Discontinuities as limits of compactly supported formulas

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1 Introduction

Here are three examples of real valued functions of a real variable that are discontinuous at $x = 0$. All three are defined piecewise, that is to say, by cases.

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

$$\chi(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

$$s(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

These three functions, together with simple combinations of them, give a fairly complete picture of the ways a function can be discontinuous at a point. Here is a list of possible misbehaviors at a point, together with an example for each.

Misbehavior	**Example**
Jump with left limit, right limit and value all distinct	$\operatorname{sgn}(x)$
Limit and value both exist, but are not equal	$\chi(x)$
Jump, but continuous from one side	$\operatorname{sgn}(x) + \chi(x)$
Limits from both sides do not exist	$s(x)$
No limit from one side, continuous from the other side	$s(x)(\operatorname{sgn}(x) - 1)$
No limit from one side, value $\neq$ other side's limit	$s(x)(\operatorname{sgn}(x) - 1) + \chi(x)$

When I first took calculus, it bothered me that whenever such a counterexample was called for, the machinery of cases was always used, often being introduced for the first time at exactly this point. These examples are satisfactory from almost

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every modern point of view, so let me try to make this vague complaint into a manageable question.

I want to work only with functions that have domain equal to all of $\mathbb{R}$. I see no way to avoid a limiting process, so the goal is to create examples as limits of sequences of functions that are given by very simple formulas. Notice that such functions would have been admissible to a seventeen century mathematician, say, for example, Euler. I will leave it to the reader to decide if my examples are "very simple."

I once wanted to put a graph of $\operatorname{sgn}(x)$ into a research paper using a graphing program that I had not sufficiently mastered to input piecewise defined functions. A colleague, Stephen Vagi, suggested that I use $\frac{2}{\pi}\arctan(100x)$. This motivates the first example: Let

$$u(x) = \frac{x}{|x|+1}. \tag{1}$$

Then

$$\operatorname{sgn}(x) = \lim_{n\to\infty} u(nx). \tag{2}$$

Write $|x|$ as $\sqrt{x^2}$, to see that there are no hidden piecewise defined objects here.

Let

$$v(x) = \frac{1}{|x|+1}. \tag{3}$$

Then

$$\chi(x) = \lim_{n\to\infty} v(nx). \tag{4}$$

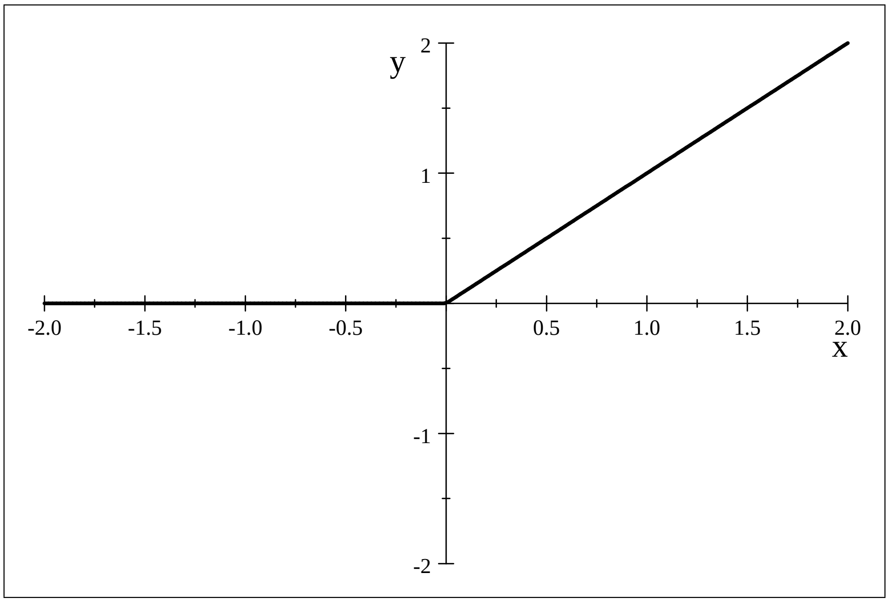
Graph $u(100x)$ and $v(100x)$ to get some intuition about these two examples.

For each positive integer n, both the functions $u(nx)$ and $v(nx)$ are immediately expressed as formulas with domain $\mathbb{R}$. A function has compact support if the set of points where it is non-zero is contained in some finite interval. We would like to add the additional constraint that the approximating functions have compact support.

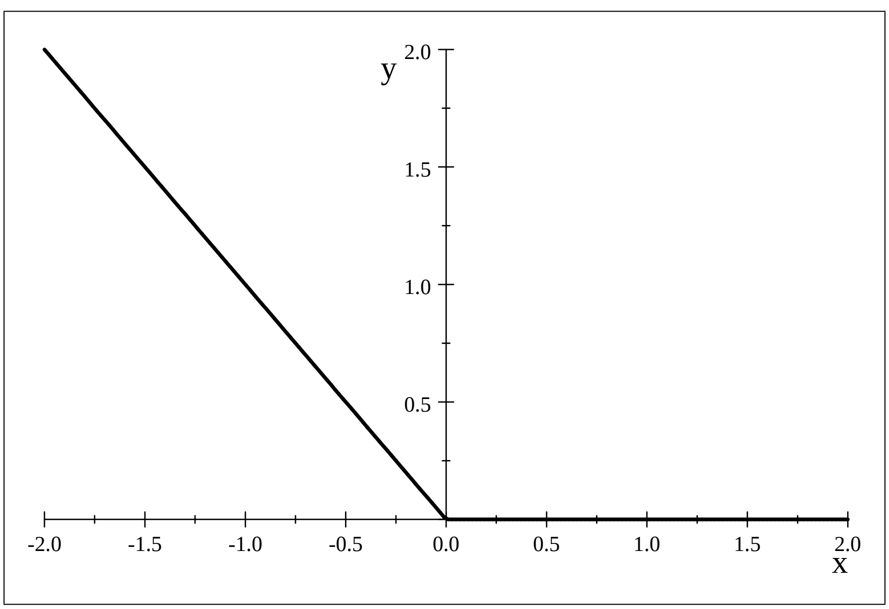
It is not even obvious that there can be *any* compactly supported function given by a formula. We begin by constructing a family of very simple such functions which I will call bumps. A bump is compactly supported and yet has a pretty formula that does not involve cases.

2 Bumps

Let $p(x) = x^{+} = \frac{1}{2}(|x| + x)$ and $n(x) = x^{-} = \frac{1}{2}(|x| - x)$. The graphs of p and n are, respectively,



and



The product $p(x-a)\,n(x-b)$ is positive on the interval (a,b) and is zero on the complement of (a,b). On (a,b) it agrees with the quadratic $(x-a)(b-x)$ which achieves a maximum value of $\left(\frac{b-a}{2}\right)^2$ at the midpoint $x = \frac{a+b}{2}$. We normalize to create a non-negative function of maximum height 1. Our bumps are the family of functions, one for each pair of real numbers a, b with $a < b$ given by

$$B_{a,b}(x) = \left(\frac{2}{b-a}\right)^2 p(x-a)\, n(x-b)$$

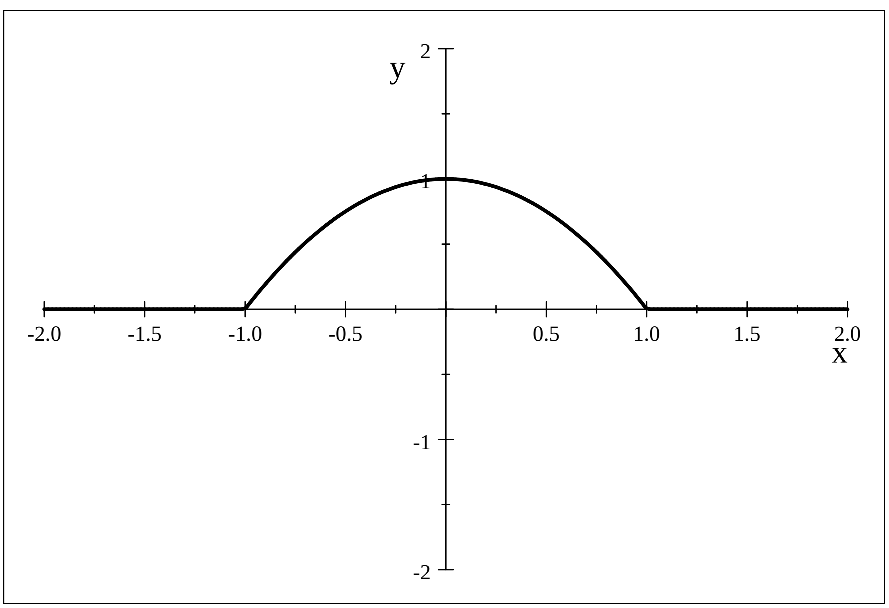
So $B_{a,b}$ can be expressed as the formula

$$B_{a,b}(x) = \left(\frac{1}{b-a}\right)^2 (|x-a| + (x-a))(|x-b| - (x-b)) \tag{5}$$

For another, more geometrically based formula: Recall that for x in the interval $[a, b]$, $B_{a,b}(x)$ is the quadratic passing through the endpoints and having maximum value 1 at the center $m = (b+a)/2$, so if we set $\delta = (b-a)/2$ to the half-length of the interval, we get

$$\begin{aligned} B_{a,b}(x) &= \left(1 - \left(\frac{x-m}{\delta}\right)^2\right)^+ \qquad (6) \\ &= \left|1 - \left(\frac{x-m}{\delta}\right)^2\right| + \left(1 - \left(\frac{x-m}{\delta}\right)^2\right). \end{aligned}$$

Here is the graph of the bump $B_{-1,1}(x)$.



For $n \geq 2$ and $|x| \leq \sqrt{n}$, $1 \geq B_{-n,n}(x) \geq B_{-n,n}(\sqrt{n}) = 1 - \frac{1}{n}$. It follows that for every real number x,

$$\lim_{n\to\infty} B_{n,n}(x) = 1. \tag{7}$$

3 Discontinuous examples

The bump functions allow us to convert any example of a function discontinuous at a point being a limit of everywhere defined formulas into a similar example where the approximating functions are also compactly supported. For example, let $U_n(x)$ be $u(nx)B_{n,n}(x)$. Formulas (1) and (6) show that for each integer n, $U_n(x)$ can be given by an everywhere defined compactly supported formula; then limits (2) and (7) lead to

$$\begin{aligned}\lim_{n\to\infty} U_n(x) &= \lim_{n\to\infty} u(nx)B_{n,n}(x) = \lim_{n\to\infty} u(nx) \lim_{n\to\infty} B_{n,n}(x) \\ &= \operatorname{sgn}(x)\cdot 1 = \operatorname{sgn}(x).\end{aligned}$$

A very similar argument using (3) and (4) in place of (1) and (2) gets the same result for the discontinuous function $\chi(x)$. Another, even faster, way of treating χ is to write

$$\chi(x) = \lim_{n\to\infty} \left(B_{-1,1}(x)\right)^n.$$

In view of the formulas expressing all six types of discontinuous functions in terms of sgn, χ, and s, it suffices to produce a function $S(x)$ that, like $s(x)$ fails to have either one sided limit at $x=0$; but that, unlike $s(x)$, is simply and naturally given as a limit of functions, each of which is compactly supported and defined everywhere by a formula.

Fix a positive integer n.

Define $C_n(x) = B_{\frac{3}{4}\frac{1}{2^n},\frac{5}{4}\frac{1}{2^n}}(x)$. Then

(1) $C_n(x)$ is zero outside of $I_n = \left(\frac{3}{4}\frac{1}{2^n}, \frac{5}{4}\frac{1}{2^n}\right)$,

(2) C_n is a quadratic bump of maximum height 1 at the center point $x = \frac{1}{2^n}$ of I_n and of height 0 at both endpoints of I_n, and

(3) C_n is given by the everywhere defined formula (5) with $a = \frac{3}{4}\frac{1}{2^n}$ and $b = \frac{5}{4}\frac{1}{2^n}$.

The sum of the first N bumps,

$$g_N(x) = \sum_{n=1}^{N} C_n(x),$$

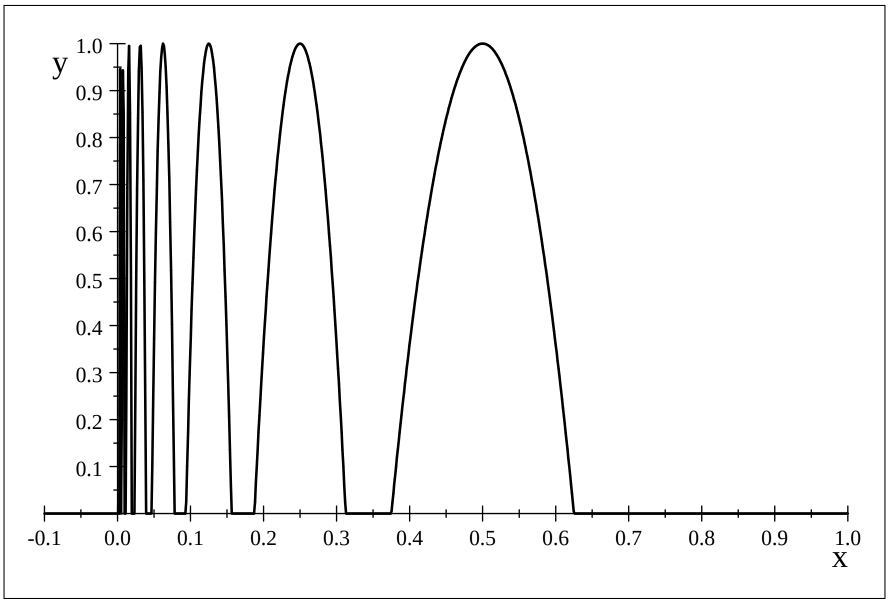
is an everywhere defined formula with support in $\left[\frac{3}{4}\frac{1}{2^N}, \frac{5}{4}\frac{1}{2^1}\right]$. The bumps do not overlap, since the sup of the set where $C_{n+1} \neq 0$ is $\frac{5}{4}\frac{1}{2^{n+1}}$ and the inf of the set where $C_n \neq 0$ is $\frac{3}{4}\frac{1}{2^n}$ and

$$\frac{5}{4}\frac{1}{2^{n+1}} = \frac{5}{8}\frac{1}{2^n} < \frac{6}{8}\frac{1}{2^n} = \frac{3}{4}\frac{1}{2^n}.$$

Now let

$$g(x) = \lim_{N\to\infty} g_N(x) = \sum_{n=1}^{\infty} C_n(x)$$

It is easy to see that $g\left(\frac{1}{2^n}\right) = 1$ and that $g\left(\frac{3}{4}\frac{1}{2^n}\right) = 0$ for $n = 1, 2, 3, \ldots$. Thus $\lim_{x\to 0^+} g(x)$ does not exist. Here is an approximation to the graph of $g(x)$.



The function $g(x)$ does satisfy $\lim_{x\to 0^-} g(x) = g(0) = 0$, since $g(x) = 0$ for all non-positive x. For a function with neither left nor right limit at $x = 0$, use $S(x) = g(x) - g(-x)$. Note that $g(x)$ itself is directly an instance of the fifth type in the misbehavior list.

Another way to create a function in the spirit of $s(x)$ without cases is this. First let

$$s_n(x) = \sin\left(\frac{1}{|x| + \frac{1}{n\pi}}\right).$$

As $n \to \infty$, at every x this sequence of functions approaches

$$\begin{cases} \sin\left(\dfrac{1}{|x|}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases},$$

which is equal to $s(|x|)$. Note that for every $n = 1, 2, \ldots$, $s_n(0) = \sin(n\pi) = 0$. One can also replace the approximating functions $\{s_n(x)\}$ by compactly supported ones using the general procedure that was illustrated above when we replaced $\{u_n(x)\}$ by $\{U_n(x)\}$.

Remark 1. *Fernando Gouvêa of Colby College showed me the bump function* $B_{a,b}(x)$*. Note that this function has corners. If we replace* $p(x)$ *by* $\frac{1}{2}\left(|x|^n + x|x|^{n-1}\right)$ *and then repeat the rest of the construction in a very similar way,the resulting analog of* $B_{a,b}(x)$ *will have a continuous derivative of order* $n-1$*. I don't see an equally simple way to make the analogue of* $B_{a,b}(x)$ *be* C^∞*. From my personal point of view, the trouble with using* $P(x) = \frac{1}{2}(|x| + x)\frac{1}{e^{\frac{1}{x^2}}}$ *and* $N(x) = \frac{1}{2}(|x| - x)\frac{1}{e^{\frac{1}{x^2}}}$ *in place of* $p(x)$ *and* $n(x)$ *is that they are undefined when* $x = 0$*. However, if we are going to use the test of being acceptable to 18th century mathematicians, then removable discontinuities may be removed:thus since* $\lim_{x\to 0} e^{-1/x^2} = 0$*, they would take* $e^{-1/0^2}$ *to be zero. Under these rules, there does exist a simple formula for a compactly supported* C^∞ *bump, namely* $P(x-a)N(x-b)$ *where* $a < b$*.*

Remark 2. *My brother, Peter Ash of Cambridge College, told me that he had seen the function* $\left(1 - x^2\right)^+$ *when he worked in computer graphics. This motivated the alternate formula (6) for the bump* $B_{a,b}$*. My colleague, Alan Berele, created* $\{s_n(x)\}$ *for me.*