

# On concentrating idempotents, a survey

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**ABSTRACT.** A sum of exponentials of the form  $f(x) = \exp(2\pi i N_1 x) + \exp(2\pi i N_2 x) + \dots + \exp(2\pi i N_m x)$ , where the  $N_k$  are distinct integers is called an *idempotent trigonometric polynomial* or, simply, an *idempotent*. It is known that for every  $p > 1$ , and every set  $S$  of the torus  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$  with  $|S| > 0$ , there are idempotents concentrated on  $S$  in the  $L^p$  sense. We sketch how this concentration phenomenon originated as a reformulation of a functional analysis problem, and, in turn, studying concentration led to some interesting questions about  $L^p$  norms of Dirichlet kernels associated with multiple trigonometric series. Some counterexamples involving linear operators not of convolution type are given.

In 1977 I was visiting Stanford on sabbatical from DePaul. It was a most productive year, both personally and professionally. On the first side, I met and married Alison who subsequently gave me the second and third of my three wonderful sons. On the mathematical side, one of the best things was discussions with Mischa Zafran concerning a question about linear operators on  $L^2(\mathbb{T})$ .

## 1. From Operators on $L^2(\mathbb{Z})$ to Concentration

**1.1. Definitions.** By  $L^2(\mathbb{Z}) = \ell^2$  we mean sequences  $C = \{\dots, c_{-1}, c_0, c_1, \dots\}$  of complex numbers such that  $\sum |c_\nu|^2 < \infty$ . We identify the sequence with its Fourier series  $C(x) = \sum c_\nu e^{2\pi i \nu x}$ . A characteristic function associated to the finite subset  $S = \{n_1, n_2, \dots, n_K\}$  of  $\mathbb{Z}$  is a sequence  $\{\dots, s_{-1}, s_0, s_1, \dots\}$  where  $s_\nu = 1$  when  $\nu \in S$  and  $s_\nu = 0$  when  $\nu \notin S$ . The product of  $C = \{c_n\}$  and  $D = \{d_n\}$  is  $CD = \{c_n d_n\}$ , while the convolution is  $C * D = \{\sum_{\nu=-\infty}^{\infty} c_{n-\nu} d_\nu\}$ . A fixed sequence  $K = \{k_\nu\}$  creates an operator on functions on  $\mathbb{Z}$  according to the rule  $T : C \rightarrow K * C$ . We identify  $TC$  with the function  $K(x)C(x)$  where  $K(x) = \sum_{n=-\infty}^{\infty} k_n e^{2\pi i n x}$ . By Plancherel's formula, we have

$$\|TC\|_2^2 = \sum_{n=-\infty}^{\infty} \left| \sum_{\nu=-\infty}^{\infty} k_{n-\nu} c_\nu \right|^2 = \int_0^1 |K(x)C(x)|^2 dx.$$

To avoid confusion we will call the above mentioned characteristic functions *idempotents* and reserve the term *characteristic function* for a function of the form

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2000 *Mathematics Subject Classification.* Primary 42A05, 42-02, 47B37; Secondary 42B99, 46B10, 47B34.

*Key words and phrases.* Concentration of idempotents, projections, weak type (2,2), restricted type (2,2), Dirichlet kernels,  $L^p$  norms, trigonometric polynomials.

This paper is in final form and no version of it will be submitted for publication elsewhere.

$\chi_E(x)$  which is 1 when  $x \in E$ ,  $E$  some measurable subset of  $\mathbb{T} = [0, 1]$ , and 0 when  $x \in \mathbb{T} \setminus E$ . For each finite subset  $S$  of  $\mathbb{Z}$ , the trigonometric polynomial associated to the idempotent associated to  $S$  is  $\iota_S(x) = \sum_{n \in S} e^{2\pi i n x}$ ;  $\iota_S(x)$  derives its name from the identity  $\iota_S \iota_S = \iota_S$ . Here and henceforth we abuse notation and write  $\iota$  for both the sequence and the associated trigonometric polynomial.

A linear operator defined on simple functions is *s. (2, 2)* or of *strong type (2, 2)* or *bounded on  $L^2(\mathbb{Z})$*  if there is a constant  $M > 0$  so that

$$\|TC\|_2 \leq M \|C\|_2$$

where  $\|C\|_2 = \sqrt{\sum |c_n|^2} = \sqrt{\int_0^1 |C(x)|^2 dx}$ . It is *r. (2, 2)* or of *restricted type (2, 2)* if there is a constant  $M > 0$  such that

$$\|T\iota\|_2 \leq M \|\iota\|_2$$

whenever  $\iota = \iota_S$  is an idempotent. It is *w. (2, 2)* or of *weak type (2, 2)* if there is a constant  $M > 0$  such that

$$\|TC\|_{2\infty}^* \leq M \|C\|_2$$

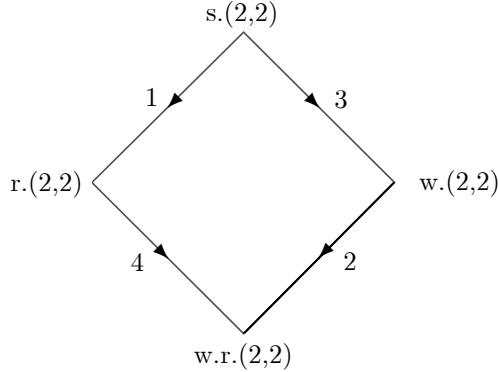
where

$$\|C\|_{2\infty}^* := \sqrt{\sup_{\alpha > 0} |\{n \in \mathbb{Z} : |C^*(n)| > \alpha\}| \alpha^2}$$

and  $C^*$  denotes the nonincreasing rearrangement of  $C$ . Finally it is *w.r. (2, 2)* or of *weak restricted type (2, 2)* if there is a constant  $M > 0$  so that for all idempotents

$$\|T\iota\|_{2\infty}^* \leq M \|\iota\|_2.$$

**1.2. Relating classes of operators.** We have four trivial implications: simply restricting the action of  $T$  to a subset of functions cannot increase the associated constant so that (1) if  $T$  is *s. (2, 2)*, then  $T$  is *r. (2, 2)* and also (2) if  $T$  is *w. (2, 2)*, then  $T$  is *w.r. (2, 2)*. By Tchebycheff's inequality, for each  $\alpha > 0$ ,  $|\{n : |C^*(n)| > \alpha\}| \alpha^2 \leq \sum |c_n|^2$ , so that (3) if  $T$  is *s. (2, 2)*, then  $T$  is *w. (2, 2)* and also (4) if  $T$  is *r. (2, 2)*, then  $T$  is *w.r. (2, 2)*.



None of these implications are reversible in general. To see this, consider the following three linear operators initially defined on idempotents.

$$\begin{aligned} T_1(\{c_n\}) &:= \left\{ \left( \sum_{\nu=-\infty}^{\infty} \frac{c_\nu}{\sqrt{|\nu|+1}} \right) \frac{1}{|n|+1} \right\}, \\ T_2(\{c_n\}) &:= \left\{ \left( \sum_{\nu=-\infty}^{\infty} \frac{c_\nu}{\sqrt{|\nu|+1}} \right) \frac{1}{\sqrt{|n|+1}} \right\}, \text{ and} \\ T_3(\{c_n\}) &:= \left\{ \left( \sum_{\nu=-\infty}^{\infty} \frac{c_\nu}{|\nu|+1} \right) \frac{1}{\sqrt{|n|+1}} \right\}. \end{aligned}$$

The operators  $T_1$  and  $T_2$  cannot be defined on all of  $\ell^2$ ; look at the sequence  $\left\{ \frac{1}{\sqrt{|n|+1} \ln(|n|+2)} \right\} \in \ell^2$  to see this. Since  $T_1 \in r.(2,2)$ , but not  $s.(2,2)$ , implication (1) is irreversible. Since  $T_2 \in w.r.(2,2)$ , but not  $w.(2,2)$ , implication (2) is irreversible. Since  $T_3 \in w.(2,2)$  and hence  $w.r.(2,2)$ , but not in  $r.(2,2)$  and hence not in  $s.(2,2)$ ; neither implication (3) nor implication (4) can be reversed. When the underlying group is the torus  $\mathbb{T} = [0,1)$  with addition mod 1, there are similar counterexamples to all four implications. (See [SW] or [As1] for examples.)

Alexander Stokolos asked me if the fact that the example  $T_1$  and  $T_2$  are not defined on all of  $L^2$  is crucial. For example, one might conjecture that a restricted  $(2,2)$  linear operator defined on all of  $L^2(\mathbb{T})$  is necessarily bounded. Paul Hagelstein and Brian Raines created the following counterexample for me.

**THEOREM 1.** *There exists a linear operator  $T : L^2(\mathbb{T}) \rightarrow \mathbb{R}$  such that  $T\chi_E = 0$  for any measurable set  $E$  and such that  $T$  is unbounded on  $L^2(\mathbb{T})$ .*

**PROOF.** Let  $\mathcal{S}$  denote the set of simple functions on  $\mathbb{T}$ . Let  $g_1 \in L^2/\mathcal{S}$  and  $[g_1] = \{ag_1 + h : a \in \mathbb{C}, h \in \mathcal{S}\}$ . Proceeding with transfinite induction, assume  $\{g_\gamma\}_{\gamma < \beta}$  and  $\{[g_\gamma]\}_{\gamma < \beta}$  have been constructed. Let  $g_\beta \in L^2 / \bigcup_{\gamma < \beta} [g_\gamma]$  and set

$$[g_\beta] = \left\{ ag_\beta + h : a \in \mathbb{C}, h \in \bigcup_{\gamma < \beta} [g_\gamma] \right\}. \text{ Note } L^2 = \bigcup_{\gamma < 2^{\omega_0}} [g_\gamma], \text{ where } 2^{\omega_0} \text{ is the}$$

cardinality of the continuum. Let  $\varphi$  be any bijection from the  $\{g_\gamma : \gamma < 2^{\omega_0}\}$  to the real numbers. Define  $Ts = 0$  for all  $s \in \mathcal{S}$ . If  $f = ag_1 + h$  with  $h \in \mathcal{S}$ , define  $Tf = a \|g_1\|_2 \varphi(g_1)$ ;  $T$  is linear on  $[g_1]$ . Extend  $T$  inductively to  $L^2$ : if  $T$  is defined and linear on  $\bigcup_{\gamma < \beta} [g_\gamma]$  and  $f = ag_\beta + h \in [g_\beta]$ , then define  $Tf = a \|g_\beta\|_2 \varphi(g_\beta) + Th$ .

Now  $T$  is defined and linear on  $[g_\beta]$ . By the principle of transfinite induction,  $T$  is defined and linear on all of  $L^2$ . The operator norm of  $T|_{\mathcal{S}}$  is zero, so  $T \in r.(2,2)$ ; but  $T$  is unbounded on  $L^2$ , since  $T$  stretches the  $L^2$  norm of  $g_\beta$  by  $|\varphi(g_\beta)|$ , and  $\varphi(g_\beta)$  can be any (arbitrarily large) real number.  $\square$

In the late 1970s several people suspected that these implications were reversible for convolution operators. I focused on trying to reverse implication (1), in other words to prove:

(1.1) For convolution operators defined on idempotents,  $r.(2,2)$  implies  $s.(2,2)$ .

Because of the following simple duality result, this would also reverse implication (3).

LEMMA 1. *If a convolution operator  $T : C \rightarrow K * C$  has the property that  $r.(2, 2) \implies s.(2, 2)$ , then by duality it has the property  $w.(2, 2) \implies s.(2, 2)$ .*

PROOF. For finite sequences  $C = \{c_n\}$  and  $D = \{d_n\}$  we have

$$(1.2) \quad (C, TD) = \sum_{n=-\infty}^{\infty} c_n \overline{(TD)_n}$$

$$(1.3) \quad = \sum_{n=-\infty}^{\infty} c_n \overline{\left( \sum_{\nu=-\infty}^{\infty} k_{n-\nu} g_\nu \right)}$$

$$(1.4) \quad = \sum_{\nu=-\infty}^{\infty} \left( \sum_{n=-\infty}^{\infty} c_n k_{\nu-n}^* \right) \overline{g_\nu} = (T^* C, D)$$

where for every  $n \in \mathbb{Z}$ ,  $k_n^* = \overline{k_{-n}}$  and  $T^*$  is the convolution operator corresponding to  $\{k_n^*\}$ . Let  $K^*(x)$  be the Fourier series associated with  $\{k_n^*\}$ .

Let  $T$  be  $w.(2, 2)$ . This means that there is a constant  $M$  so that

$$(1.5) \quad \|TC\|_{2\infty}^* \leq M \|C\|_2.$$

We show that this implies that  $T^*$  is  $r.(2, 2)$ .

Let  $\iota_S$  be the idempotent associated to a finite set  $S$ ,  $S \subset \mathbb{Z}$ . By the definition of adjoint and Lorentz's generalization of Holder's inequality (see page 261 of [Hu]) we have

$$\begin{aligned} \|T^* \iota_S\|_2 &= \sup_{\|D\|_2 \leq 1} |(K^* * \iota_S, D)| \\ &= \sup_{\|D\|_2 \leq 1} |(\iota_S, K * D)| \\ &\leq 2 \sup_{\|D\|_2 \leq 1} \|\iota_S\|_{21}^* \|K * D\|_{2\infty}^*. \end{aligned}$$

Applying inequality 1.5 yields

$$(1.6) \quad \|T^* \iota_S\|_2 \leq M \|\iota_S\|_{21}^*.$$

Letting  $C^*$  denotes the nonincreasing rearrangement of a sequence  $C$ , the following calculations hold

$$(1.7) \quad \begin{aligned} \|\iota_S\|_{21}^* &:= \frac{1}{2} \int \iota_S^*(x) \frac{dx}{\sqrt{x}} = \frac{1}{2} \int_0^{|S|} \frac{dx}{\sqrt{x}} = \sqrt{|S|} \text{ and} \\ \|\iota_S\|_2 &= \sqrt{\int (\iota_S^*(x))^2 dx} = \sqrt{\int_0^{|S|} dx} = \sqrt{|S|}. \end{aligned}$$

From this and inequality 1.6 we have

$$(1.8) \quad \|T^* \iota_S\|_2 \leq M \|\iota_S\|_2,$$

so that  $T^*$  is  $r.(2, 2)$ . Our hypothesis now yields that  $T^*$  is  $s.(2, 2)$ . Finally if  $T^*$  is  $s.(2, 2)$ , so is  $T$ .  $\square$

**1.3. A surprising connection.** We say that  $L^2$  interval concentration occurs if there is an absolute constant  $a > 0$  such that for each interval  $I \subset [0, 1]$  there is an idempotent  $\iota(x) = \iota_I(x) = \sum_{j=1}^K e^{2\pi n_j x} \in L^2(\mathbb{Z})$  so that

$$\frac{\int_I |\iota(x)|^2 dx}{\int_0^1 |\iota(x)|^2 dx} > a$$

and that  $L^2$  set concentration occurs if there is an absolute constant  $b > 0$  such that for each set of positive measure  $E \subset [0, 1]$  there is an idempotent  $\iota(x) = \iota_E(x)$  so that

$$(1.9) \quad \frac{\int_E |\iota(x)|^2 dx}{\int_0^1 |\iota(x)|^2 dx} > b.$$

Note that if  $b$  exists we may take  $a$  to be  $b$ .

When I was studying implication (1.1) in the late 1970s, I was only able to find an equivalent formulation in terms of concentration. Here is that equivalence.

**THEOREM 2.** *If  $L^2$  concentration for sets holds, then  $r.(2, 2)$  implies  $s.(2, 2)$  when the underlying group is  $\mathbb{Z}$ .*

*If  $r.(2, 2)$  implies  $s.(2, 2)$  when the underlying group is  $\mathbb{Z}$ , then  $L^2$  concentration for intervals holds.*

**PROOF.** Assume that  $L^2$  concentration holds for sets and that  $T$  is  $r.(2, 2)$ . Letting  $T$  correspond to convolution with  $\{k_n\}$  so that with  $K(x) = \sum k_n e^{2\pi i n x}$  for a certain positive constant  $A$  the inequality

$$(1.10) \quad \int_0^1 |K(x) \iota(x)|^2 dx \leq A \int_0^1 |\iota(x)|^2 dx$$

holds for every idempotent  $\iota$ . We also know that there is a positive number  $b$  so that inequality (1.9) holds. Our goal is to show that  $K(x)$  is an essentially bounded function, since this is well known to be the necessary and sufficient condition for a multiplier operator to be bounded on  $L^2$ . Assume that the multiplier  $K(x)$  exceeds  $A/b$  on a set  $E$  of positive measure. Find an idempotent  $\iota$  so that

$$\int_E |\iota(x)|^2 dx > b \int_0^1 |\iota(x)|^2 dx$$

Applying first this and then inequality (1.10), we find

$$\begin{aligned} \int_0^1 |\iota(x)|^2 dx &< \frac{1}{b} \int_E |\iota(x)|^2 dx \\ &= \frac{1}{A} \frac{A}{b} \int_E |\iota(x)|^2 dx \\ &\leq \frac{1}{A} \int_E |K(x) \iota(x)|^2 dx \\ &\leq \frac{1}{A} A \int_0^1 |\iota(x)|^2 dx, \end{aligned}$$

which is a contradiction.

The converse implication is Theorem 7 of [As1]: Assume the failure of  $L^2$  concentration for intervals. Then there is a sequence of intervals  $\{I_i\}_{i=1,2,\dots}$  so that

for every  $i$  and every idempotent  $\iota$ ,

$$\int_{I_i} |\iota(x)|^2 dx \leq 2^{-2i} \int_0^1 |\iota(x)|^2 dx.$$

Then  $K(x) = \sum i \chi_{I_i}(x)$  is unbounded so that the operator corresponding to the multiplier  $K$  is not bounded on  $L^2(\mathbb{Z})$ . However, for any idempotent  $\iota$ , by Minkowski's inequality we have

$$\begin{aligned} \|K(x)\iota(x)\|_2 &\leq \sum \left( \int_0^1 |i \chi_{I_i}(x) \iota(x)|^2 dx \right)^{1/2} \\ &\leq \sum i \left( \int_{I_i} |\iota(x)|^2 dx \right)^{1/2} \\ &\leq \left( \sum i 2^{-i} \right) \left( \int_0^1 |\iota(x)|^2 dx \right)^{1/2} \\ &= 2 \|\iota(x)\|_2 \end{aligned}$$

so that the operator is  $r.(2, 2)$ .  $\square$

**1.4. Results for  $L^2$  Concentration.** Just about the time of the formulation of the equivalence theorem, Michael Cowling proved that when the underlying group is  $\mathbb{Z}$ ,  $r.(2, 2)$  implies  $s.(2, 2)$ . [Co] Actually, he proved much, much more than this. He proved that if the underlying group is any amenable group, then  $w.r.(2, 2)$  implies  $s.(2, 2)$ . An amenable group is a topological group  $G$  carrying a kind of averaging operation, that is invariant under translations by group elements. In the case where  $G$  is not an abelian group, that means translation on a fixed side (left- or right-translation). For our purposes, it is enough to know that  $\mathbb{Z}$  is an amenable group. So by the equivalence theorem above, it followed that  $L^2$  concentration for intervals was true!

But this result is exceedingly non constructive. Define the absolute constant  $C_2$  as the largest real number such that for *every* set  $E \subset \mathbb{T}$  with  $|E| > 0$ , and every  $\epsilon > 0$ , there is an idempotent  $\iota = \iota_{E, \epsilon}$  satisfying the inequality

$$(1.11) \quad \frac{\sqrt{\int_E |\iota(x)|^2 dx}}{\sqrt{\int_{\mathbb{T}} |\iota(x)|^2 dx}} \geq C_2 - \epsilon.$$

Thus  $C_2$  is the amount of  $L^2$  norm that can be concentrated on any set, no matter how small, nor no matter how inconveniently situated. Obviously,  $C_2 \leq 1$ . So far we know that  $C_2 > 0$ , but our information is neither quantitative nor constructive, we have no idea of its size, no lower bound, nor any effective procedure for finding one.

**1.5. Quantitative results for  $L^2$  concentration.** (1) The referee of paper [As1] pointed out that  $C_2$  must be at least  $1/8 = .125$ . This follows from Cowling's Theorem. The assumption of Cowling's Theorem, that  $T$  is a convolution operator of type  $w.r.(2, 2)$  means that there is a constant  $A > 0$  such that for every idempotent  $\iota$ , we have

$$(1.12) \quad \|T\iota\|_{2\infty}^* \leq A \|\iota\|_2 = A \|\iota\|_{21}^*.$$

There are two steps to the proof that  $T$  is of strong type; first we show that inequality (1.12) can be extended to hold for simple functions and hence for all functions in  $L^{21}$  - this step results in the operator norm being stretched by at most 8. In other words, producing the inequality

$$\|TC\|_{2\infty}^* \leq 8A \|C\|_{21}^*$$

for every sequence  $C \in L^{21}$ . (This step is shown in detail on page 682 of [As1].) The second step gets from this to the final result

$$(1.13) \quad \|TC\|_2 \leq 8A \|C\|_2$$

for every  $C \in L^2$  with no further increase in operator norm.

Assume that the  $L^2$  concentration constant  $C_2$  satisfies  $C_2 < 1/8$ . This means that there is a set  $E \subset \mathbb{T}$  of positive measure so that for every idempotent  $\iota$ ,

$$(1.14) \quad \|\chi_E(x) \iota(x)\|_2 \leq C_2 \|\iota(x)\|_2.$$

Let  $T$  be the linear operator associated with the multiplier function  $\chi_E(x)$ . Then by Tchebycheff's inequality, for any idempotent  $\iota$

$$(1.15) \quad \|T\iota\|_{2\infty}^* \leq \|T\iota\|_2 = \|\chi_E(x) \iota(x)\|_2.$$

Concatenating inequalities (1.14) and (1.15) gives

$$\|T\iota\|_{2\infty}^* \leq C_2 \|\iota(x)\|_2.$$

Thus by the remark above that the strong  $(2, 2)$  constant is at most 8 times the weak restricted  $(2, 2)$  constant we have that for every sequence  $C$ ,

$$\|TC\|_2 \leq 8C_2 \|C\|_2.$$

But  $8C_2 < 1$ , which contradicts the fact that the strong  $(2, 2)$  norm of the operator  $T$  must be 1, since 1 is the essential supremum of the multiplier function  $\chi_E$ .

Define a constant  $C_2^*$  which is for intervals what  $C_2$  is for sets. In other words, the absolute constant  $C_2^*$  is the largest real number such that for *every* interval  $J \subset \mathbb{T}$  with  $|J| > 0$ , and every  $\epsilon > 0$ , there is an idempotent  $\iota = \iota_{J,\epsilon}$  satisfying the inequality

$$(1.16) \quad \frac{\sqrt{\int_J |\iota(x)|^2 dx}}{\sqrt{\int_{\mathbb{T}} |\iota(x)|^2 dx}} \geq C_2^* - \epsilon.$$

Of course  $C_2^* \geq C_2$ .

(2) S. Pichorides [Pi] obtained  $C_2^* \geq .14$ .

(3) H. L. Montgomery [Mo], and

(4) J.-P. Kahane [Ka2] obtained several better lower bounds. (The ideas of H. L. Montgomery were “deterministic” while those of J.-P. Kahane used probabilistic methods from [Ka1].)

**1.6. The best possible  $L^2$  concentration constant.** Finally, in [AJS], together with Roger Jones and Bahman Saffari, I achieved this lower bound for  $C_2$  :

$$(1.17) \quad \gamma_2 := \max_{x>0} \frac{\sin x}{\sqrt{\pi x}} = .4802...;$$

which, in [DPQ1], was proved to be best possible, thus  $C_2 = \gamma_2$ . (See [DPQ2] for a more detailed exposition of the contents of [DPQ1].)

## 2. A Paper 20 Years in the Making

In 1982, I began to consider the  $L^p$  concentration question for values of  $p$  other than 2. This question represents a move away from the functional analysis issues naturally connected to the  $L^2$  concentration question in Theorem 2 for two reasons. First, only for  $p = 2$  do we have the very simple characterization of a convolution operator being bounded if and only if the corresponding multiplier function is in  $L^\infty$ . Second, Misha Zafran has shown that even when the underlying group is  $\mathbb{T}$  so that things are as simple as possible, there are convolution operators of type  $w.(p, p)$  but not  $s.(p, p)$  when  $1 < p < 2$  and thus by duality there are also convolution operators of type  $r.(p, p)$  but not  $s.(p, p)$  when  $2 < p < \infty$ . [Za] Consequently,  $w.r.(p, p)$  implies  $s.(p, p)$  only when  $p = 2$ .

Misha Zafran was a talented mathematician. When I was spending 1977 at Stanford, Misha selflessly and patiently shared many ideas. The subject discussed here owes much of its origin to Misha. I profoundly regret his untimely death.

**2.1. The early years.** In the fall of 1982, Roger Jones and I submitted a grant proposal to the National Science Foundation centered around the question of whether  $L^p$  concentration was valid for any  $p < 2$ . One proposal reviewer wrote “...This [ $L^p$  concentration] is a very specific problem. They [Ash and Jones] mention several ways it has been done for  $p > 2$ . Doing it for  $p < 2$  doesn’t seem very difficult either; a product of Dirichlet kernels is likely to work.” Needless to say, the proposal was not funded. Intrigued by the above comment, I communicated a desire to the NSF analysis director John Ryff that the referee divest anonymity and collaborate on the question. A few months later, I received a letter from Dan Rider stating “...Enclosed is a sketch of what I think works for your problem for  $1 < p \leq 2$ . It turned out to be messier than I had anticipated...”

Here, in very heuristic terms, is Rider’s idea. We fix  $p > 1$  and explain how to find an idempotent which has a goodly percentage of its  $L^p$  mass near  $k/q$ , where  $q$  is a large prime and  $1 \leq k \leq q - 1$ . First let  $k = 1$ . We concentrate  $L^p$  mass on the interval  $\left[\frac{1}{q} - \frac{1}{q^2}, \frac{1}{q} + \frac{1}{q^2}\right]$  by considering the idempotent

$$I(x) = D_{q^2}(qx) D_{\frac{q-1}{2}}(x),$$

where  $D_n(x) = \sum_{\nu=0}^{n-1} e^{2\pi i \nu x}$ . The first factor is concentrated near  $x = 0$  and has period  $1/q$  so we think of it as being roughly

$$c \sum_{j=0}^{q-1} \chi_{\left[\frac{j}{q} - \frac{1}{q^2}, \frac{j}{q} + \frac{1}{q^2}\right]}(x),$$

in other words as a series of  $q$  equal pulses. The second factor we think of as being even and having decay like  $1/x$  on  $[1/q, 1/2]$ . This gives  $L^p$  concentration since

$$\frac{\int_{\frac{1}{q} - \frac{1}{q^2}}^{\frac{1}{q} + \frac{1}{q^2}} |I(x)|^p dx}{\int_0^1 |I(x)|^p dx} \approx \frac{\sum_{j=1}^1 \frac{1}{j^p}}{\sum_{j=1}^{(q-1)/2} \frac{1}{j^p}} \geq \frac{1}{\sum_{j=1}^\infty \frac{1}{j^p}}.$$

If  $k > 1$ , since the set  $\{1, 2, \dots, q-1\}$  is a group under multiplication modulo  $q$ , we may find  $a$  so that  $ka \equiv 1 \pmod{q}$  and use

$$D_{q^2}(qx) D_{\frac{q-1}{2}}(ax)$$

as our concentrated idempotent. Notice that  $a \left( \frac{k}{q} + \delta \right) \equiv \frac{1}{q} + a\delta \pmod{1}$  so that this idempotent behaves at  $k/q$  in a way that is similar to how  $I(x)$  behaved at  $1/q$ .

**2.2. On the virtues of procrastination.** Our early work on the  $L^p$  concentration problem is summarized in [AAJRS1]. This paper lists the results of what we were able to prove developing what was mentioned above. We had established  $L^p$  concentration for intervals for  $1 < p < \infty$  and  $L^p$  concentration for sets for  $2 \leq p < \infty$ . A good thing was that the method was both quantitative and constructive. But there was a 17 year gap between the  $L^2$  result in [AJS] and this and there would be another 7 year gap between this and our next paper [AAJRS2]. Two reasons for this time gap were our desire to find out whether  $L^p$  concentration for sets held when  $1 < p < 2$  and whether there was any  $L^1$  concentration. We had a little negative evidence for the latter and more interesting (to us) of these questions, namely that if one considers the “enemy” interval  $\left[ \frac{1}{q} - \frac{1}{q^2}, \frac{1}{q} + \frac{1}{q^2} \right]$ , the fraction of the  $L^1$  mass of  $I(x)$  concentrated here is roughly

$$\frac{1}{\sum_{j=1}^{(q-1)/2} \frac{1}{j}} \approx \frac{1}{\ln q}.$$

So if  $I(x)$  were about as concentrated as an idempotent can be, then  $C_1$  would be less than  $\frac{1}{\ln q}$  for arbitrarily large  $q$  and  $L^1$  concentration would be false.

We finally gave up on resolving these two questions and sent the paper to Annal. Inst. Fourier where an excellent referee showed us how to solve the former question, that is how to establish concentration for sets when  $1 < p < 2$ . What was surprising to me was that doing this involved using the following two dimensional result.

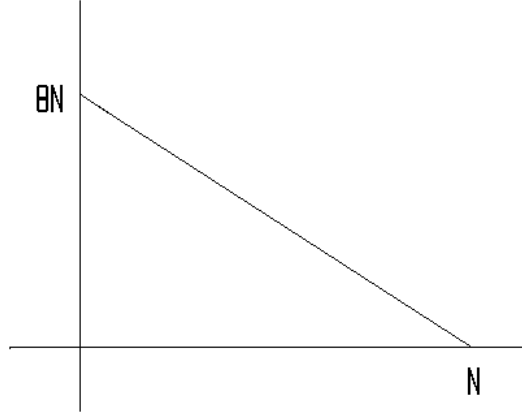
LEMMA 2 (Triangle Lemma). *Let  $p > 1$ ,  $0 < \theta < 1$ , and*

$$K_{\theta,N} = \{(x, y) \in \mathbb{Z}^2 : x + \theta^{-1}y \leq N, x \geq 0, y \geq 0\}.$$

*Then for arbitrary  $N \geq 4$ ,*

$$(2.1) \quad \int_0^1 \int_0^1 \left| \sum_{(m,n) \in K_{\theta,N}} e^{2\pi i(mx+ny)} \right|^p dx dy \leq C_p N^{2p-2}$$

uniformly with respect to  $\theta$  and  $N$ .



But this was still not the end of the line for paper [AAJRS2]. We resubmitted the paper with the Triangle Lemma linked in, but the referee then pointed out that the proof of the Triangle Lemma was just fine, but that the linkage was defective. We now needed a number theory fact that none of us knew, namely:

LEMMA 3. *Almost every point  $\xi$  has the property that there are infinitely many primes  $q$  and integers  $k$  for which*

$$\left| \xi - \frac{k}{q} \right| < \frac{1}{q^2}.$$

But paper [AAJRS2] had taken so long to evolve that the internet was now equipped with strong enough search engines to allow a one day wrap up. Searching for "Approximation to Irrational Number by Rational Numbers with Prime Denominators" led me directly to Chao-Hua Jia in China. His immediate response to my email in turn led me to Glyn Harman in England. Harman immediately emailed me that Lemma 3 was on page 27 of his book! [Ha] One last point: we were able to guess the identity of the referee, from the combination of his previously displayed expertise in moving from intervals to sets of positive measure, his involvement with two dimensional estimates like the Triangle Lemma, and his extreme generosity. Since Fedja Nazarov would not let us make him a coauthor, I thank him here.

### 3. The Future

**3.1. A segue.** The Triangle Lemma, Lemma 2 was a natural conjecture since

$$\begin{aligned}
& \int_0^1 \int_0^1 \left| \sum_{|m|, |n| \leq N} e^{2\pi i(mx+ny)} \right|^p dx dy \\
&= \int_0^1 \int_0^1 \left| \sum_{|m| \leq N} e^{2\pi imx} \sum_{|n| \leq N} e^{2\pi iny} \right|^p dx dy \\
&= \left\{ \int_0^1 \left| \sum_{|m| \leq N} e^{2\pi imx} \right|^p dx \right\}^2 \\
&= \{O(N^{p-1})\}^2 = O(N^{2p-2}).
\end{aligned}$$

Furthermore,

$$\int_0^1 \int_0^1 \left| \sum_{(m,n) \in K_{\theta,N}} e^{2\pi i(mx+ny)} \right| dx dy \leq C_p \ln^2 N,$$

(which is again obvious when  $K_{\theta,N}$  is replaced by the square  $\{|m|, |n| \leq N\}$ ) had been proved by Yudin and Yudin.[YY] Their proof involved a number theoretic argument and luckily extended directly to  $p > 1$ . (See [As2].)

Thinking further about the Triangle Lemma led me to think about generalizations. First of all, every convex polygon can be decomposed into a finite number of triangles, so it is almost immediate that an estimate like (2.1) holds for convex polygons in general. The next thought was to try to push the result up to three dimensions. Unfortunately, the number theory required to extend the method to a higher dimension appeared formidable.

The one dimensional integral

$$\int_0^1 \left| \sum_{|k| \leq N} e^{2\pi ikx} \right| dx \asymp \frac{4}{\pi^2} \ln N$$

is called the “Lebesgue constant.” The  $L^p$  Lebesgue constant is

$$\int_0^1 \left| \sum_{|k| \leq N} e^{2\pi ikx} \right|^p dx \asymp \delta_p N^{p-1}.$$

Since  $\{k \in \mathbb{Z} : |k| \leq N\} = N[-1, 1]$  is the set of lattice points in the dilate of the one dimensional convex polygon  $[-1, 1]$  by  $N$ , it is natural to let

$$L_N(D) = \int_{T^d} \left| \sum_{k \in ND \cap \mathbb{Z}^d} e^{2\pi ik \cdot x} \right|^p dx$$

be a  $d$ -dimensional  $L^p$  Lebesgue constant for any bounded set  $D \subset \mathbb{R}^d$ . This raised the following two questions. (1) If  $p > 1$  and  $D$  is a  $d$ -dimensional polyhedron, do there exist constants  $c = c(p, D)$  and  $C = C(p, D)$  so that

$$cN^{d(p-1)} \leq L_N(D) \leq CN^{d(p-1)}?$$

(2) What can be said for more general bounded  $d$ -dimensional sets with nonempty interior?

The first question became much more accessible when Elijah Liflyand told me that Belinsky had proved the  $p = 1$  analogue of this with a methodology that avoided the delicate number theory completely.[BT] With respect to the second question, here is a conjecture.

CONJECTURE 1. *If  $D$  is a bounded  $d$ -dimensional set with nonempty interior then*

$$cN^{d(p-1)} \leq L_N(D) \leq \begin{cases} CN^{d(p-1)} & \text{if } p > \frac{2d}{d+1} \\ CN^{\frac{d-1}{2}p} \ln^{\frac{d+1}{2d}p} N & \text{if } p = \frac{2d}{d+1} \\ CN^{\frac{d-1}{2}p} & \text{if } 1 < p < \frac{2d}{d+1} \end{cases}.$$

Laura De Carli and I have affirmed the first question and have some partial progress that points toward the conjecture in a recently submitted paper.[AD]

**3.2. The  $L^1$  concentration question.** Aline Bonami and Szilárd Gy. Révész appear to have disproved the main conjecture of [AAJRS2]. Evidentially  $L^p$  concentration *does* occur for  $p = 1$  and even for some  $p < 1$  as well. These results should appear soon. To keep abreast of developments, look at Révész's website, <http://www.renyi.hu/~revesz/preprints.html>.

**3.3. A conjecture about operators.** A long standing question about convolution operators is this.

CONJECTURE 2. *Let  $p > 2$ . If a convolution operator is  $w.(p, p)$ , then it must also be  $s.(p, p)$ .*

I have no feelings either way concerning the truth of this conjecture. Misha Zafran warned me of difficulty in 1977, so I am not too surprised to see that it is still unsolved.

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