cdot 3^k \pm 126\right)|h|^k.$$

Taking $\varphi(k)$ as the expression in the parenthesis, the only choices for $\varphi(k)$ with the property that $\varphi(7)\varphi(9) < 0$ are $\varphi(k) = \pm(9^k - 9\cdot 7^k + 36\cdot 5^k - 84\cdot 3^k - 126)$. Unfortunately, no choice for $a, b, c \in \{0, 1\}$ leads to either expression, and so the method in Theorem 3.1 does not extend to the $n = 9$ case. In this way, part (ii) of Conjecture C remains open for $n \geq 9$.

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