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The classification of complex generalized Riemann derivatives ☆

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ABSTRACT

This article completes the more than a half a century old problem of finding the equivalences between generalized Riemann derivatives. The real functions case is studied in a recent paper by the authors. The complex functions case developed here is more general and comes with numerous applications.

We say that a complex generalized Riemann derivative $\mathcal{A}$ implies another complex generalized Riemann derivative $\mathcal{B}$ if whenever a measurable complex function is $\mathcal{A}$-differentiable at z then it is $\mathcal{B}$-differentiable at z. We characterize all pairs $(\Delta_{\mathcal{A}}, \Delta_{\mathcal{B}})$ of complex generalized Riemann differences of any orders for which $\mathcal{A}$-differentiability implies $\mathcal{B}$-differentiability, and those for which $\mathcal{A}$-differentiability is equivalent to $\mathcal{B}$-differentiability. We show that all m points based generalized Riemann difference quotients of order n that Taylor approximate the ordinary nth derivative to highest rank form a projective variety of dimension $m-n$ for which an explicit parametrization is given.

One application provides an infinite number of equivalent ways to define analyticity. For example, a function f is analytic on a region Ω if and only if at each z in Ω, the limit

$$\lim_{h\to 0}\frac{f(z+h)+f(z+ih)-f(z-h)+f(z-ih)-2f(z)}{2h}$$

exists and is a finite number. Four more applications relate the classification of complex generalized Riemann derivatives to analyticity and the Cauchy-Riemann equations, and to the theory of best approximations.

A complex function f is n times *Peano differentiable* at z if there exist complex numbers $f_0(z), f_1(z), \dots,$ $f_n(z)$ such that, as h tends to 0,

$$f(z+h)=f_0(z)+f_1(z)h+\cdots+f_n(z)\frac{h^n}{n!}+o(h^n).$$

Taylor expansion shows that if f is n times ordinary differentiable at z then it is n times Peano differentiable

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at z and the nth *Peano derivative* $f_n(z)$ equals the nth ordinary derivative $f^{(n)}(z)$. The converse is in general false. Indeed, for $n \geq 2$, the function f defined as $f(z) = z^{n+1}$ for z rational, and $f(z) = 0$ for any other complex number z is n times Peano differentiable at 0, with $f_n(0) = 0$, and is not twice ordinary differentiable at 0.

Let $\mathcal{A} = \{A_1, \ldots, A_m; a_1, \ldots, a_m\}$ be a $2m$-vector of complex numbers such that $a_1, \ldots, a_m$ are distinct and satisfy the Vandermonde conditions $\sum_{i=1}^m A_i a_i^j = \delta_{jn} n!$, for $j = 0, 1, \ldots, n$. A complex function f is n times *generalized Riemann differentiable* in the sense of $\mathcal{A}$ at z, or f is $\mathcal{A}$-*differentiable* at z, if the limit

$$D_{\mathcal{A}} f(z) = \lim_{h \to 0} \frac{A_1 f(z + a_1 h) + A_2 f(z + a_2 h) + \cdots + A_m f(z + a_m h)}{h^n}$$

exists and is a finite number. The numerator $\Delta_{\mathcal{A}} f(z, h)$ is called an nth generalized Riemann difference. The Vandermonde conditions imply that $m > n$. Furthermore, if the nth Peano derivative exists at a point, then every nth generalized Riemann derivative exists and agrees with it.

Real generalized Riemann derivatives were introduced by Denjoy in [22]. Special cases are the nth Riemann derivative that has $\Delta_{\mathcal{A}} f(x, h) = \sum_{i=0}^n (-1)^i \binom{n}{i} f(x + (n - i)h)$, which is the ordinary derivative when $n = 1$; and the nth symmetric Riemann derivative that has $\Delta_{\mathcal{A}} f(x, h) = \sum_{i=0}^n (-1)^i \binom{n}{i} f(x + (\frac{n}{2} - i)h)$, which is the first symmetric derivative when $n = 1$, and is the Schwarz derivative when $n = 2$. Stein and Zygmund have shown in [47,51] that the generalized Riemann derivatives have numerous applications in the theory of trigonometric series.

The first classification of real generalized Riemann derivatives is due to Ash in [3], where it was shown that if f is $\mathcal{A}$-differentiable of order n on a measurable set E, then a.e. on E, f has n Peano derivatives. This means that all nth generalized Riemann derivatives are a.e. equivalent to the nth Peano derivative, and consequently to one another. In other words, all generalized Riemann derivatives classify, according to a.e. equivalence, as only one equivalence class for each order of differentiation. Ash's result followed an earlier result of Marcinkiewicz and Zygmund in [37] that the nth Riemann derivative is a.e. equivalent to the nth Peano derivative. This in turn was preceded by a result of Khintchine in [34] on the a.e. equivalence between the symmetric and the ordinary first derivatives.

Pointwise results on generalized Riemann derivatives are relatively less common in the literature. Basic examples like $f(x) = |x|$, which is symmetrically differentiable but not differentiable at 0, had suggested until very recently that pointwise generalized Riemann derivatives are far from the ordinary derivatives, hence from each other. Then, a remarkable phenomenon observed in [12], that the first generalized Riemann differentiation

$$D_{\mathcal{A}} f(x) = \lim_{h \to 0} \frac{2f(x + h) + f(x - h) - 3f(x)}{h}$$

is equivalent to ordinary differentiation, led to the classification of all first order generalized Riemann derivatives at x that are equivalent to the ordinary first order derivative [12, Theorem 1]. Restated later on as Theorem 3.4, this theorem provides the equivalence class of the ordinary first order derivative at x under the relation "$\mathcal{A}$ is equivalent to $\mathcal{B}$" if, "for all measurable functions f and points x, f is $\mathcal{A}$-differentiable at x is equivalent to f is $\mathcal{B}$-differentiable at x". This equivalence class is much smaller than all first order generalized Riemann derivatives. The same theorem shows that no generalized Riemann derivative can ever be pointwise equivalent to any ordinary derivative of order > 1.

Description of results. This article has five main results: Theorems 2.4 and 2.8 on the classification of complex $\mathcal{A}$-derivatives, Theorem 3.1 on $\mathcal{A}$-differentiation and ℓ-grading, Theorem 4.3 on the numerical analysis of complex $\mathcal{A}$-derivatives, and Theorem 6.6 on the connection between group algebras and $\mathcal{A}$-derivatives. In addition, there are five applications: Propositions 3.5, 3.8 and 3.9 on $\mathcal{A}$-differentiation and analyticity, and Theorems 5.2 and 5.3 on best approximations and the classification of complex $\mathcal{A}$-derivatives.

We now list general descriptions of the contents of each of the six sections that comprise this paper.

Section 1. The classification of all real generalized Riemann derivatives of all orders according to pointwise equivalence is given in [13]. A detailed description of the real classification is included in this section. The classification is given in terms of even and odd components of a generalized Riemann difference, and its proof involves the group algebra of the multiplicative group of the real numbers over the real field, whose torsion subgroup, or the subgroup of all elements of finite order, is $G_1 = \{\pm 1\} = U_2$.

Section 2. Both the almost everywhere and pointwise classifications of real generalized Riemann derivatives, and their proofs extend naturally to the more complicated complex case. The proof of the pointwise case uses the group algebra of the multiplicative group G of complex numbers over the complex field. The torsion subgroup G_1 of G is the more complicated union $G_1 = \bigcup_{\ell=2}^{\infty} U_\ell$ of subgroups $U_\ell = \{1, \omega, \ldots, \omega^{\ell-1}\}$, where $\omega = e^{2\pi i/\ell}$. Therefore, the U_2-grading for the real differences, that is, the writing of every difference as a unique sum of even and odd differences, will become a U_ℓ-grading (or simply an ℓ-grading) for all ℓ, that is, every difference $\Delta_{\mathcal{A}} f(z,h)$ is uniquely a sum of ℓ-graded components

$$\Delta_{\mathcal{A}} f(z,h) = \sum_{k=0}^{\ell-1} \Delta_{\mathcal{A}}^{(k,\ell)} f(z,h), \text{ where } \Delta_{\mathcal{A}}^{(k,\ell)} f(z,h) = \frac{1}{\ell} \sum_{j=0}^{\ell-1} \omega^{-kj} \Delta_{\mathcal{A}}^{(k,\ell)} f(z, \omega^j h).$$

This is consistent with the decomposition of a complex function as a sum of ℓ-graded components, given in [16]. As an example, if $\Delta_{\mathcal{B}} f(z,h) = f(x+h) - f(x)$ is the difference corresponding to the ordinary first derivative, then the $\ell = 2$-components

$$\Delta_{\mathcal{B}}^{(0,2)} f(z,h) = \frac{1}{2}\{f(z+h) - 2f(z) + f(z-h)\} \text{ and } \Delta_{\mathcal{B}}^{(1,2)} f(z,h) = \frac{1}{2}\{f(z+h) - f(z-h)\}$$

are its even and odd components, and the $\ell = 3$-components of the same difference are

$$\Delta_{\mathcal{B}}^{(0,3)} f(z,h) = \frac{1}{3}\{f(z+h) + f(z+\omega h) + f(z+\omega^2 h)\} - f(z),$$
$$\Delta_{\mathcal{B}}^{(1,3)} f(z,h) = \frac{1}{3}\{f(z+h) + \omega^2 f(z+\omega h) + \omega f(z+\omega^2 h)\},$$
$$\Delta_{\mathcal{B}}^{(2,3)} f(z,h) = \frac{1}{3}\{f(z+h) + \omega f(z+\omega h) + \omega^2 f(z+\omega^2 h)\},$$

where $\omega = e^{2\pi i/3}$. *The first main theorem*, Theorem 2.4, classifies all pairs $(\Delta_{\mathcal{A}}, \Delta_{\mathcal{B}})$ of generalized Riemann differences for which pointwise $\mathcal{A}$-differentiation is equivalent to pointwise $\mathcal{B}$-differentiation, for all measurable functions f at z. Specifically, $\mathcal{A}$-differentiation of order n is equivalent to $\mathcal{B}$-differentiation of order ν if and only if $n = \nu$ and, for a fixed ℓ and variable $k = 0, \ldots, \ell-1$, $\Delta_{\mathcal{A}}^{(k,\ell)} f(z,h) = R\Delta_{\mathcal{B}}^{(k,\ell)} f(z, rh)$, for some $R, r \neq 0$, depending on k, ℓ, that satisfy the normalizing condition $Rr^n = 1$ when $k = n \mod \ell$. For example, when $\ell = 2$, the generalized Riemann derivatives $\mathcal{A}$ equivalent to ordinary differentiation $\mathcal{B}$, that is $\Delta_{\mathcal{B}} f(z,h) = f(z+h) - f(z)$, are those for which

$$\begin{aligned} \Delta_{\mathcal{A}}^{(0,2)} f(z,h) &= R\{f(z+rh) - 2f(z) + f(z-rh)\} \\ \Delta_{\mathcal{A}}^{(1,2)} f(z,h) &= \frac{1}{s}\{f(z+sh) - f(z-sh)\} \end{aligned} \quad \text{, for } R, r, s \neq 0,$$

where the right sides are $R\Delta_{\mathcal{B}}^{(0,2)} f(z, rh)$ and $\frac{1}{s}\Delta_{\mathcal{B}}^{(1,2)} f(z, sh)$. For $\ell = 3$ and the same $\mathcal{B}$,

$$\Delta_{\mathcal{A}}^{(0,3)} f(z,h) = R\{f(z+rh) + f(z+\omega rh) + f(z+\omega^2 rh) - 3f(z)\},$$
$$\Delta_{\mathcal{A}}^{(1,3)} f(z,h) = \frac{1}{s}\{f(z+sh) + \omega^2 f(z+\omega sh) + \omega f(z+\omega^2 sh)\},$$

$$\Delta_{\mathcal{A}}^{(2,3)} f(z,h) = T\{f(z+th) + \omega f(z+\omega th) + \omega^2 f(z+\omega^2 th)\},$$

where $R, T, r, s, t \neq 0$ and the right sides are $R\Delta_{\mathcal{B}}^{(0,3)} f(z,rh)$, $\frac{1}{s}\Delta_{\mathcal{B}}^{(1,3)} f(z,sh)$ and $T\Delta_{\mathcal{B}}^{(2,3)} f(z,th)$, respectively.

A similar description is given in *the second main theorem*, Theorem 2.8, to characterize all pairs of generalized Riemann differences $(\Delta_{\mathcal{A}}, \Delta_{\mathcal{B}})$ for which $\mathcal{A}$-differentiation implies $\mathcal{B}$-differentiation.

Section 3. There are many meanings of the notion of smoothness in analysis. The most common is that a function f is smooth at z if its derivative f' is continuous at z. A. Zygmund defines a different kind of smoothness on Page 43 of his book [51]. This is related to the notions of modulus of continuity and generalized Lipschitz conditions, and can be rephrased as follows: f is *Z-smooth* at z if

$$\lim_{h\to 0} \frac{f(z+h) - 2f(z) + f(z-h)}{h} = 0.$$

Note that the difference in the numerator is a second difference, while the denominator is degree one. Moreover, Z-smoothness does not change when the numerator in its defining limit is replaced by any nonzero scalar multiple of itself.

The notion of Z-smoothness extends naturally to generalized Riemann Z-smoothness. Throughout this paper, smooth means Z-smooth. If $\mathcal{A}$ is the data vector of a difference $\Delta_{\mathcal{A}}$, which is a scalar multiple of a generalized Riemann difference of order $> n$, then a function f is n times $\mathcal{A}$-*smooth* at z if

$$\lim_{h\to 0} \frac{\Delta_{\mathcal{A}} f(z,h)}{h^n} = 0.$$

Section 3 is about $\mathcal{A}$-differentiability and $\mathcal{A}$-smoothness and their relation with ℓ-grading and analyticity. *The third main theorem*, Theorem 3.1, relates the $\mathcal{A}$-differentiability and $\mathcal{A}$-smoothness with the ℓ-grading. For the earlier example with $\ell = 2$ and $\mathcal{B}$-differentiation is the first ordinary differentiation, Theorem 3.1 says in Corollary 3.2 that a function f is first order ordinary differentiable at z if and only if there exists a complex number L such that the following two limit equations hold at z:

$$\lim_{h\to 0} \frac{f(z+h) - 2f(z) + f(z-h)}{2h} = 0 \text{ and } \lim_{h\to 0} \frac{f(z+h) - f(z-h)}{2h} = L.$$

In other words, if $\mathcal{A}_1 = \{1/2, -1, 1/2; 1, 0, -1\}$ and $\mathcal{A}_2 = \{1/2, -1/2; 1, -1\}$, the even part of f is 1 time $\mathcal{A}_1$-smooth, while the odd part of f is $\mathcal{A}_2$-differentiable of order 1. ($\mathcal{A}_1$ can be replaced with $\{1, -2, 1; 1, 0, -1\}$ or any other rescale of the first three entries; whereas because of the definition of generalized Riemann differentiation, the first two entries in $\mathcal{A}_2$ cannot be rescaled.)

In the case when $\ell = 3$ and the same $\mathcal{B}$, the same theorem says that f is differentiable at z if and only if there exists a complex number L such that the following three limit equations hold at z:

$$\lim_{h\to 0} \frac{f(z+h) + f(z+\omega h) + f(z+\omega^2 h) - 3f(z)}{3h} = 0,$$
$$\lim_{h\to 0} \frac{f(z+h) + \omega^2 f(z+\omega h) + \omega f(z+\omega^2 h)}{3h} = L,$$
$$\lim_{h\to 0} \frac{f(z+h) + \omega f(z+\omega h) + \omega^2 f(z+\omega^2 h)}{3h} = 0.$$

These amount to two conditions of first order smoothness and one condition of first order generalized Riemann differentiability. Since these two systems are equivalent to the same $\mathcal{B}$-differentiation, Remark 3.3 says that the two systems must be equivalent via elementary row operations and $h \mapsto rh$ dilations. The direct proof of this equivalence is left as an exercise to the reader. The case $\ell = 2$ vs. $\ell = 4$ is worked out completely within the same remark.

Section 3.1 ends with an application of the classification Theorem 2.4 to analytic functions: Proposition 3.5 provides infinitely many definitions of analyticity.

Section 3.2 has two applications of Theorem 3.1 to analytic functions: by expressing the fundamental theorem on analytic functions and Cauchy-Riemann in terms of the symmetric Cauchy-Riemann relations in Proposition 3.8, or the generalized Cauchy-Riemann relations in Proposition 3.9. These involve either symmetric partial derivatives or generalized partial derivatives in place of ordinary partial derivatives of the real and imaginary parts of an analytic function.

Section 4. This is devoted to the numerical analysis of generalized Riemann derivatives. For example, Taylor expansion around z of the Schwarz second symmetric difference $\Delta_1 f(z,h) = f(z+h) - 2f(z) + f(z-h)$ yields

$$\begin{aligned}\Delta_1 f(z,h) &= [f(z) + \frac{f'(z)}{1!}h + \frac{f''(z)}{2!}h^2 + \frac{f'''(z)}{3!}h^3 + \frac{f^{(4)}(z)}{4!}h^4 + \cdots] - 2f(z)\\ &\quad + [f(z) - \frac{f'(z)}{1!}h + \frac{f''(z)}{2!}h^2 - \frac{f'''(z)}{3!}h^3 + \frac{f^{(4)}(z)}{4!}h^4 - \cdots]\\ &= 2\frac{f''(z)}{2!}h^2 + 2\frac{f^{(4)}(z)}{4!}h^4 + \cdots\end{aligned}$$

and division by h^2 in the equality of the first and last terms produces

$$\frac{\Delta_1 f(z,h)}{h^2} = f''(z) + 2\frac{f^{(4)}(z)}{4!}h^2 + \cdots.$$

Read this as saying that the second difference quotient $\Delta_1 f(z,h)/h^2$ approximates $f''(z)$ with error of magnitude $O(h^2)$. Similar work done for the three points based second difference, $\Delta_2 f(z,h) = \frac{2}{3}[f(z+h) + \omega f(z+\omega h) + \omega^2 f(z+\omega^2 h)]$, where $\omega = e^{2\pi i/3}$, yields

$$\frac{\Delta_2 f(z,h)}{h^2} = f''(z) + 2!\frac{f^{(5)}(z)}{5!}h^3 + \cdots,$$

so the second difference quotient $\Delta_2 f(z,h)/h^2$ approximates $f''(z)$ with an error of magnitude $O(h^3)$. In particular, this approximation is better than the Schwarz approximation.

In general, an m points based generalized difference quotient of order n approximates the nth derivative $f^{(n)}(z)$ with an error of magnitude $O(h^r)$, where r is the rank of the first non-zero term in the Taylor approximation. Theorem 4.1 says that, for all such approximations, $r \le m =$ the number of base points and, for some, equality is attained. Those m points based generalized Riemann difference quotients of order n for which $r = m$ are called *highest rank approximations*.

The fourth main theorem, Theorem 4.3, shows that these highest rank approximations form a symmetric projective variety of dimension $m-n$ for which an explicit parametrization is given. Back to our last example, all three points based generalized Riemann difference quotients that approximate the second derivative $f''(z)$ to highest rank $m = 3$ form a variety of dimension $m - n = 3 - 2 = 1$. These are all rescales $\frac{1}{t^2}\Delta_2 f(z,th)$ of $\Delta_2 f(z,h)$, for a non-zero complex parameter t.

Section 5. The main classification theorems, Theorems 2.4 and 2.8, are applied in this section to the theory of highest rank approximations developed in Section 4. Specifically, for any two $\mathcal{A}$-derivatives that are either equivalent or imply each other (hence they have the same order n), and at least one of them is a highest rank approximation of the nth ordinary derivative, we relate the expressions of the error terms in their corresponding approximations.

Section 6. This contains the proofs of the first two main theorems, Theorem 2.4 and Theorem 2.8. These are based on a *fifth main theorem*, Theorem 6.6, which translates the implication and equivalence of $\mathcal{A}$-derivatives into the containment and equality of principal ideals of the group algebra of the multiplicative group of complex numbers over the field of complex numbers. The properties of this group algebra, which is a more complicated object of abstract algebra than its real analog, are outlined in the first half of the section. Other instances where the abstract algebra theory of ideals is applied to (functional) analysis are provided in [20,33,42,46].

The present paper is not the first instance where complex $\mathcal{A}$-derivatives have been investigated. For example, the nth roots of unity derivative of Section 4.1 first appeared in [36]. In the real case, an application to continuity of the classification of real $\mathcal{A}$-derivatives in [13] is given in [2]. Equivalences between Peano and sets of generalized Riemann derivatives are studied in [7,14,15,21,29]. Some general properties of ordinary differentiation, such as the mean value theorem, convexity or monotonicity, have been shown to hold to a certain extent for $\mathcal{A}$-differentiation; see [9,28,31,32,38,44,48–50]. Quantum Riemann derivatives were investigated in [11] and [5], and multidimensional Riemann derivatives in [6]. Sufficient conditions that make certain first order $\mathcal{A}$-differentiations imply the ordinary first order differentiation appeared in [17] and [43]. Best approximations real $\mathcal{A}$-derivatives of orders $n = 1, 2$ have been investigated in [8,10] and [45]. For more on Peano derivatives, see [19,24–27,30,35,40]. Reviews on generalized Riemann and Peano derivatives are found in [4] and [23].

1. Previous work: the real case

This section outlines the results in [13] on the classification of real generalized Riemann derivatives. These will be extended to the complex domain in Section 2.

1.1. Even and odd functions and differences

Recall that a real function f is even if $f(-h) = f(h)$, for all h, and is odd if $f(-h) = -f(h)$, for all h. In addition, every function f is expressed uniquely as a sum

$$f = f_0 + f_1$$

of an even function f_0 and an odd function f_1. The unique expressions of f_0 and f_1 are

$$f_0(x) = \frac{f(x) + f(-x)}{2} \quad \text{and} \quad f_1(x) = \frac{f(x) - f(-x)}{2}.$$

A difference $\Delta_{\mathcal{A}} f(x, h) = \sum_{i=1}^{m} A_i f(x + a_i h)$ is even if $\Delta_{\mathcal{A}} f(x, -h) = \Delta_{\mathcal{A}} f(x, h)$ and is odd if $\Delta_{\mathcal{A}} f(x, -h) = -\Delta_{\mathcal{A}} f(x, h)$.

The following property is proved within text in [13, Section 2].

Proposition 1.1. *Let $\Delta_{\mathcal{A}} f(x, h)$ be a generalized Riemann difference of order n.*
(i) *If $\Delta_{\mathcal{A}} f(x, h)$ is an even difference, then n must be even.*
(ii) *If $\Delta_{\mathcal{A}} f(x, h)$ is an odd difference, then n must be odd.*

Every difference $\Delta_{\mathcal{A}} f(x, h)$ is expressed uniquely as a sum

$$\Delta_{\mathcal{A}} f(x, h) = \Delta_{\mathcal{A}}^{ev} f(x, h) + \Delta_{\mathcal{A}}^{odd} f(x, h)$$

of an even difference $\Delta_{\mathcal{A}}^{ev} f(x, h)$ and an odd difference $\Delta_{\mathcal{A}}^{odd} f(x, h)$. Their expressions are

$$\Delta_{\mathcal{A}}^{ev} f(x,h) = \frac{\Delta_{\mathcal{A}} f(x,h) + \Delta_{\mathcal{A}} f(x,-h)}{2},\ \Delta_{\mathcal{A}}^{odd} f(x,h) = \frac{\Delta_{\mathcal{A}} f(x,h) - \Delta_{\mathcal{A}} f(x,-h)}{2}.$$

Proposition 1.2. *[13, Theorem 4] Let $\Delta_{\mathcal{A}} f(x,h)$ be a generalized Riemann difference of order n and let $\Delta_{\mathcal{A}}^{\epsilon} f(x,h)$ and $\Delta_{\mathcal{A}}^{\epsilon'} f(x,h)$ be the odd/even components of $\Delta_{\mathcal{A}} f(x,h)$ that have the same or opposite parity as n. Then*

(i) *$\Delta_{\mathcal{A}}^{\epsilon} f(x,h)$ is a generalized Riemann difference of order n;*
(ii) *$\Delta_{\mathcal{A}}^{\epsilon'} f(x,h)$ is a scalar multiple of a generalized Riemann difference of order $> n$.*

1.2. The classification of real generalized Riemann derivatives

A *rescale* by r of an nth generalized Riemann difference $\Delta_{\mathcal{A}} f(x,h) = \sum_i A_i f(x + a_i h)$ of a function f at x is the difference $\Delta_{\mathcal{A}_r} f(x,h) = r^{-n} \sum_i A_i f(x + a_i r h)$. This is an nth generalized Riemann difference with data vector $\mathcal{A}_r = \{r^{-n} A_i; a_i r\}$. One can easily check that a measurable function f is $\mathcal{A}$-differentiable at x if and only if it is $\mathcal{A}_r$-differentiable at x and, if this is the case, then $D_{\mathcal{A}} f(x) = D_{\mathcal{A}_r} f(x)$.

The following theorem is a rephrase of [13, Theorem 2]. It classifies all pairs $(\Delta_{\mathcal{A}}, \Delta_{\mathcal{B}})$ of generalized Riemann differences for which pointwise $\mathcal{A}$-differentiability is equivalent to pointwise $\mathcal{B}$-differentiability for measurable functions f.

Theorem 1.3 *(The equivalence of real generalized Riemann derivatives). Let $\mathcal{A}$ and $\mathcal{B}$ be data vectors corresponding to generalized Riemann differences of orders m and n. For measurable functions f, the following are equivalent:*

(i) *f is $\mathcal{A}$-differentiable at x if and only if f is $\mathcal{B}$-differentiable at x;*
(ii) *$m = n$ and*

$$\Delta_{\mathcal{B}} f(x,h) = r^{-n} \Delta_{\mathcal{A}}^{\epsilon} f(x, rh) + A \Delta_{\mathcal{A}}^{\epsilon'} f(x, sh),$$

for some non-zero real numbers A, r, s.

Part (ii) of the above theorem says that the components of $\Delta_{\mathcal{A}} f(x,h)$ and $\Delta_{\mathcal{B}} f(x,h)$ satisfy the system of identities

$$\Delta_{\mathcal{B}}^{\epsilon} f(x,h) = \Delta_{\mathcal{A}}^{\epsilon} f(x,h) \text{ up to a nonzero rescale, and}$$
$$\Delta_{\mathcal{B}}^{\epsilon'} f(x,h) = A \Delta_{\mathcal{A}}^{\epsilon'} f(x,h),\ A \neq 0,\ \text{up to a nonzero dilate.}$$

The following theorem classifies all pairs $(\Delta_{\mathcal{A}}, \Delta_{\mathcal{B}})$ of generalized Riemann differences for which pointwise $\mathcal{A}$-differentiability implies pointwise $\mathcal{B}$-differentiability for measurable functions f. It is a rephrase of [13, Theorem 3].

Theorem 1.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be data vectors of generalized Riemann differences of orders m and n. For measurable functions f, the following statements are equivalent:*

(i) *f is $\mathcal{A}$-differentiable at $x \Longrightarrow f$ is $\mathcal{B}$-differentiable at x;*
(ii) *$m = n$ and the component differences $\Delta_{\mathcal{B}}^{\epsilon} f(x,h)$ and $\Delta_{\mathcal{B}}^{\epsilon'} f(x,h)$ are finite linear combinations*

$$\Delta_{\mathcal{B}}^{\epsilon} f(x,h) = \sum_i U_i \Delta_{\mathcal{A}}^{\epsilon} f(x, u_i h) \text{ and } \Delta_{\mathcal{B}}^{\epsilon'} f(x,h) = \sum_i V_i \Delta_{\mathcal{A}}^{\epsilon'} f(x, v_i h)$$

of non-zero u_i-dilates of $\Delta_{\mathcal{A}}^{\epsilon} f(x,h)$ and v_i-dilates of $\Delta_{\mathcal{A}}^{\epsilon'} f(x,h)$.

2. The classification of complex generalized Riemann derivatives

Theorem 1.3 above can be easily modified to also give a classification of complex generalized Riemann derivatives into equivalence classes for pointwise differentiations: Simply allow the nonzero numbers A, r and s to be complex. This is the $\ell = 2$ case of our main theorem, Theorem 2.4, below.

What is interesting about Theorem 2.4 is that it gives an infinite number of ways to obtain the same equivalence class decomposition. These classifications, corresponding to the $\ell = 3, 4, \dots$ cases of Theorem 2.4, allow us to begin to understand how the symmetries of the much larger torsion subgroup of the multiplicative group of nonzero complex numbers leads to new interesting generalized Riemann derivatives. For example, as we shall show in Section 5 below, some of these new derivatives have a role in complex approximation theory.

2.1. Generalized even and odd functions and differences

The notions of even and odd functions were generalized in [16]. Fix an integer $\ell > 1$, and let $\omega = e^{2\pi i/\ell}$ be a primitive ℓth root of unity. A complex function f is *type* (k, ℓ), for $k = 0, 1, \dots, \ell - 1$, if

$$f(\omega z) = \omega^k f(z), \text{ for all } z.$$

Each complex function f is expressed uniquely as a sum

$$f(z) = \sum_{k=0}^{\ell-1} f_k(z)$$

of type (k, ℓ) functions f_k, for $k = 0, 1, \dots, \ell - 1$. The component function f_k has the expression

$$f_k(z) = \frac{1}{\ell} \sum_{j=0}^{\ell-1} \omega^{-kj} f(\omega^j z).$$

When $z = x$ and $\ell = 2$, $\omega = -1$ and the two terms $f_0(x)$ and $f_1(x)$ are the even and odd components of $f(x)$ that were discussed at the beginning of the previous section.

By analogy with the previous section, a difference $\Delta_{\mathcal{A}} f(z, h)$ is a *type* (k, ℓ) *difference*, for $k = 0, 1, \dots, \ell - 1$, if

$$\Delta_{\mathcal{A}} f(z, \omega h) = \omega^k \Delta_{\mathcal{A}} f(z, h), \text{ for all } z \text{ and } h.$$

The following is the extension to complex numbers of the result of Proposition 1.1.

Proposition 2.1. *If an nth complex generalized Riemann difference* $\Delta_{\mathcal{A}} f(z, h)$ *is a type* (k, ℓ) *difference, for some* $k = 0, 1, \dots, \ell - 1$*, then* $k = n \mod \ell$*.*

Proof. Suppose $\Delta_{\mathcal{A}} f(z, h) = \sum A_i f(z + a_i h)$ and recall that this is a generalized Riemann difference if $\sum A_i a_i^s = \delta_{ns} n!$, for $s = 0, 1, \dots, n$. The condition making $\Delta_{\mathcal{A}} f(z, h)$ a type (k, ℓ) difference is

$$\sum A_i f(z + a_i \omega h) = \omega^k \sum A_i f(z + a_i h).$$

The equality of the nth Vandermonde expressions in both sides,

$$\sum A_i (a_i \omega)^n = \omega^k \sum A_i a_i^n,$$

when simplified by the nonzero factor $\sum A_i a_i^n = n!$, leads to $\omega^n = \omega^k$, which is equivalent to $k = n \mod \ell$. □

Each difference $\Delta_{\mathcal{A}} f(z, h)$ is expressed uniquely as a sum

$$\Delta_{\mathcal{A}} f(z, h) = \sum_{k=0}^{\ell-1} \Delta_{\mathcal{A}}^{(k,\ell)} f(z, h),$$

where $\Delta_{\mathcal{A}}^{(k,\ell)} f(z, h)$ is a type (k, ℓ) difference, for $k = 0, \dots, \ell - 1$. The type (k, ℓ) component $\Delta_{\mathcal{A}}^{(k,\ell)} f(z, h)$ of $\Delta_{\mathcal{A}} f(z, h)$ has the expression

$$\Delta_{\mathcal{A}}^{(k,\ell)} f(z, h) = \frac{1}{\ell} \sum_{j=0}^{\ell-1} \omega^{-kj} \Delta_{\mathcal{A}} f(z, \omega^j h).$$

The following is the extension to complex numbers of the result of Proposition 1.2.

Proposition 2.2. *Let $\Delta_{\mathcal{A}} f(z, h)$ be an nth complex generalized Riemann difference and let $\Delta_{\mathcal{A}}^{(k,\ell)} f(x, h)$ be its type (k, ℓ) component, for some $k = 0, 1, \dots, \ell - 1$.*

(i) *If $k = n \mod \ell$, then $\Delta_{\mathcal{A}}^{(k,\ell)} f(z, h)$ is an nth generalized Riemann difference.*

(ii) *If $k \neq n \mod \ell$, then $\Delta_{\mathcal{A}}^{(k,\ell)} f(z, h)$ is a scalar multiple of a generalized Riemann difference of order $> n$.*

Proof. The type (k, ℓ) component of $\Delta_{\mathcal{A}} f(z, h) = \sum A_i f(z + a_i h)$ is

$$\Delta_{\mathcal{A}}^{(k,\ell)} f(z, h) = \frac{1}{\ell} \sum_{j=0}^{\ell-1} \omega^{-kj} \Delta_{\mathcal{A}} f(z, \omega^j h) = \frac{1}{\ell} \sum_i \sum_{j=0}^{\ell-1} A_i \omega^{-kj} f(z + a_i \omega^j h).$$

Its sth Vandermonde condition is

$$\frac{1}{\ell} \sum_i \sum_{j=0}^{\ell-1} A_i \omega^{-kj} (a_i \omega^j)^s = \frac{1}{\ell} \sum_i A_i a_i^s \sum_{j=0}^{\ell-1} \omega^{-kj} \omega^{js} = \frac{1}{\ell} \delta_{sn} n! \sum_{j=0}^{\ell-1} (\omega^{s-k})^j.$$

This equals zero when $s < n$, and is $\frac{1}{\ell} n! \sum_{j=0}^{\ell-1} (\omega^{n-k})^j$ when $s = n$. The last expression is $n!$ when $k = n \mod \ell$, making $\Delta_{\mathcal{A}}^{(k,\ell)} f(z, h)$ an nth generalized Riemann difference, and is zero when $k \neq n \mod \ell$, making $\Delta_{\mathcal{A}}^{(k,\ell)} f(z, h)$ a scalar multiple of a generalized Riemann difference of order $> n$. □

Let $\mathcal{A}$ be the data vector of any difference $\Delta_{\mathcal{A}} f(z, h)$ and let $\mathcal{A}^{(k,\ell)}$ be the data vector of $\Delta_{\mathcal{A}}^{(k,\ell)} f(z, h)$, for $\ell \geq 2$ and $k = 0, 1, \dots, \ell - 1$. Note that $\mathcal{A}^{(k,\ell)} = \varnothing$ when $\Delta_{\mathcal{A}}^{(k,\ell)} f(z, h) = 0$.

Corollary 2.3. *Fix positive integers ℓ and n, with $\ell \geq 2$. The following statements are equivalent:*

(i) *$\mathcal{A} = \{A_i; a_i\}$ satisfies $\sum A_i a_i^j = 0$, for $j = 0, 1, \dots, n$.*

(ii) *$\mathcal{A}^{(k,\ell)} = \{A_i^{(k,\ell)}; a_i^{(k,\ell)}\}$ satisfies $\sum A_i^{(k,\ell)} (a_i^{(k,\ell)})^j = 0$, for $j = 0, 1, \dots, n$.*

Proof. The hypothesis (i) means that $\Delta_{\mathcal{A}} f(z, h)$ is a scalar multiple of a generalized Riemann difference of order $> n$. By Proposition 2.2, so are the component differences $\Delta_{\mathcal{A}}^{(k,\ell)} f(z, h)$, for $k = 0, 1, \dots, \ell - 1$, hence (ii). Conversely, for each j, the jth condition in (i) is obtained by adding the jth conditions in (ii) for $k = 0, 1, \dots, \ell - 1$. □

2.2. The classification of complex generalized Riemann derivatives

In this section we extend the classification of real generalized Riemann derivatives in Theorems 1.3 and 1.4 to the classification of complex generalized Riemann derivatives. This is given in Theorems 2.4 and 2.8. We shall see that the complex results are much more involved.

The first complex classification theorem generalizes the real result of Theorem 1.3 to the complex domain: it characterizes the pairs $(\Delta_{\mathcal{A}}, \Delta_{\mathcal{B}})$ of complex generalized Riemann differences for which $\mathcal{A}$-differentiation is equivalent to $\mathcal{B}$-differentiation.

Theorem 2.4 *(The equivalence of complex generalized Riemann derivatives). Let $\mathcal{A}$ and $\mathcal{B}$ be the data vectors of complex generalized Riemann differences of orders m and n. For measurable complex functions f, the following statements are equivalent:*

(i) *f is $\mathcal{A}$-differentiable at $z \Longleftrightarrow f$ is $\mathcal{B}$-differentiable at z;*

(ii) *$m = n$ and, for each $\ell \geq 2$ and $k = 0, \dots, \ell - 1$, there exist non-zero constants R_k, r_k, with $R_k = r_k^{-n}$ when $k = n \mod \ell$, such that*

$$\Delta_{\mathcal{B}}^{(k,\ell)} f(z, h) = R_k \Delta_{\mathcal{A}}^{(k,\ell)} f(z, r_k h).$$

The proof, given in Section 6, uses a tool from abstract algebra: the group algebra of the multiplicative group $\mathbb{C}^\times$ of complex numbers over the base field $\mathbb{C}$.

The simplest possible example of a pair of equivalent generalized Riemann derivatives is one where $\mathcal{A} = \{A_j; a_j\}$ and $\mathcal{B} = \{s^{-n} A_j; s a_j\}$ is an s-rescale of $\mathcal{A}$. In this case, both parts of Theorem 2.4 easily hold: (i) by change of variable $h \mapsto sh$; and (ii) happens with every R_k being s^{-n}.

Example 2.5. Fix $\ell > 1$ and let $\mathcal{A}$ be the ordinary first order differentiation, i.e. $\Delta_{\mathcal{A}} f(z, h) = f(z+h) - f(z)$. For each $k = 0, \dots, \ell - 1$, let

$$\Delta_k f(z, h) = \begin{cases} \frac{1}{\ell} \sum_{i=0}^{\ell-1} f(z + \omega^i h) - f(z) & \text{if } k = 0, \\ \frac{1}{\ell} \sum_{i=0}^{\ell-1} \omega^{-ik} f(z + \omega^i h) & \text{if } k > 0. \end{cases}$$

Then $k! \Delta_k f(z, h)$ is a generalized Riemann difference of order k when $k \neq 0$, and of order ℓ when $k = 0$. Moreover, for each k, $\Delta_k f(z, h)$ is a type (k, ℓ) difference and

$$f(z + h) - f(z) = \sum_{k=0}^{\ell-1} \Delta_k f(z, h).$$

The uniqueness of the decomposition of a difference as a sum of type (k, ℓ) differences makes $\Delta_{\mathcal{A}}^{(k,\ell)} f(z, h) = \Delta_k f(z, h)$, for $k = 0, \dots, \ell - 1$, a fact that we could have also checked directly. By the main classification Theorem 2.4, the complex generalized Riemann derivatives $\mathcal{B}$ equivalent to $\mathcal{A}$ are those with

$$\Delta_{\mathcal{B}} f(z, h) = \sum_{k=0}^{\ell-1} R_k \Delta_k f(z, r_k h),$$

for some non-zero constants R_k, r_k, for $k = 0, \dots, \ell - 1$, with $R_1 = r_1^{-1}$.

Example 2.6. Set $\ell = 3$ and $\omega = e^{2\pi i/3}$. Then

$$\Delta_0 f(z,h) = \frac{1}{3}\left\{f(z+h) + f(z+\omega h) + f(z+\omega^2 h)\right\} - f(z),$$
$$\Delta_1 f(z,h) = \frac{1}{3}\left\{f(z+h) + \omega^2 f(z+\omega h) + \omega f(z+\omega^2 h)\right\},$$
$$\Delta_2 f(z,h) = \frac{1}{3}\left\{f(z+h) + \omega f(z+\omega h) + \omega^2 f(z+\omega^2 h)\right\},$$

are generalized Riemann differences of respective orders $3, 1, 2$. By Theorem 2.4, the differences of the form

$$A\Delta_0 f(z, rh) + \frac{1}{s}\Delta_1 f(z, sh) + B\Delta_2 f(z, th),$$

for nonzero constants A, B, r, s, t, are those equivalent to ordinary differentiation.

Example 2.7. Let $\mathcal{B}$ be the first order generalized derivative with

$$\Delta_{\mathcal{B}} f(z,h) = \Delta_1 f(z,h) = \frac{1}{3}\left\{f(z+h) + \omega^2 f(z+\omega h) + \omega f(z+\omega^2 h)\right\}$$

as mentioned above. Then $\Delta_{\mathcal{B}}$ is a type $(1,3)$ difference. In particular, both its type $(0,3)$ and type $(2,3)$ components are zero, so by the previous example, $\mathcal{B}$-differentiation is not equivalent to ordinary differentiation.

The second main classification theorem characterizes the pointwise implication relation between complex generalized Riemann derivatives.

Theorem 2.8 *(The implication relation on complex generalized Riemann derivatives). Let ℓ, m, n be positive integers, with $\ell \geq 2$, and let $\mathcal{A}$ and $\mathcal{B}$ be complex generalized Riemann derivatives of orders m and n, respectively. Then the following statements are equivalent for all measurable functions f and points z:*

(i) *f is $\mathcal{A}$-differentiable at $z \Longrightarrow f$ is $\mathcal{B}$-differentiable at z;*

(ii) *$m = n$ and, for each $k = 0, 1, \ldots, \ell - 1$, $\Delta_{\mathcal{B}}^{(k,\ell)} f(z,h)$ is a linear combination*

$$\Delta_{\mathcal{B}}^{(k,\ell)} f(z,h) = \sum_i R_i^{(k,\ell)} \Delta_{\mathcal{A}}^{(k,\ell)} f(z, r_i^{(k,\ell)} h),$$

of $r_i^{(k,\ell)}$-dilates of $\Delta_{\mathcal{A}}^{(k,\ell)} f(z,h)$ such that $\sum_i R_i^{(k,\ell)} \left(r_i^{(k,\ell)}\right)^n = 1$ when $k = n \mod \ell$.

Remark 2.9. (i) Part (i) in Theorem 2.8 is independent of ℓ, while part (ii) is ℓ-dependent. This means that when part (ii) holds true for an ℓ, it holds true for all ℓ, $\ell \geq 2$.

(ii) If $\ell = 2$ then $\omega = -1$. In this case, the type $(0,2)$ differences are the even differences, and the type $(1,2)$ differences are the odd differences. Moreover, when $\ell = 2$, the complex classification Theorems 2.4 and 2.8 are obtained from the real classification Theorems 1.3 and 1.4 by extending scalars. On the other hand, by looking at the cases when $\ell > 2$, it is clear that the complex classification is more than just scalar extending the real classification.

Since any equivalence is the compound of two implications, it is quite clear how Part (i) of Theorem 2.8 can be used to prove Part (i) of Theorem 2.4. However, when extrapolating this to Parts (ii), it is not obvious how the compound of two complicated expressions has a very easy expression. Indeed, this is achieved using a powerful theorem of group algebras, Lemma 6.4, which is a significant extension of the following basic abstract algebra property of polynomials in one indeterminate over any field.

It is well known that the algebra $F[x]$ of polynomials in one indeterminate x and with coefficients in a field F is a principal ideal domain. Let $f(x)$ and $g(x)$ be two polynomials in $F[x]$. Then the inclusion of ideals $(f(x)) \subseteq (g(x))$, means that $f(x) = g(x)p(x)$ for some $p(x) \in F[x]$. On the other hand, by abstract algebra, the equality of ideals $(f(x)) = (g(x))$ amounts to $f(x) = Ag(x)$, for some nonzero scalar A. This is a much easier expression than the equivalent compound of $f(x) = g(x)p(x)$ and $g(x) = f(x)q(x)$ coming from the double inclusion of ideals.

3. Graded generalized Riemann differentiation and Cauchy-Riemann

By Proposition 2.2, the property of a difference $\Delta_{\mathcal{A}} f(z, h)$ of a function f at z that makes it an nth generalized Riemann difference is not a graded property relative to the standard ℓ-grading

$$\Delta_{\mathcal{A}} f(z, h) = \sum_{k=0}^{\ell-1} \Delta_{\mathcal{A}}^{(k,\ell)} f(z, h).$$

This means that the difference $\Delta_{\mathcal{A}} f(z, h)$ being an nth generalized Riemann difference is not equivalent to all $\Delta_{\mathcal{A}}^{(k,\ell)} f(z, h)$, $k = 0, 1, \ldots, \ell$, being nth generalized Riemann differences. Instead, this is graded-intertwined with being a scalar multiple of a higher order generalized Riemann difference, for some of the graded components $\Delta_{\mathcal{A}}^{(k,\ell)} f(z, h)$. Nevertheless, it is an easy exercise to prove that the property of a difference that makes it a scalar multiple of a generalized Riemann difference of order $> n$ is graded, and so is the compound notion of either an nth generalized Riemann or a scalar multiple of a generalized Riemann difference of order $> n$.

Dividing by h^n and taking limit as $h \to 0$, the main result of the section, Theorem 3.1 implicitly says that the existence of the difference quotient limit is graded, i.e.,

$$\lim_{h\to 0} \frac{\Delta_{\mathcal{A}} f(z, h)}{h^n} = \sum_{k=0}^{\ell-1} \lim_{h\to 0} \frac{\Delta_{\mathcal{A}}^{(k,\ell)} f(z, h)}{h^n},$$

for each difference $\Delta_{\mathcal{A}} f(z, h)$ of order at least n. Using the concept of generalized Riemann smoothness, which was defined in the Introduction and will be made explicit in Section 3.1 below, Theorem 3.1(i) says that the notion of nth generalized Riemann differentiability of a function f at z gets graded-intertwined with the one of nth generalized Riemann smoothness of f at z. In addition, the nth generalized Riemann smoothness of f at z is a graded property (Theorem 3.1(ii)), and the same is true about the compound notion of either nth generalized Riemann differentiability or nth generalized Riemann smoothness of f at z (both parts of the theorem).

As a consequence of Theorem 3.1, we show in Corollary 3.2 that the ordinary first order differentiability of f at z is equivalent to both symmetric differentiability and Z-smoothness of f at z. This applied to an analytic function f on a region Ω (resp. to the real components of an analytic function f on Ω) in Proposition 3.5 (resp. Propositions 3.8 and 3.9) produces infinitely many equivalent ways to define analyticity.

3.1. Grading generalized Riemann differentiation and smoothness

Suppose a difference $\Delta_{\mathcal{A}} f(z, h)$ of a function f at z and h is a scalar multiple of a generalized Riemann difference of order $> n$. Recall that f is n times $\mathcal{A}$-*smooth* at z if

$$\lim_{h\to 0} \frac{\Delta_{\mathcal{A}} f(z, h)}{h^n} = 0.$$

In this way, higher order $\mathcal{A}$-smoothness of a function f at z implies any lower order $\mathcal{A}$-smoothness of f at z. This is in general not true for $\mathcal{A}$-differentiability. On the other hand, if $m > n$ and f is m times $\mathcal{A}$-differentiable at z, then f is n times $\mathcal{A}$-smooth at z. Indeed,

$$\lim_{h\to 0} \frac{\Delta_{\mathcal{A}} f(z,h)}{h^n} = \lim_{h\to 0} \frac{\Delta_{\mathcal{A}} f(z,h)}{h^m} \cdot \frac{h^m}{h^n} = D_{\mathcal{A}} f(z) \cdot 0 = 0.$$

Thus higher order $\mathcal{A}$-differentiability of f at z implies any lower order $\mathcal{A}$-smoothness of f at z and, as a consequence, higher order compound notion of either $\mathcal{A}$-differentiability or $\mathcal{A}$-smoothness of f at z implies any lower order of the same compound notion of f at z.

The following theorem characterizes the pointwise notions of nth $\mathcal{A}$-differentiability and n times $\mathcal{A}$-smoothness of a function f at z in terms of the ℓ-grading. The component vector $\mathcal{A}^{(k,\ell)}$ of $\mathcal{A}$, for $k = 0, 1, \ldots, \ell - 1$, is the data vector of the component difference $\Delta_{\mathcal{A}}^{(k,\ell)} f(z,h)$ defined in Section 2.1. The vector $\mathcal{A}$ in Part (i) of the theorem corresponds to an nth generalized Riemann difference, and in Part (ii) corresponds to a scalar multiple of a generalized Riemann difference of order $> n$.

Theorem 3.1. *Let f be a measurable function, let ℓ, n be positive integers with $\ell \geq 2$, and let $\mathcal{A} = \{A_i; a_i\}$ be a vector of $2m$ complex numbers, with the A_i non-zero and the a_i distinct. The following statements hold true at z:*

(i) *f is nth $\mathcal{A}$-differentiable $\iff$ f is nth $\mathcal{A}^{(k,\ell)}$-differentiable, for $k = n \mod \ell$, and is n times $\mathcal{A}^{(k,\ell)}$-smooth, for every other k, with $k = 0, 1, \ldots, \ell - 1$. In addition,*

$$D^n_{\mathcal{A}^{(k,\ell)}} f(z,h) := \lim_{h\to 0} \frac{\Delta_{\mathcal{A}^{(k,\ell)}} f(z,h)}{h^n} = \delta_{k,n \mod \ell} D_{\mathcal{A}} f(z).$$

(ii) *f is n times $\mathcal{A}$-smooth $\iff$ f is n times $\mathcal{A}^{(k,\ell)}$-smooth, for $k = 0, 1, \ldots, \ell - 1$.*

Proof. (i) Suppose f is n times $\mathcal{A}$-differentiable at z, and let k, ℓ be integers with $\ell \geq 2$ and $0 \leq k < \ell$. Then

$$\begin{aligned} D^n_{\mathcal{A}^{(k,\ell)}} f(z) &= \lim_{h\to 0} \frac{\Delta_{\mathcal{A}^{(k,\ell)}} f(z,h)}{h^n} = \lim_{h\to 0} \frac{\frac{1}{\ell}\sum_{j=0}^{\ell-1} \omega^{-kj} \Delta_{\mathcal{A}} f(z, \omega^j h)}{h^n} \\ &= \frac{1}{\ell} \sum_{j=0}^{\ell-1} \omega^{-kj} \lim_{h\to 0} \frac{\Delta_{\mathcal{A}} f(z, \omega^j h)}{\omega^{nj} h^n} \cdot \omega^{nj} = \frac{1}{\ell} \sum_{j=0}^{\ell-1} \omega^{(n-k)j} D_{\mathcal{A}} f(z) \\ &= \delta_{k,n \mod \ell} D_{\mathcal{A}} f(z). \end{aligned}$$

The equality of the first and last terms in the above chain of equalities proves the desired identity and the direct implication. The reverse implication is clear.

(ii) Recall from Corollary 2.3 that n times $\mathcal{A}$-smoothness makes sense if and only if so does n times $\mathcal{A}^{(k,\ell)}$-smoothness, for $k = 0, 1, \ldots, \ell - 1$. Working as in the proof of Part (i), we deduce $D^n_{\mathcal{A}^{(k,\ell)}} f(z) = \delta_{k,n \mod \ell} D^n_{\mathcal{A}} f(z)$. The desired equivalence follows from here, since f is n times $\mathcal{A}$-smooth at z means $D^n_{\mathcal{A}} f(z) = 0$, and f is n times $\mathcal{A}^{(k,\ell)}$-smooth at z means $D^n_{\mathcal{A}^{(k,\ell)}} f(z,h) = 0$. □

Theorem 3.1(i) says that f is $\mathcal{A}$-differentiable at z if and only if there exists a complex number L such that the system of ℓ limit equations

$$\lim_{h\to 0} \frac{\Delta_{\mathcal{A}^{(k,\ell)}} f(z,h)}{h^n} = \delta_{k,n \mod \ell} L, \text{ for } k = 0, \ldots, \ell - 1, \tag{3.1}$$

holds true at z. When this happens, then $L = D_{\mathcal{A}} f(z)$. All these limit equations are smoothness conditions of order n for f at z, except for the one corresponding to $k = n \mod \ell$, which is an nth generalized Riemann differentiation condition for f at z.

The next corollary writes the above system of limits explicitly in the case when $\ell = 2$ and $\mathcal{A}$ corresponds to the first order ordinary differentiation of f at z.

Corollary 3.2. *A complex function f is ordinary differentiable at z if and only if the following two conditions hold:*

(i) $\lim_{h\to 0} \dfrac{f(z+h) - 2f(z) + f(z-h)}{2h} = 0$;

(ii) $\lim_{h\to 0} \dfrac{f(z+h) - f(z-h)}{2h}$ *exists and is a complex number.*

Proof 1. This comes from Theorem 3.1(i) with $n = 1$, $\ell = 2$ and $\mathcal{A} = \{1, -1; 1, 0\}$, that is $\Delta_{\mathcal{A}} f(z, h) = f(z+h) - f(z)$. Indeed, $\Delta^{(0,2)}_{\mathcal{A}} f(z, h) = \frac{1}{2}[f(z+h)+f(z-h)] - f(z)$ and $\Delta^{(1,2)}_{\mathcal{A}} f(z, h) = \frac{1}{2}[f(z+h) - f(z-h)]$. □

Proof 2. Suppose f is differentiable at z. Then the limit in (i) is

$$\frac{1}{2} \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} - \frac{f(z-h) - f(z)}{-h} = \frac{1}{2}\left[f'(z) - f'(z)\right] = 0,$$

and the limit in (ii) is

$$\frac{1}{2} \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} + \frac{f(z-h) - f(z)}{-h} = \frac{1}{2}\left[f'(z) + f'(z)\right] = f'(z).$$

Conversely, if both (i) and (ii) hold, then the addition of the two limits makes the ordinary derivative $f'(z)$ exist as a finite number. □

Recall that a function f satisfying the condition in Part (i) of Corollary 3.2 is *Z-smooth* at z, and one satisfying only the condition in Part (ii) is *symmetric differentiable* at z; in this case, the value of the limit, or the symmetric derivative of f at z, is denoted by $f'_s(z)$. Corollary 3.2 says that f is differentiable at z if and only if it is both Z-smooth and symmetrically differentiable at z. If this is the case, then $f'(z) = f'_s(z)$.

Remark 3.3. Corollary 3.2 makes the ordinary differentiation equivalent to a system of two limits. It reflects the result of Theorem 3.1(i) for $\ell = 2$. Working in the same way for $\ell = 4$, ordinary differentiation of f at z is equivalent to the following system of limits

$$\begin{aligned}
&\lim_{h\to 0} \frac{f(z+h) + f(z+ih) + f(z-h) + f(z-ih) - 4f(z)}{4h} = 0,\\
&\lim_{h\to 0} \frac{f(z+h) - if(z+ih) - f(z-h) + if(z-ih)}{4h} = L < \infty,\\
&\lim_{h\to 0} \frac{f(z+h) - f(z+ih) + f(z-h) - f(z-ih)}{4h} = 0,\\
&\lim_{h\to 0} \frac{f(z+h) + if(z+ih) - f(z-h) - if(z-ih)}{4h} = 0.
\end{aligned} \tag{3.2}$$

We expect this system to be equivalent to the one of Corollary 3.2. Indeed, by taking sums and differences of the same parity equations, the system (3.2) is reduced to

$$\lim_{h\to 0}\frac{f(z+h)+f(z-h)-2f(z)}{2h}=0,\qquad \lim_{h\to 0}\frac{f(z+h)-f(z-h)}{2h}=L<\infty,$$
$$\lim_{h\to 0}\frac{f(z+ih)+f(z+ih)-2f(z)}{2h}=0,\quad \lim_{h\to 0}\frac{f(z+ih)-if(z-ih)}{2ih}=L.$$

As the second row limits are equivalent to rescales $h\mapsto ih$ of the first row limits, the system is reduced to only the first row equations, which is the result of Corollary 3.2.

In general, the result of Theorem 3.1(i) makes the system of limits (3.1) for general ℓ equivalent to the same system for $\ell=2$. Since the limit operator is linear and invariant under dilations $h\mapsto rh$, our observation is that the same equivalence is also achieved by elementary row operations and dilations. In particular, for each $\mathcal{A}$, the system of limits (3.1) for general ℓ has row-rank ≤ 2, up to dilations. It has rank 1 precisely when $\mathcal{A}$ corresponds to either an even or an odd difference.

Corollary 3.2 also holds true for real functions instead of complex functions. We leave as an exercise to the reader proving that the real version of Corollary 3.2 implies Part A of the theorem we state below. This theorem is the main motivation for the classification of real generalized Riemann derivatives in [13], and implicitly for the classification of complex generalized Riemann derivatives in this article. We revisit this in Example 6.7 where a second proof of the whole theorem is provided.

Theorem 3.4. *[12, Theorem 1]* **A:** *The first order real $\mathcal{A}$-derivatives which are dilates ($h\to sh$, for some $s\neq 0$) of*

$$\lim_{h\to 0}\frac{A[f(x+rh)+f(x-rh)-2f(x)]+f(x+h)-f(x-h)}{2h},$$

where $Ar\neq 0$, are equivalent to ordinary differentiation.

B: *Given any other $\mathcal{A}$-derivative of any order $n=1,2,\dots$, there is a measurable function $f(x)$ such that $D_{\mathcal{A}}f(0)$ exists, but the Peano derivative $f_n(0)$ does not.*

Theorem 3.4 is also true in the complex case. This is the particular case of Theorem 2.4 where $\mathcal{A}=\{A_1=1,A_2=-1;a_1=1,a_2=0\}$ corresponds to ordinary first order differentiation.

A *region* Ω in the complex plane is an open connected set. A function $f(z)$ is *analytic* on Ω if $f'(z)$ exists at each point $z=x+iy$ on Ω.

The following proposition is our first application of the classification of generalized Riemann derivatives to analytic functions. It provides an infinite number of equivalent definitions of analyticity.

Proposition 3.5. *The following statements are equivalent for a complex function f defined on a region Ω:*

(i) *f is analytic on Ω;*

(ii) *for each z in Ω, there exist non-zero complex numbers A_z and r_z such that the limit*

$$\lim_{h\to 0}\frac{A_z[f(z+r_zh)+f(z-r_zh)-2f(z)]+f(z+h)-f(z-h)}{2h}$$

exists and is a finite number.

Proof. Simply apply the complex version of Theorem 3.4 at each point z in Ω. □

The example provided in the Abstract is obtained by taking $A_z=1$ and $r_z=i$ at each point z in Ω.

We close this section with a lemma concerning the behavior of generalized Riemann differentiations under taking linear combinations of dilates.

Lemma 3.6. (i) *If $\Delta_{\mathcal{A}_j} f(z,h)$, for $j = 1, \ldots, s$, are extended nth generalized Riemann differences, then so is $\sum_j R_j \Delta_{\mathcal{A}_j} f(z, r_j h)$, for any nonzero complex numbers R_j, r_j.*

(ii) *If $\Delta_{\mathcal{A}_j} f(z,h)$, for $j = 1, 2, \ldots, s$, are nth generalized Riemann differences, then a linear combination of dilates $\sum_j R_j \Delta_{\mathcal{A}_j} f(z, r_j h)$, where the $R_j r_j \neq 0$ for all j, is an nth generalized Riemann difference if and only if $\sum_j R_j r_j^n = 1$.*

Proof. This follows from the explicit writing of the Vandermonde conditions. □

3.2. Cauchy-Riemann equations and generalized Riemann derivatives

For a complex function f of variable $z = x + iy$, we denote $u(z) = u(x,y)$ and $v(z) = v(x,y)$ as the real and imaginary parts of $f(z)$, so that $f(z) = u(x,y) + iv(x,y)$.

The following is a fundamental theorem on analyticity and the Cauchy-Riemann equations; see Ahlfors [1, p.68].

Theorem 3.7 *(Analyticity and Cauchy-Riemann equations). The following statements are equivalent for a complex function f defined on a region Ω:*

(i) *f is analytic on Ω;*

(ii) *the components u and v admit continuous partial derivatives on Ω that satisfy the Cauchy-Riemann differential equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Its proof is based on first showing that the component functions u and v are harmonic, that is, $\Delta u = 0 = \Delta v$. Ahlfors moves on to comment in Chapter II of his book that while the continuity of the partial derivatives is a strong condition, basic complex analysis "is not the place to discuss the weakest conditions of regularity which can be imposed to harmonic functions." The goal of this section is to see whether generalized Riemann differentiation can be used in weakening the regularity conditions on u and v in Theorem 3.7(ii).

The components u and v of the function $f(z)$ are *partially Z-smooth* (resp. *partially symmetric differentiable*) at (x,y) if the partial real maps $x \mapsto u(x,y)$, $y \mapsto u(x,y)$, $x \mapsto v(x,y)$ and $y \mapsto v(x,y)$ are Z-smooth (resp. symmetric differentiable) at (x,y). The symmetric partial derivatives of u at (x,y) are defined by the real limits

$$\frac{\partial_s u}{\partial x} := \lim_{h\to 0} \frac{u(x+h,y) - u(x-h,y)}{2h},$$
$$\frac{\partial_s u}{\partial y} := \lim_{h\to 0} \frac{u(x,y+h) - u(x,y-h)}{2h}.$$

Similar definitions are given for the symmetric partial derivatives of v.

The existence of the symmetric partial derivatives is strictly weaker than the existence of the ordinary partial derivatives. For example, the function $u(x,y) = \max(|x|,|y|)$ has both symmetric partial derivatives at $(0,0)$ equal to 0, while both ordinary partial derivatives at the origin do not exist.

Our second application of the classification of generalized Riemann derivatives characterizes analyticity in terms of the symmetric Cauchy-Riemann equations. It is a translation of Theorem 3.7 in the language of symmetric partial derivatives and partial Z-smoothness. Weaker conditions such as the existence of the symmetric partial derivatives or the symmetric Cauchy-Riemann equations are balanced by stronger conditions such as the continuity of the symmetric partial derivatives.

Proposition 3.8 *(Analyticity and symmetric Cauchy-Riemann equations). The following statements are equivalent for a complex function f defined on a region Ω:*

(i) *f is analytic on Ω;*

(ii) *the components u and v are both partially Z-smooth and admit continuous symmetric partial derivatives on Ω that satisfy the symmetric Cauchy-Riemann equations*

$$\frac{\partial_s u}{\partial x} = \frac{\partial_s v}{\partial y}, \qquad \frac{\partial_s u}{\partial y} = -\frac{\partial_s v}{\partial x}.$$

Proof. It suffices to show that Part (ii) is equivalent to Theorem 3.7(ii). Indeed, by Corollary 3.2, u and v are partially differentiable at (x, y) if and only if u and v are both Z-smooth and partially symmetric differentiable at the same point. If this is the case, the symmetric partial derivatives of u and v are equal to the ordinary ones. This in turn identifies the two continuity conditions and the two Cauchy-Riemann systems. □

Let $\mathcal{A}$ be the data vector of a first order real generalized Riemann differentiation. A two-variable real function $g(x, y)$ is partially $\mathcal{A}$-differentiable with respect to x (resp. y) at (x, y) if the partial map $g_y : x \mapsto g(x, y)$ (resp. $g_x : y \mapsto g(x, y)$) is $\mathcal{A}$-differentiable at x (resp. y). If this is the case, then

$$\frac{\partial_{\mathcal{A}} g}{\partial x}(x, y) := D_{\mathcal{A}} g_y(x) \text{ and } \frac{\partial_{\mathcal{A}} g}{\partial y}(x, y) := D_{\mathcal{A}} g_x(y)$$

are the $\mathcal{A}$-partial derivatives of g at (x, y).

Recall from Theorem 3.4 that the real generalized Riemann differentiations equivalent to ordinary differentiation are those for which the data vector $\mathcal{A}$ is of the form $\mathcal{A} = \mathcal{A}(A, r)$, where $\mathcal{A}(A, r) = \{\frac{A}{2}, \frac{A}{2}, -A, \frac{1}{2}, -\frac{1}{2}; r, -r, 0, 1, -1\}$, for some nonzero real numbers A and r.

The following proposition is the third application of the classification of generalized Riemann derivatives to analyticity. It provides infinitely many equivalent ways of defining analyticity of a complex function $f(z) = u(x, y) + iv(x, y)$ in terms of the partial generalized Riemann differentiabilities of the real components u and v and the generalized Cauchy-Riemann equations satisfied by them.

Proposition 3.9 *(Analyticity and generalized Cauchy-Riemann equations). The following statements are equivalent for a complex function f defined on a region Ω:*

(i) *f is analytic on Ω;*

(ii) *There exist non-zero real numbers $A_1, A_2, A_3, A_4, r_1, r_2, r_3, r_4$ such that the partial generalized Riemann derivatives*

$$\frac{\partial_{\mathcal{A}(A_1,r_1)} u}{\partial x}, \frac{\partial_{\mathcal{A}(A_2,r_2)} u}{\partial y}, \frac{\partial_{\mathcal{A}(A_3,r_3)} v}{\partial x}, \frac{\partial_{\mathcal{A}(A_4,r_4)} v}{\partial y}$$

of u and v exist on Ω, are continuous, and satisfy the generalized Cauchy-Riemann equations

$$\frac{\partial_{\mathcal{A}(A_1,r_1)} u}{\partial x} = \frac{\partial_{\mathcal{A}(A_4,r_4)} v}{\partial y}, \qquad \frac{\partial_{\mathcal{A}(A_2,r_2)} u}{\partial y} = -\frac{\partial_{\mathcal{A}(A_3,r_3)} v}{\partial x}.$$

Proof. This follows from Theorem 3.7 and Theorem 3.4 applied to the partial functions u_y, u_x, v_y, v_x. □

4. Numerical analysis of complex generalized Riemann derivatives

The main result here is Theorem 4.3 of Section 4.3, providing an explicit parametrization for the $m - n$ dimensional symmetric projective variety of highest rank generalized Riemann derivatives of order n based at m points. The main theorem is preceded by the computation of the highest rank in Theorem 4.1 of Section 4.1, and a discussion on best numerical estimates in Section 4.2. The last notable result, Theorem 4.7 of Section 4.4 computes the first significant term in the Taylor approximation of the nth derivative by a highest rank generalized Riemann difference quotient of order n based at m points. This is followed by a discussion on normalizing the same significant term.

4.1. The rank of an m points based generalized Riemann derivative of order n

Fix an order of differentiation n and an integer m, with $m \geq n+1$. Let $\omega = e^{2\pi i/m}$ be a primitive mth root of 1. The linear system

$$\sum_{j=0}^{m-1} A_j \omega^{jk} = \begin{cases} 0 & \text{if } k = 0, \dots, n-1 \\ n! & \text{if } k = n \\ 0 & \text{if } k = n+1, \dots, m-1 \end{cases} \tag{4.1}$$

whose coefficient matrix $\left(\omega^{jk}\right)$ is non-singular has a unique solution $A_0, A_1, \dots, A_{m-1}$. Then $\mathcal{A} = \left\{A_j; \omega^j\right\}$ corresponds to an nth generalized Riemann derivative. We call it the *mth roots of unity derivative of order n*. Observe that if $k = m, m+1, \dots, m+n-1$, then

$$\sum_{j=0}^{m-1} A_j \omega^{jk} = \sum_{j=0}^{m-1} A_j \omega^{j(m+\ell)} = \sum_{j=0}^{m-1} A_j \omega^{jm} \omega^{j\ell}, \tag{4.2}$$

where ℓ runs from 0 to $n-1$. But $\omega^{jm} = (\omega^m)^j = 1$, so by the first n equations of the system (4.1), all the n quantities

$$\sum_{j=0}^{m-1} A_j \omega^{jk}, \text{ for } k = m, m+1, \dots, m+n-1,$$

are also 0. Similarly,

$$\sum_{j=0}^{m-1} A_j \omega^{j(m+n)} = \sum_{j=0}^{m-1} A_j \left(\omega^m\right)^j \omega^{jn} = \sum_{j=0}^{m-1} A_j \omega^{jn} = n!,$$

so if f has $m+n$ Peano derivatives, then

$$\begin{aligned} \sum_{j=0}^{m-1} A_j f\left(z + \omega^j h\right) &= \sum_{j=0}^{m-1} A_j \sum_{k=0}^{m+n} \omega^{jk} \frac{f_k(z)}{k!} h^k + o\left(h^{m+n}\right) \\ &= \sum_{k=0}^{m+n} \left(\sum_{j=0}^{m-1} A_j \omega^{jk}\right) \frac{f_k(z)}{k!} h^k + o\left(h^{m+n}\right) \\ &= n! \frac{f_n(z)}{n!} h^n + n! \frac{f_{m+n}(z)}{(m+n)!} h^{m+n} + o\left(h^{m+n}\right). \end{aligned} \tag{4.3}$$

Divide the equation of the above first and last terms by h^n and get

$$\frac{\sum_{j=0}^{m-1} A_j f\left(z + \omega^j h\right)}{h^n} = f_n(z) + n! \frac{f_{m+n}(z)}{(m+n)!} h^m + o\left(h^m\right). \tag{4.4}$$

This gives a clear picture of how the mth roots of unity derivative of order n approximates the nth Peano derivative. Observe that there are $m-1$ "missing" terms, those of order $h^1, h^2, \dots, h^{m-1}$, and any one of these terms not being 0 would have resulted in a main error term that would have been infinitely large asymptotically as $h \to 0$ when compared with the actual main error term $n!\frac{f_{m+n}(z)}{(m+n)!}h^m$. The identities

$$\sum_{j=0}^{m-1} A_j \left(\omega^j\right)^{n+1} = \sum_{j=0}^{m-1} A_j \left(\omega^j\right)^{n+2} = \cdots = \sum_{j=0}^{m-1} A_j \left(\omega^j\right)^{m+n-1} = 0 \tag{4.5}$$

are the reason for the missing terms.

Motivated by this example, we create a scale for ranking m points based generalized Riemann derivatives of order n by saying that the *rank* of $\mathcal{B} = \{B_j; b_j\}$ is the natural number r, if $\sum_{j=0}^{m-1} B_j b_j^{n+r} \neq 0$ and $\sum_{j=0}^{m-1} B_j b_j^{n+1} = \sum_{j=0}^{m-1} B_j b_j^{n+2} = \cdots = \sum_{j=0}^{m-1} B_j b_j^{n+r-1} = 0$. If $\mathcal{B}_1, \mathcal{B}_2$ are m points based generalized Riemann derivatives of order n whose ranks r_1, r_2 satisfy $r_1 < r_2$, then $\mathcal{B}_2$ is a better approximation of the nth Peano derivative than $\mathcal{B}_1$ is. This means that the difference quotient $\Delta_{\mathcal{B}_2} f(z,h)/h^n$ approximates $f_n(z)$ to higher h-order when compared to $\Delta_{\mathcal{B}_1} f(z,h)/h^n$.

An m points based generalized Riemann derivative of order n is a *highest rank approximation* of the nth Peano derivative if its rank $r(m,n)$ is the largest among the ranks of all m points based generalized Riemann derivatives of order n. The above work shows that the mth roots of unity derivative of order n has rank $r = m$, so $r(m,n) \geq m$. The next theorem shows that $r(m,n) = m$; that is, the largest possible rank of an m points based generalized Riemann derivative of order n is m.

Theorem 4.1. *If an nth generalized Riemann derivative is a highest rank approximation, then its rank coincides with the number m of base points.*

Proof. Suppose an m points based generalized derivative $\mathcal{B} = \{B_j; b_j\}$ of order n, has rank $> m$. This would mean that

$$\sum_{j=0}^{m-1} B_j b_j^{n+i} = 0, \text{ for } i = 1, 2, \dots, m. \tag{4.6}$$

If all $b_j \neq 0$, then $\det\left(b_j^{n+i}\right) = b_0^{n+1} \cdots b_{m-1}^{n+1} \cdot \det\left(b_j^i\right)_{i=0,\dots,m-1; j=0,\dots,m-1} \neq 0$ since $\det\left(b_j^i\right)$ is a Vandermonde determinant. Thus the matrix $\left(b_j^{n+i}\right)$ is non-singular, so that the system (4.6) has the unique solution of all $B_j = 0$, contradicting $\sum_{j=0}^{m-1} B_j b_j^n = n!$.

If some $b_j = 0$, say $b_{m-1} = 0$, then the system (4.6) can be written in matrix block form as

$$\begin{pmatrix} & & & 0 \\ & B & & \vdots \\ & & & \vdots \\ b_0^{m+n} & \cdots & b_{m-2}^{m+n} & 0 \end{pmatrix} \begin{pmatrix} B_0 \\ \vdots \\ \vdots \\ B_{m-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}$$

where B, the $m-1 \times m-1$ block matrix $\left(b_j^i\right)_{i=n+1,\dots,n+m-1; j=0,\dots,m-2}$, is non-singular. Block multiplication gives in particular that $B \begin{pmatrix} B_0 \\ \vdots \\ B_{m-2} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$, whence as above $B_0 = \cdots = B_{m-2} = 0$. But since $\sum_{j=0}^{m-1} B_j = 0$, also $B_{m-1} = 0$ and we arrive at the same contradiction. □

4.2. A first refinement of the classification by rank

Next we present a method of determining the relative merits for numerical approximation of m points based generalized derivatives of different orders n and highest rank m. Let $\mathcal{B} = \{B_j; b_j\}$ be such a rank m derivative and compute the number $s = \min_{i \neq j} |b_i - b_j|$. If $s = 1$, then $\mathcal{B}$ is *normalized*. In general, we assign to $\mathcal{B}$ a scaled version of itself, $\mathcal{B}' = \{B'_j; b'_j\}$, where $B'_j = s^n B_j$ and $b'_j = b_j/s$, for each j, and observe that $\mathcal{B}'$ is normalized, since $s' = \min_{i \neq j} |b'_i - b'_j| = 1$. Call $\mathcal{B}'$ the *normalized version* of $\mathcal{B}$. Finally, Taylor expand the numerical estimate for $f_n(z)$ given by $\mathcal{B}'$ arriving at

$$\frac{\sum_{j=0}^{m-1} B'_j f\left(z + b'_j h\right)}{h^n} = f_n(z) + E(\mathcal{B}') \frac{f_{m+n}(z)}{(m+n)!} h^m + o(h^m),$$

where

$$E(\mathcal{B}') = \sum_{j=0}^{m-1} B'_j \left(b'_j\right)^{m+n} = \frac{1}{s^m} \sum_{j=0}^{m-1} B_j b_j^{m+n}, \text{ where } s = \min_{i \neq j} |b_i - b_j|.$$

The smaller E is, the better the estimate for $f_n(z)$. If there is a rank m derivative with smallest E, then we say that derivative provides the *best numerical estimate* for $f_n(z)$. This method is given, together with some justification, in reference [8]. Keep in mind, however, that since the rank of the error term in a Taylor approximation is much more important than the exact value of its scalar factor, *all* m points based generalized derivatives of order n and rank m are quite good approximations for the nth Peano derivative.

We illustrate the above method by deriving the normalized version of the nth order, mth roots of unity derivative $\mathcal{A} = \{A_j; \omega^j\}$ we studied in Section 4.1. This has $s = \min_{i \neq j} |\omega^i - \omega^j| = |\omega^0 - \omega^1| = |1 - e^{2\pi i/m}| = 2\sin\frac{\pi}{m}$. The normalized nth order, mth roots of unity derivative is $\mathcal{A}' = \left\{A'_j; \frac{\omega^j}{s}\right\}$, where $s = 2\sin\frac{\pi}{m}$ and each $A'_j = A_j s^n$. This has base points $\left\{\frac{\omega^0}{s}, \frac{\omega^1}{s}, \ldots, \frac{\omega^{m-1}}{s}\right\}$, so that the adjacent ones are distance 1 apart. To compute $E(\mathcal{A}')$, start from equation (4.4) and replace each A_j by $A'_j s^{-n}$ to get

$$\frac{1}{s^n} \sum_{j=0}^{m-1} A'_j f\left(z + \frac{\omega^j}{s}(sh)\right) = \frac{1}{s^n} f_n(z)(sh)^n + \frac{1}{s^n} \frac{n!}{s^m} \frac{f_{m+n}(z)}{(m+n)!}(sh)^{m+n} + o\left(h^{m+n}\right).$$

Note that as $h \to 0$, the variable sh also tends to 0 and $o(h^{m+n}) = o\left((sh)^{m+n}\right)$, so after division by h^n and replacement of sh by h we have

$$\frac{\sum_{j=0}^{m-1} A'_j f\left(z + \frac{\omega^j}{s} h\right)}{h^n} = f_n(z) + \frac{n!}{s^m} \frac{f_{m+n}(z)}{(m+n)!} h^m + o(h^m), \tag{4.7}$$

where $s = 2\sin\frac{\pi}{m}$, so that $E(\mathcal{A}') = \frac{n!}{(2\sin\frac{\pi}{m})^m}$. A glance at the approximations in (4.7) and (4.4) shows that the normalized version (4.7) differs by a factor of $1/s^m$ when compared with the original version (4.4).

In [8], the case of $m = 3, n = 1$ is worked out completely. The normalized first order, cube roots of unity derivative is the best possible approximating derivative. It has $s = 2\sin\frac{\pi}{3} = \sqrt{3}$ and $E = \frac{1!}{(\sqrt{3})^3} = \frac{\sqrt{3}}{9}$, which is the smallest possible value of E. Furthermore, there are also real-valued generalized Riemann derivatives (this means that the $\{b_j\}$, and consequently also the $\{B_j\}$, are real) of order 1 and rank 3 and the best of these has $E = 2 \cdot \frac{\sqrt{3}}{9}$.

4.2.1. Questions

We close this discussion on best approximations with a list of seven questions, of which the first two give insight into the material in Sections 4.3 and 4.4, and the last five remain open.

1. Is the mth roots of unity derivative of order n always the unique highest rank complex generalized Riemann derivative? The answer to this question is YES when $n = m - 1$ and NO when $n < m - 1$; see Examples 4.5 and 4.6 below. When all base points have modulus one, the answer to the same question is YES, for each n, $n = 1, 2, \dots, m - 1$; see Theorem 4.10.

2. Since all best numerical estimates are among all rank m generalized Riemann derivatives based at m points, what do all rank m generalized Riemann derivatives based at m points look like? This question is answered in Theorem 4.3.

3. What do the best numerical estimates look like in all cases when the answer to Question 2 is NO?

The same examples mentioned in Question 1 show that the highest rank of a real m points based generalized Riemann derivative of order n sometimes is m and some other times is $< m$. The next four questions refer to real generalized Riemann derivatives.

4. Given m and n, what is the highest rank of a real generalized Riemann derivative of order n based at m points?

5. Given m and n, what do all highest rank real generalized Riemann derivatives of order n based at m points look like? In other words, what is the variety of real highest rank approximations?

6. In particular, what do all m points based real Generalized Riemann derivatives of order n and rank m look like? These are the real generalized Riemann derivatives that count as highest rank complex generalized Riemann derivatives.

7. What are the best numerical estimates real generalized Riemann derivatives of order n and based at m points?

4.3. Explicit parametrization for the variety of all highest rank approximations

The main result, Theorem 4.3, shows that all rank m generalized Riemann derivatives of order n based at m points form a symmetric projective variety that is explicitly given by $m - n$ parameters.

Let $W_m = W_m(a_1, \dots, a_m)$ be the determinant of the $m \times m$ Vandermonde matrix (a_j^{i-1}), for $1 \le i \le m$ and $1 \le j \le m$. For each $k = 1, \dots, m - 1$, denote $w_{m,k} = w_{m,k}(a_1, \dots, a_m)$ as the determinant of the k-shifted $m \times m$ Vandermonde matrix (β_{ij}), where

$$\beta_{ij} = \begin{cases} a_j^{i-1} & \text{if } i = 1, \dots, k, \\ a_j^{i} & \text{if } i = k + 1, \dots, m. \end{cases}$$

In particular, this is $(-1)^{m+1+k+1}$ times the $(k + 1, m + 1)$-cofactor of the $m + 1$ Vandermonde determinant $W_{m+1}(a_1, \dots, a_{m+1})$. This interpretation allows the definition of $w_{m,k}$ to extend naturally to $k = m$ and $k = 0$ by setting $w_{m,m} = W_m$ and $w_{m,0} = a_1 a_2 \cdots a_m W_m$. If $W_{\ell j}$ is the determinant of the matrix obtained from the one of W_m by removing the ℓth row and j-th column, then

$$W_{\ell j} = w_{m-1,\ell-1}(a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_m) = w_{m-1,\ell-1}(a_1, \dots, \widehat{a}_j, \dots, a_m).$$

The following lemma establishes the relationship between the k-shifted Vandermonde determinant and the regular Vandermonde determinant. Thomas Muir in [39, Vol.1, Chapt.XII] attributes it to Prony (1795), despite a general consensus that this result marked Cauchy's (1812) invention of alternating symmetric functions. Its proof is omitted.

Lemma 4.2. *If $0 \le k \le m$, then*

$$w_{m,k}(a_1,\dots,a_m) = e_{m-k}(a_1,\dots,a_m)\cdot W_m(a_1,\dots,a_m),$$

where $e_r(a_1,\dots,a_m)$, for $1 \le r \le m$, is the rth elementary symmetric polynomial in variables $a_1,\dots,a_m$ and $e_0 = 1$.

The goal for this section is to determine all $2m$-vectors $\mathcal{A} = \{A_j; a_j \mid j = 1,\dots,m\}$ for which the difference quotient $\Delta_{\mathcal{A}} f(x,h)/h^n$ approximates the nth derivative $f_n(x)$ with error of magnitude $O(h^m)$. This condition is accounted in the Vandermonde system

$$\sum_{j=1}^{m} A_j a_j^{i-1} = \begin{cases} 0 & \text{if } i \neq n+1, \\ n! & \text{if } i = n+1, \end{cases}$$

for $i = 1,2,\dots,m+n$. The system of the first m equations has an $m \times m$ Vandermonde coefficient matrix, hence it can be solved uniquely for variables A_j. Cramer's rule yields

$$A_j = \frac{\Delta_j}{\Delta}, \qquad \text{for } j = 1,\dots,m,$$

where $\Delta = W_m$ and $\Delta_j = (-1)^{n+1+j}\cdot n!\cdot W_{n+1,j}$. Consequently,

$$\begin{aligned} A_j &= \frac{(-1)^{n+1+j}\cdot n!\cdot W_{n+1,j}}{W_m} = \frac{(-1)^{n+1+j}\cdot n!\cdot w_{m-1,n}(a_1,\dots,\widehat{a}_j,\dots,a_m)}{W_m} \\ &= \frac{(-1)^{n+1+j}\cdot n!\cdot e_{m-n-1}(a_1,\dots,\widehat{a}_j,\dots,a_m)\cdot W_{m-1}(a_1,\dots,\widehat{a}_j,\dots,a_m)}{W_m} \\ &= \frac{(-1)^{n}\cdot n!\cdot e_{m-n-1}(a_1,\dots,\widehat{a}_j,\dots,a_m)}{\prod_{i\neq j}(a_i - a_j)}. \end{aligned}$$

We regard the system of the last $m-1$ Vandermonde equations

$$\sum_{j=1}^{m} A_j a_j^{i-1} = 0, \qquad \text{for } i = n+2,\dots,n+m$$

as the linear homogeneous system

$$\sum_{j=1}^{m} A_j a_j^{i-1} x_j = 0, \qquad \text{for } i = 1,\dots,m-1$$

in variables $x_j = a_j^{n+1}$, for $j = 1,\dots,m$. Passing the x_m-terms to the right side, the remaining left side of the system has a non-singular coefficient matrix, hence it can be solved by Cramer's rule as

$$x_j = \frac{\Delta'_j}{\Delta'}, \qquad \text{for } j = 1,\dots,m-1,$$

where

$$\Delta' = \begin{vmatrix} A_1 & A_2 & \cdots & A_{m-1} \\ A_1 a_1 & A_2 a_2 & \cdots & A_{m-1} a_{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_1 a_1^{m-2} & A_2 a_2^{m-2} & \cdots & A_{m-1} a_{m-1}^{m-2} \end{vmatrix} = A_1 \cdots A_{m-1} W_{mm}$$

and

$$\Delta'_j = \begin{vmatrix} A_1 & \cdots & -A_m x_m & \cdots & A_{m-1} \\ A_1 a_1 & \cdots & -A_m a_m x_m & \cdots & A_{m-1} a_{m-1} \\ \vdots & & \vdots & \ddots & \vdots \\ A_1 a_1^{m-2} & \cdots & -A_m a_m^{m-2} x_m & \cdots & A_{m-1} a_{m-1}^{m-2} \end{vmatrix} = \frac{A_1 \cdots A_m x_m}{A_j} (-1)^{m-j} W_{mj}.$$

This makes

$$\begin{aligned} x_j &= \frac{\frac{A_1 \cdots A_m x_m}{A_j}(-1)^{m-j} W_{mj}}{A_1 \cdots A_{m-1} W_{mm}} = (-1)^{m-j} \cdot \frac{A_m x_m}{A_j} \cdot \frac{W_{mj}}{W_{mm}} \\ &= (-1)^{m-j} \cdot \frac{(-1)^{n+1+m} W_{n+1,m} x_m}{(-1)^{n+1+j} W_{n+1,j}} \cdot \frac{W_{mj}}{W_{mm}} = \frac{W_{n+1,m} x_m}{W_{n+1,j}} \cdot \frac{W_{mj}}{W_{mm}} \\ &= \frac{w_{m-1,n}(a_1, \ldots, a_{m-1}) x_m}{w_{m-1,n}(a_1, \ldots, \widehat{a}_j, \ldots, a_m)} \cdot \frac{W_{m-1}(a_1, \ldots, \widehat{a}_j, \ldots, a_m)}{W_{m-1}(a_1, \ldots, a_{m-1})} \\ &= \frac{e_{m-n-1}(a_1, \ldots, a_{m-1}) W_{m-1}(a_1, \ldots, a_{m-1}) x_m \cdot W_{m-1}(a_1, \ldots, \widehat{a}_j, \ldots, a_m)}{e_{m-n-1}(a_1, \ldots, \widehat{a}_j, \ldots, a_m) W_{m-1}(a_1, \ldots, \widehat{a}_j, \ldots, a_m) \cdot W_{m-1}(a_1, \ldots, a_{m-1})}. \end{aligned}$$

Thus

$$a_j^{n+1} = \frac{e_{m-n-1}(a_1, \ldots, a_{m-1})}{e_{m-n-1}(a_1, \ldots, \widehat{a}_j, \ldots, a_m)} a_m^{n+1}, \text{ for } j = 1, \ldots, m-1. \tag{4.8}$$

A condition that makes m complex numbers $a_1, \ldots, a_m$ distinct is that the discriminant $D = \prod_{1 \le i < j \le m} (a_j - a_i)^2$ of the degree m equation $x^m + \alpha_1 x^{m-1} + \cdots + \alpha_m = 0$ that has $a_1, \ldots, a_m$ as its roots is non-zero. This expression D is symmetric in variables $a_1, \ldots, a_m$ and can be written in terms of the elementary symmetric polynomials in the same variables, or in terms of the coefficients $\alpha_1, \ldots, \alpha_m$ of the above equation. The expression of D can be made more explicit as $D = (\det W_m)(\det W_m^T) = \det(W_m W_m^T) =$

$$\left| \begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_m \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{m-1} & a_2^{m-1} & \cdots & a_m^{m-1} \end{pmatrix} \begin{pmatrix} 1 & a_1 & \cdots & a_1^{m-1} \\ 1 & a_2 & \cdots & a_2^{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_m & \cdots & a_m^{m-1} \end{pmatrix} \right| = \begin{vmatrix} m & t_1 & \cdots & t_{m-1} \\ t_1 & t_2 & \cdots & t_m \\ \vdots & \vdots & \ddots & \vdots \\ t_{m-1} & t_m & \cdots & t_{2m-2} \end{vmatrix} \tag{4.9}$$

where $t_r = a_1^r + a_2^r + \cdots + a_m^r$, for $r \ge 1$, are Newton's polynomials. These are expressed in terms of the elementary symmetric polynomials $e_1, \ldots, e_m$ using Newton's identities

$$t_r = \begin{cases} e_1 t_{r-1} - e_2 t_{r-2} + \cdots + (-1)^{r-2} e_{r-1} t_1 + (-1)^{r-1} r e_r & \text{if } r \le m, \\ e_1 t_{r-1} - e_2 t_{r-2} + \cdots + (-1)^{m-1} e_m t_{r-m} & \text{if } r > m. \end{cases} \tag{4.10}$$

The next theorem shows that all m points based generalized Riemann derivatives of order n and rank m, the highest rank approximates of the ordinary (Peano) nth derivative, form a projective variety parametrized by $m - n$ complex numbers $\alpha_1, \ldots, \alpha_{m-n-1}, \alpha_m$.

Theorem 4.3 *(Parametrizing the variety of highest rank approximations). Let m, n be integers with $0 < n < m$ and $\mathcal{A} = \{A_1, \ldots, A_m; a_1, \ldots, a_m\}$ is a vector of $2m$ complex numbers, where the A_j are non-zero and the a_j are distinct. The following statements are equivalent:*

(i) *$\mathcal{A}$ corresponds to a highest rank approximation generalized Riemann derivative of order n, that is, $\mathcal{A}$ satisfies the Vandermonde system of $m + n$ equations*

$$\sum_{j=1}^{m} A_j a_j^i = \begin{cases} 0 & \textit{if } i \neq n, \\ n! & \textit{if } i = n, \end{cases}$$

for $i = 0, 1, \ldots, m+n-1$.

(ii) *The numbers* $a_1, \ldots, a_m$ *are all* m *roots of the family of degree* m *monic equations*

$$x^m + \alpha_1 x^{m-1} + \cdots + \alpha_{m-n-1} x^{n+1} + \alpha_m = 0 \tag{4.11}$$

parametrized by complex numbers $\alpha_1, \ldots, \alpha_{m-n-1}, \alpha_m$ *that satisfy the condition* $D \neq 0$, *where* $D = \prod_{1 \le i < j \le m} (a_j - a_i)^2$ *is expressed in terms of the coefficients* $\alpha_j = (-1)^j e_j$ *using relations (4.9) and (4.10). The numbers* $A_1, \ldots, A_m$ *are uniquely expressed in terms of* $a_1, \ldots, a_m$ *as follows:*

$$\begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix} = W_m^{-1}(a_1, \ldots, a_m) \begin{bmatrix} 0 \\ \vdots \\ n! \\ \vdots \\ 0 \end{bmatrix}, \tag{4.12}$$

where the nonzero entry of the last column vector is in the $n+1$*st row.*

Proof. (i) $\Rightarrow$ (ii): Suppose $\mathcal{A}$ satisfies the Vandermonde system of $m+n$ equations. Multiplication of (4.8) by $e_{m-n-1}(a_1, \ldots, \widehat{a}_j, \ldots, a_m)$ clears the denominator. The identity

$$e_{m-n-1}(a_1, \ldots, \widehat{a}_j, \ldots, a_m) = \sum_{k=n+1}^{m} (-1)^{k-n-1} e_{m-k} a_j^{k-n-1}$$

applied to the resulting equation leads to

$$\sum_{k=n+1}^{m} (-1)^{k-n-1} e_{m-k} a_j^k = \sum_{k=n+1}^{m} (-1)^{k-n-1} e_{m-k} a_m^k. \tag{4.13}$$

When divided by $(-1)^{m-n-1} e_0 = (-1)^{m-n-1}$, this equation says that $a_1, \ldots, a_m$ are roots of the degree m equation

$$x^m + \alpha_1 x^{m-1} + \cdots + \alpha_{m-n-1} x^{n+1} + \alpha_m = 0,$$

where $\alpha_{m-k} = (-1)^{(k-n-1)-(m-n-1)} e_{m-k}$ for $k = n+1, \ldots, m-1$ and $(-1)^{m-n} \alpha_m$ is the common value of both sides in (4.13). The numbers $A_1, \ldots, A_m$ are computed using the first m equations of the Vandermonde system.

(ii) $\Rightarrow$ (i): Suppose (ii) is satisfied. Then multiplication of (4.12) by the Vandermonde matrix $W_m(a_1, \ldots, a_m)$ is the system of the first m Vandermonde equations in matrix form. Then, inductively, it suffices to show that for $i = 1, \ldots, n$ the system of the first $m+i-1$ Vandermonde equations implies the $m+i$th Vandermonde equation. To see this, multiply equation (4.11) for $x = a_k$ by $A_k a_k^{i-1}$ and sum up over k. The resulting equation,

$$\sum A_k a_k^{m+i-1} + \alpha_1 \sum A_k a_k^{m+i-2} + \cdots + \alpha_{m-n-1} \sum A_k a_k^{n+i} + \alpha_m \sum A_k a_k^{i-1} = 0,$$

has the terms $\sum A_k a_k^{m+i-2}, \ldots, \sum A_k a_k^{n+i}, \sum A_k a_k^{i-1}$ all equal to zero by the system of the first $m+i-1$ Vandermonde equations. The leftover equation $\sum A_k a_k^{m+i-1} = 0$ is the needed $m+i$th Vandermonde equation. $\square$

Corollary 4.4. *Suppose $\mathcal{A} = \{A_1, \dots, A_m; a_1, \dots, a_m\}$ is the data vector of a highest rank approximation m points based generalized Riemann derivative of order n, where m, n are positive integers. Then each of $a_1, \dots, a_m$ is a non-zero number.*

Proof. Assuming the contrary, we have $(-1)^m a_1 \cdots a_m = 0$, so equation (4.11), whose roots are $a_1, \dots, a_m$, has no constant term. The hypothesis $n > 0$ makes $m - n - 1 < m - 1$, and the same equation has no degree 1 term either. Then the equation has $x = 0$ as a multiple root, a contradiction with the hypothesis that $a_1, \dots, a_m$ are distinct. □

Example 4.5. When $n = m - 1$ in Theorem 4.3, equation (4.11) becomes $x^m + \alpha_m = 0$. Since $a_1, \dots, a_m$ are distinct, without loss, we may assume that $a_m \neq 0$, hence $\alpha_m = -a_m^m \neq 0$. Then $(a_j/a_m)^m = (-\alpha_m)/(-\alpha_m) = 1$, for all j, and the distinct numbers $a_1/a_m, \dots, a_m/a_m$ are precisely all mth roots of unity. We may further assume without loss that, up to a dilation by $1/a_m$, $a_j = \omega^j$, for $j = 1, \dots, m$, where $\omega = e^{2\pi i/m}$ is a primitive mth root of unity. Theorem 4.3(ii) computes the A_j's by inverting the Vandermonde system

$$\begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix} = W_m^{-1}(\omega, \dots, \omega^m) \begin{bmatrix} 0 \\ \vdots \\ n! \\ \vdots \\ 0 \end{bmatrix} = \frac{W_m(\omega^{-1}, \dots, \omega^{-m})}{m} \begin{bmatrix} 0 \\ \vdots \\ n! \\ \vdots \\ 0 \end{bmatrix} = \frac{n!}{m} \begin{bmatrix} 1 \\ \omega^{-n} \\ \omega^{-2n} \\ \vdots \\ \omega^{(1-m)n} \end{bmatrix}.$$

We conclude that, when $n = m - 1$, the mth roots of unity derivative, as a highest rank approximation of the nth derivative, is unique up to a rescaling and a permutation of base points.

Example 4.6. When $n < m - 1$, the mth roots of unity derivative is not a unique highest rank approximation of the nth derivative. For example, when $m = 3$ and $n = 1$, the cubic $x^3 - 7x^2 + 36 = 0$ has roots $a_1 = 3$, $a_2 = -2$ and $a_3 = 6$. By Theorem 4.3(ii), we compute $A_1 = \frac{4}{15}$, $A_2 = -\frac{9}{40}$ and $A_3 = -\frac{1}{24}$. The difference quotient

$$\frac{\frac{4}{15} f(z + 3h) - \frac{9}{40} f(z - 2h) - \frac{1}{24} f(z + 6h)}{h}$$

approximates the first Peano derivative $f_1(z)$ to highest rank without being a rescale of the third roots of unity first derivative. This derivative was considered in [8].

4.4. Error estimation and normalizing the highest rank approximations

We are now ready to generalize the error estimation given in (4.4) for the mth roots of unity derivative of order n. For any m points based difference quotient $\Delta_{\mathcal{A}} f(x, h)/h^n$ of order n that approximates the nth derivative $f_n(x)$ to highest rank, the next theorem computes exactly the first significant term in the Taylor approximation.

Theorem 4.7 *(Error estimation). Suppose $\mathcal{A} = \{A_1, \dots, A_m; a_1, \dots, a_m\}$ is the data vector corresponding to a difference quotient $\Delta_{\mathcal{A}} f(x, h)/h^n$ that approximates $f_n(z)$ to highest rank, for any $m + n$ times (Peano) differentiable function f. Then*

$$\frac{\Delta_{\mathcal{A}} f(z, h)}{h^n} - f_n(z) = (-1)^{m+1} a_1 \cdots a_m \cdot n! \cdot \frac{f_{m+n}(z)}{(m + n)!} h^m + o(h^m).$$

Proof. Using Taylor expansion and the $m+n$ Vandermonde relations, we compute

$$\begin{aligned}\frac{\Delta_{\mathcal{A}} f(z,h)}{h^n} &= \frac{1}{h^n}\sum_{j=1}^{m} A_j f(z+a_jh) = \frac{1}{h^n}\sum_{j=1}^{m} A_j\left(\sum_{k=0}^{m+n}\frac{f_k(z)}{k!}(a_jh)^k + o(h^{m+n})\right)\\ &= \frac{1}{h^n}\sum_{k=0}^{m+n}\left(\sum_{j=1}^{m} A_j a_j^k\right)\frac{f_k(z)}{k!}h^k + o(h^m)\\ &= f_n(z) + \left(\sum_{j=1}^{m} A_j a_j^{m+n}\right)\frac{f_{m+n}(z)}{(m+n)!}h^m + o(h^m).\end{aligned}$$

To finish the proof, it suffices to show that $\sum_{j=1}^{m} A_j a_j^{m+n} = (-1)^{m+1}a_1\cdots a_m$. Indeed, the equation (4.11) allows the replacement of a_j^m by $-\sum_{i=1}^{m-n-1}\alpha_i a_j^{m-i} - \alpha_m$ as follows:

$$\begin{aligned}\sum_{j=1}^{m} A_j a_j^{m+n} &= \sum_{j=1}^{m} A_j a_j^{n}\Big(-\sum_{i=1}^{m-n-1}\alpha_i a_j^{m-i} - \alpha_m\Big)\\ &= -\sum_{i=1}^{m-n-1}\alpha_i\sum_{j=1}^{m} A_j a_j^{m+n-i} - \alpha_m\sum_{j=1}^{m} A_j a_j^n\\ &= -\sum_{i=1}^{m-n-1}\alpha_i\cdot 0 - \alpha_m\cdot n! = -\alpha_m\cdot n!\\ &= (-1)^{m+1}a_1\cdots a_m\cdot n!. \quad\square\end{aligned}$$

Example 4.8. The symmetric derivative $D_s f(z) = \lim_{h\to 0}\frac{f(z+h/2)-f(z-h/2)}{h}$ is a first generalized Riemann derivative. Its data vector $\mathcal{A} = \{1,-1;1/2,-1/2\}$ satisfies the $m+n$ Vandermonde relations $(1)(1/2)^i + (-1)(-1/2)^i = \delta_{in}\cdot n!$, $i = 0,1,\ldots,m+n-1$, for $m=2$ and $n=1$, where m is the number of base points in $\mathcal{A}$. Then the associated first difference quotient is a highest rank approximation of the first derivative of any three times differentiable function f at z. Theorem 4.7 provides the error term in this approximation:

$$\frac{f(z+\frac{h}{2}) - f(z-\frac{h}{2})}{h} - f_1(z) = (-1)^3 a_1a_2\cdot 1!\cdot\frac{f_3(z)}{3!}h^2 + o(h^2) = \frac{f_3(z)}{24}h^2 + o(h^2).$$

Let $\mathcal{A} = \{A_j; a_j \mid j = 1,\ldots,m\}$ be the data vector of a nth generalized Riemann difference $\Delta_{\mathcal{A}}$. This means that for an n times differentiable function f the difference quotient $\Delta_{\mathcal{A}} f(z,h)/h^n$ approximates the nth Peano derivative $f_n(z)$. When this is a highest rank approximation, Theorem 4.7 provides the error for each $m+n$ times differentiable function f. Recall that the rescale of $\Delta_{\mathcal{A}}$ by a nonzero complex number s is the difference $\Delta_{\mathcal{A}_s}$, where $\mathcal{A}_s = \{s^d A_j; a_j/s \mid j = 1,\ldots,m\}$.

The next corollary studies the behavior of the estimate given in Theorem 4.7 under rescaling the corresponding highest rank generalized Riemann difference.

Corollary 4.9. *Let $\Delta_{\mathcal{A}}$ be a generalized Riemann difference of order n based at m points $a_1,\ldots,a_m$, and let $\Delta_{\mathcal{A}_s}$ be its rescale by a non-zero complex number s. Then*

(i) *$\Delta_{\mathcal{A}_s}$ is also a generalized Riemann difference of order n.*

(ii) *If $\Delta_{\mathcal{A}} f(z,h)/h^n$ approximates $f_n(z)$ to highest rank for any $m+n$ times differentiable function f, then so does $\Delta_{\mathcal{A}_s} f(z,h)/h^n$ and*

$$\frac{\Delta_{\mathcal{A}_s} f(z,h)}{h^n} - f_n(z) = s^{-m}\cdot(-1)^{m+1}a_1\cdots a_m\cdot n!\cdot\frac{f_{m+n}(z)}{(m+n)!}h^m + o(h^m).$$

Proof. Part (i) and the first half in Part (ii) is standard Vandermonde computation. The error estimation comes from Theorem 4.7 applied to $\mathcal{A}_s$. □

In particular, a rescale by an mth root of unity s leaves the error term unchanged, and one by a complex number s of modulus one leaves the magnitude of the error term unchanged. The best numerical estimate (m points based generalized Riemann derivative of order n) in Section 4.2 is then unique up to a rescale by a complex number s on the unit circle. By the results in this section, the problem of finding the best numerical estimate derivative translates into the following formal problem:

(P1) For all m-tuplets $(a_1, \ldots, a_m)$ of complex numbers satisfying Theorem 4.3(ii), minimize $|a_1 \cdots a_m|$ subject to $\min_{i \neq j} |a_i - a_j| = 1$.

This is equivalent to the problem

(P2) For all m-tuplets $(a_1, \ldots, a_m)$ of complex numbers satisfying Theorem 4.3(ii), maximize $\min_{i \neq j} |a_i - a_j|$ subject to $|a_1 \cdots a_m| = 1$.

It is not hard to prove that the additional condition that $(a_1, \ldots, a_m)$ is closest to the origin, coupled with $|a_1 \cdots a_m| = 1$, makes $|a_1| = \cdots = |a_m| = 1$. Problem (P2) is then reduced to the following easier problem:

(P2′) For all m-tuplets $(a_1, \ldots, a_m)$ of complex numbers of modulus one satisfying Theorem 4.3(ii), maximize $\min_{i \neq j} |a_i - a_j|$.

Solution to Problem (P2′). The minimum distance between any two of m points on the unit circle is the minimum distance between two consecutive points. This corresponds to the minimum of the m central angles corresponding to consecutive points on the circle. The largest such minimum is attained when the points $a_1, \ldots, a_m$ determine a regular polygon, that is, when they are base points of a scale of an mth roots of unity derivative. By Section 4.1, this is a highest rank approximation, so $a_1, \ldots, a_m$ satisfy the conditions in Theorem 4.3(ii). The maximum of $2\sin(\pi/m)$ is computed in Section 4.2. □

We have then proved the following theorem, which gives a different answer to the second question in Section 4.2.1 when the base points are required to be on the unit circle.

Theorem 4.10. *When the base points are required to be on the unit circle, for each $n = 1, 2, \ldots, m-1$, the mth roots of unity derivative of order n is the best numerical estimate m points based derivative of order n. This is unique up to a rescale and a permutation of base points.*

5. Numerical analysis and the classification of generalized Riemann derivatives

In this section we put together the classification of generalized Riemann derivatives of Section 2 and the numerical analysis of generalized Riemann derivatives of Section 4. Specifically, given any two nth generalized Riemann differentiations $\mathcal{A}$ and $\mathcal{B}$ that are either equivalent or imply each other, one obviously suspects that the first significant terms in the approximations of the nth derivative by the nth difference quotients corresponding to $\mathcal{A}$ and $\mathcal{B}$ are related to each other. This question has a nice answer, given in Theorems 5.2 and 5.3, in the case when either $\mathcal{A}$ or both $\mathcal{A}$ and $\mathcal{B}$ correspond to highest rank approximations.

By Theorem 4.7, if $\mathcal{A}$ is the data vector of a highest rank m points based nth generalized Riemann difference then, for an $m + n$ times differentiable function f,

$$\Delta_{\mathcal{A}} f(z,h) - f_n(z)h^n = (-1)^{m+1} a_1 \cdots a_m \cdot n! \cdot \frac{f_{m+n}(z)}{(m+n)!} h^{m+n} + o(h^{m+n}). \tag{5.1}$$

Fix a positive integer ℓ, and recall that $\Delta_{\mathcal{A}} f(z,h)$ is expressed uniquely as the sum

$$\Delta_{\mathcal{A}} f(z,h) = \sum_{k=0}^{\ell-1} \Delta_{\mathcal{A}}^{(k,\ell)} f(z,h)$$

of ℓ-components $\Delta_{\mathcal{A}}^{(k,\ell)} f(z,h)$, for $k = 0, 1, \ldots, \ell - 1$, where

$$\Delta_{\mathcal{A}}^{(k,\ell)} f(z,h) = \frac{1}{\ell} \sum_{j=0}^{\ell-1} \omega^{-kj} \Delta_{\mathcal{A}} f(z, \omega^j h)$$

is a type (k, ℓ) difference of f at z and h. We average the products of the form ω^{-kj} times (5.1) evaluated at z and $\omega^j h$, for $j = 0, 1, \ldots, \ell - 1$, to deduce that

$$\begin{aligned} &\Delta_{\mathcal{A}}^{(k,\ell)} f(z,h) - \left(\frac{1}{\ell} \sum_{j=0}^{\ell-1} \omega^{(n-k)j} \right) f_n(z) h^n \\ &= \left(\frac{1}{\ell} \sum_{j=0}^{\ell-1} \omega^{(m+n-k)j} \right) (-1)^{m+1} a_1 \cdots a_m \cdot n! \cdot \frac{f_{m+n}(z)}{(m+n)!} h^{m+n} + o(h^{m+n}). \end{aligned} \tag{5.2}$$

Basic complex roots of unity computation show that

$$\frac{1}{\ell} \sum_{j=0}^{\ell-1} \omega^{(n-k)j} = \delta_{0,n-k \mod \ell} \text{ and } \frac{1}{\ell} \sum_{j=0}^{\ell-1} \omega^{(m+n-k)j} = \delta_{0,m+n-k \mod \ell}. \tag{5.3}$$

The following lemma adds numerical perspective to the result of Theorem 3.1.

Lemma 5.1. *Let $\Delta_{\mathcal{A}} f(z,h)/h^n$ be an m points based nth generalized Riemann difference quotient that approximates $f_n(z)$ to highest rank, for each $m + n$ times (Peano) differentiable function f. Then*

(i) *If $n \mod \ell = k = m + n \mod \ell$, then*

$$\frac{\Delta_{\mathcal{A}}^{(k,\ell)} f(z,h)}{h^n} - f_n(z) = (-1)^{m+1} a_1 \cdots a_m \cdot n! \cdot \frac{f_{m+n}(z)}{(m+n)!} h^m + o(h^m).$$

(ii) *If $n \mod \ell = k \neq m + n \mod \ell$, then*

$$\frac{\Delta_{\mathcal{A}}^{(k,\ell)} f(z,h)}{h^n} - f_n(z) = o(h^m).$$

(iii) *If $n \mod \ell \neq k = m + n \mod \ell$, then*

$$\frac{\Delta_{\mathcal{A}}^{(k,\ell)} f(z,h)}{h^n} = (-1)^{m+1} a_1 \cdots a_m \cdot n! \cdot \frac{f_{m+n}(z)}{(m+n)!} h^m + o(h^m).$$

(iv) *If $n \mod \ell \neq k \neq m + n \mod \ell$, then*

$$\frac{\Delta_{\mathcal{A}}^{(k,\ell)} f(z,h)}{h^n} = o(h^m).$$

Proof. All parts follow from (5.2) and (5.3). □

Recall that each n times differentiable function f is $\mathcal{A}$-differentiable, for each generalized Riemann derivative $\mathcal{A}$ of order n. In particular, the difference quotient $\Delta_{\mathcal{A}} f(z,h)/h^n$ approximates the ordinary nth (Peano) derivative $f_n(z)$. Higher order of differentiability for f allows the computation of the error by Taylor expansion. For highest rank approximations, this error is computed in Theorem 4.7.

The next theorem is the numerical application of Theorem 2.4. It relates the error terms of any two generalized Riemann derivatives $\mathcal{A}$ and $\mathcal{B}$ of order n for which $\mathcal{A}$-differentiation is equivalent to $\mathcal{B}$-differentiation, when at least one of $\mathcal{A}$ and $\mathcal{B}$ corresponds to a highest rank approximation.

Theorem 5.2. *Let $\mathcal{A} = \{A_1, \dots, A_m, a_1, \dots, a_m\}$, $\mathcal{B} = \{B_1, \dots, B_\mu, b_1, \dots, b_\mu\}$ be the data vectors of two nth generalized Riemann derivatives, for $1 \le n < \min\{m, \mu\}$. Suppose $\mathcal{A}$-differentiability $\Longleftrightarrow$ $\mathcal{B}$-differentiability, and $\mathcal{A}$ is a highest rank approximation, or*

$$\frac{\Delta_{\mathcal{A}} f(z,h)}{h^n} - f_n(z) = (-1)^{m+1} a_1 \cdots a_m \cdot n! \cdot \frac{f_{m+n}(z)}{(m+n)!} h^m + o(h^m),$$

for each $m+n$ times (Peano) differentiable function f. Then

(i) *There exists a nonzero constant C such that*

$$\frac{\Delta_{\mathcal{B}} f(z,h)}{h^n} - f_n(z) = C(-1)^{m+1} a_1 \cdots a_m \cdot n! \cdot \frac{f_{m+n}(z)}{(m+n)!} h^m + o(h^m),$$

for each $m+n$ times differentiable function f.

(ii) *If $\mathcal{B}$ is also of highest rank, then $\mu = m$ and $b_1 b_2 \dots b_m = C a_1 a_2 \dots a_m$.*

Proof. (i) Let k, ℓ be integers with $0 \le k < \ell$. By Theorem 2.4,

$$\Delta_{\mathcal{B}}^{(k,\ell)} f(z,h) = \begin{cases} r_k^{-n} \Delta_{\mathcal{A}}^{(k,\ell)} f(z, r_k h) & \text{if } k = n \mod \ell \\ R_k \Delta_{\mathcal{A}}^{(k,\ell)} f(z, r_k h) & \text{otherwise.} \end{cases} \tag{5.4}$$

When $m \mod \ell = 0$ and $k = n \mod \ell$, the above expression and Lemma 5.1(i) yield

$$\begin{aligned} \frac{\Delta_{\mathcal{B}}^{(k,\ell)} f(z,h)}{h^n} - f_n(z) &= \frac{\Delta_{\mathcal{A}}^{(k,\ell)} f(z, r_k h)}{(r_k h)^n} - f_n(z) \\ &= (-1)^{m+1} a_1 \cdots a_m \cdot n! \cdot \frac{f_{m+n}(z)}{(m+n)!} (r_k h)^m + o(h^m) \end{aligned} \tag{5.5}$$

When $m \mod \ell = 0$ and $k \ne n \mod \ell$, the same expression and Lemma 5.1(iv) yield

$$\frac{\Delta_{\mathcal{B}}^{(k,\ell)} f(z,h)}{h^n} = R_k r_k^n \cdot \frac{\Delta_{\mathcal{A}}^{(k,\ell)} f(z, r_k h)}{(r_k h)^n} = o(h^m) \tag{5.6}$$

The desired relation with $C = r_{n \bmod \ell}^m$ is obtained by adding (5.5) and all $\ell - 1$ (5.6)s. When $m \mod \ell \ne 0$ and $k = n \mod \ell$, the expression (5.5) and Lemma 5.1(ii) yield

$$\frac{\Delta_{\mathcal{B}}^{(k,\ell)} f(z,h)}{h^n} - f_n(z) = \frac{\Delta_{\mathcal{A}}^{(k,\ell)} f(z, r_k h)}{(r_k h)^n} - f_n(z) = o(h^m) \tag{5.7}$$

When $m \mod \ell \ne 0$ and $k \ne n \mod \ell$, the expression (5.4) and Lemma 5.1(iii)(iv) yield

$$\frac{\Delta_{\mathcal{B}}^{(k,\ell)} f(z,h)}{h^n} = R_k r_k^n \cdot \frac{\Delta_{\mathcal{A}}^{(k,\ell)} f(z,r_k h)}{(r_k h)^n} = \tag{5.8}$$
$$\begin{cases} R_k r_k^n \cdot (-1)^{m+1} a_1 \cdots a_m n! \frac{f_{m+n}(z)}{(m+n)!}(r_k h)^m + o(h^m) & \text{if } k = m+n \mod \ell, \\ o(h^m) & \text{otherwise.} \end{cases}$$

The desired relation with $C = R_k r_k^{m+n}$ and $k = m+n \mod \ell$ is obtained by adding (5.7) and all $\ell - 1$ (5.8)s.

(ii) The hypothesis that $\mathcal{B}$ is a highest rank approximation translates into

$$\frac{\Delta_{\mathcal{B}} f(z,h)}{h^n} - f_n(z) = (-1)^{\mu+1} b_1 \cdots b_\mu \cdot n! \cdot \frac{f_{n+\mu}(z)}{(n+\mu)!} h^\mu + o(h^\mu).$$

The result comes by identifying the rank and magnitude of the error term on the right side here and the one on the right side in Part (i). □

The hypothesis in Theorem 5.2 does not imply the hypothesis in its Part (ii). This means $\mathcal{A}$-differentiability equivalent to $\mathcal{B}$-differentiability and $\mathcal{A}$ is a highest rank approximation do not imply that $\mathcal{B}$ is also a highest rank approximation. Indeed, suppose $\mathcal{A} = \{A_i; a_i \,|\, i = 1, \ldots, m\}$ corresponds to a highest rank approximation nth generalized Riemann derivative, and take $\mathcal{B} = (\mathcal{A} + \mathcal{A}_s)/2$, where $\mathcal{A}_s = \{s^{-n} A_i; s a_i \,|\, i = 1, \ldots, m\}$ is the s-rescale of $\mathcal{A}$. More explicitly, $\mathcal{B}$ is the $4m$-vector

$$\mathcal{B} = \{A_i/2; a_i \,|\, i = 1, \ldots, m\} \cup \{s^{-n} A_i/2; s a_i \,|\, i = 1, \ldots, m\}, \tag{5.9}$$

subject to the reduction

$$\{A_i/2; a_i\} \cup \{s^{-n} A_j/2; s a_j\} = \{(A_i + s^{-n} A_j)/2; a_i\} \text{ when } s a_j = a_i. \tag{5.10}$$

Since $\mathcal{A}$-differentiation is equivalent to $\mathcal{A}_s$-differentiation (see the paragraph following Theorem 2.4), this clearly implies $\mathcal{A}$-differentiation equivalent to $\mathcal{B}$-differentiation. The choice $a_1 a_2 \ldots a_m \neq 0$ and $s > (\max |a_i|)/(\min |a_i|)$ makes $\mathcal{B}$ precisely the $4m$-vector given by (5.9), with no reductions (5.10). By Theorem 5.2(i) and its proof, the rank of the approximation of $f_n(z)$ by the nth difference quotient determined $\mathcal{B}$ is m. Since $\mathcal{B}$ is based at $2m > m$ points, this is not a highest rank approximation generalized Riemann derivative of order n.

The next theorem is the numerical application of Theorem 2.8.

Theorem 5.3. *Let $\mathcal{A} = \{A_1, \ldots, A_m, a_1, \ldots, a_m\}$, $\mathcal{B} = \{B_1, \ldots, B_\mu, b_1, \ldots, b_\mu\}$ be the data vectors of two nth generalized Riemann derivatives, for $1 \le n < \min\{m, \mu\}$. Suppose $\mathcal{A}$-differentiability $\Longrightarrow$ $\mathcal{B}$-differentiability, and $\mathcal{A}$ is a highest rank approximation, or*

$$\frac{\Delta_{\mathcal{A}} f(z,h)}{h^n} - f_n(z) = (-1)^{m+1} a_1 \cdots a_m \cdot n! \cdot \frac{f_{m+n}(z)}{(m+n)!} h^m + o(h^m),$$

for each $m+n$ times (Peano) differentiable function f. Then

(i) *There exists a nonzero constant C such that*

$$\frac{\Delta_{\mathcal{B}} f(z,h)}{h^n} - f_n(z) = C(-1)^{m+1} a_1 \cdots a_m \cdot n! \cdot \frac{f_{m+n}(z)}{(m+n)!} h^m + o(h^m),$$

for each $m+n$ times differentiable function f.

(ii) *If $\mathcal{B}$ is also a highest rank approximation and $\mu \ge m$, then either $\mu > m$ and $C a_1 \ldots a_m = 0$ or $\mu = m$ and $C a_1 \ldots a_m = b_1 \ldots b_m$.*

Proof. (i) Let k, ℓ be integers with $0 \leq k < \ell$. By Theorem 2.8,

$$\Delta_{\mathcal{B}}^{(k,\ell)} f(z,h) = \sum_i R_i^{(k,\ell)} \Delta_{\mathcal{A}}^{(k,\ell)} f(z, r_i^{(k,\ell)} h), \tag{5.11}$$

of $r_i^{(k,\ell)}$-dilates of $\Delta_{\mathcal{A}}^{(k,\ell)} f(z,h)$ such that $\sum_i R_i^{(k,\ell)} (r_i^{(k,\ell)})^n = 1$ when $k = n \mod \ell$. When $m \mod \ell = 0$ and $k = n \mod \ell$, the above expression and Lemma 5.1(i) yield

$$\begin{aligned} \frac{\Delta_{\mathcal{B}}^{(k,\ell)} f(z,h)}{h^n} - f_n(z) &= \sum_i R_i^{(k,\ell)} (r_i^{(k,\ell)})^n \cdot [\frac{\Delta_{\mathcal{A}}^{(k,\ell)} f(z, r_i^{(k,\ell)} h)}{(r_i^{(k,\ell)} h)^n} - f_n(z)] \\ &= \sum_i R_i^{(k,\ell)} (r_i^{(k,\ell)})^n \cdot (-1)^{m+1} a_1 \cdots a_m \cdot n! \cdot \frac{f_{m+n}(z)}{(m+n)!} (r_i^{(k,\ell)} h)^m + o(h^m) \end{aligned} \tag{5.12}$$

When $m \mod \ell = 0$ and $k \neq n \mod \ell$, the same expression and Lemma 5.1(iv) yield

$$\frac{\Delta_{\mathcal{B}}^{(k,\ell)} f(z,h)}{h^n} = \sum_i R_i^{(k,\ell)} (r_i^{(k,\ell)})^n \cdot \frac{\Delta_{\mathcal{A}}^{(k,\ell)} f(z, r_i^{(k,\ell)} h)}{(r_i^{(k,\ell)} h)^n} = o(h^m) \tag{5.13}$$

The desired relation with $C = \sum_i R_i^{(k,\ell)} (r_i^{(k,\ell)})^{m+n}$ and $k = n \mod \ell$ is obtained by adding (5.12) and all $\ell - 1$ (5.13)s.

When $m \mod \ell \neq 0$ and $k = n \mod \ell$, the expression (5.11) and Lemma 5.1(ii) yield

$$\frac{\Delta_{\mathcal{B}}^{(k,\ell)} f(z,h)}{h^n} - f_n(z) = \sum_i R_i^{(k,\ell)} (r_i^{(k,\ell)})^n \cdot [\frac{\Delta_{\mathcal{A}}^{(k,\ell)} f(z, r_i^{(k,\ell)} h)}{(r_i^{(k,\ell)} h)^n} - f_n(z)] = o(h^m) \tag{5.14}$$

When $m \mod \ell \neq 0$ and $k \neq n \mod \ell$, the equation (5.11) and Lemma 5.1(iii)(iv) yield

$$\begin{aligned} &\frac{\Delta_{\mathcal{B}}^{(k,\ell)} f(z,h)}{h^n} = \sum_i R_i^{(k,\ell)} (r_i^{(k,\ell)})^n \cdot \frac{\Delta_{\mathcal{A}}^{(k,\ell)} f(z, r_i^{(k,\ell)} h)}{(r_i^{(k,\ell)} h)^n} = \\ &\begin{cases} \sum_i R_i^{(k,\ell)} (r_i^{(k,\ell)})^n (-1)^{m+1} a_1 \cdots a_m n! \frac{f_{m+n}(z)}{(m+n)!} (r_i^{(k,\ell)} h)^m + o(h^m) & \text{if } \ell | m+n-k, \\ o(h^m) & \text{otherwise.} \end{cases} \end{aligned} \tag{5.15}$$

The desired relation with $C = \sum_i R_i^{(k,\ell)} (r_i^{(k,\ell)})^{m+n}$ and $k = m+n \mod \ell$ is obtained by adding (5.14) and all $\ell - 1$ (5.15)s.

(ii) The hypothesis that $\mathcal{B}$ is a highest rank approximation translates into

$$\frac{\Delta_{\mathcal{B}} f(z,h)}{h^n} - f_n(z) = (-1)^{\mu+1} b_1 \cdots b_\mu \cdot n! \cdot \frac{f_{n+\mu}(z)}{(n+\mu)!} h^\mu + o(h^\mu).$$

The result comes by identifying the rank and magnitude of the error term on the right side here and the ones on the right side in Part (i). □

6. Divisible groups, group algebras, and the proof of the main classification results

The proofs of the main classification theorems, Theorems 2.4 and 2.8 of Section 2, are given in Section 6.5. These are based on Theorem 6.6 of Section 6.4, which reduces the equivalence and implication relations on generalized Riemann differentiations to equality and containment of principal ideals of the group algebra of the multiplicative group of complex numbers over the complex field. Properties of divisible groups, a class of groups related to the multiplicative group of complex numbers, are reviewed in Section 6.1, and those of group algebras are studied in Sections 6.2 and 6.3.

6.1. Divisible groups

The standard way of writing the operation in an abelian group is additive. An abelian group G is *divisible* if for any $g \in G$ and any positive integer n, there exists $h \in G$ such that $nh = g$. For example, the additive groups $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are divisible, and the same is true about the factor group $\mathbb{Q}/\mathbb{Z}$ and its quasicyclic subgroup

$$C_{p^\infty} = \left\{ \frac{a}{p^t} + \mathbb{Z} : t \geq 0, a = 0, 1, \ldots, p^t - 1 \right\},$$

where p is a prime number. Multiplicative examples of divisible groups include the group $\mathbb{C}^\times$ of non-zero complex numbers and its subgroup $U = \{z \in \mathbb{C} : |z| = 1\}$. If U_n is the group of all nth complex roots of unity, then both $U_{\text{fin}} = \bigcup_{n=1}^\infty U_n = \{e^{2\pi q i} : q \in \mathbb{Q}\}$ and $U_{p^\infty} = \bigcup_{k=1}^\infty U_{p^k}$ are divisible groups. The map $q \mapsto e^{2\pi q i} : \mathbb{Q} \to \mathbb{C}^\times$ induces isomorphisms $\mathbb{Q}/\mathbb{Z} \cong U_{\text{fin}}$ and $C_{p^\infty} \cong U_{p^\infty}$. The multiplicative group $\mathbb{R}^\times$ of non-zero real numbers is not divisible, since -1 is not the square of any real number. By contrast, the multiplicative group $\mathbb{R}^+$ of positive real numbers is a divisible group, since it is isomorphic to $\mathbb{R}$ via exponentiation.

An abelian group is *indecomposable* if it is not the direct sum (product) of two non-trivial subgroups. For example, $\mathbb{Z}_3$ is indecomposable, while $\mathbb{Z}_6 = 3\mathbb{Z}_6 \oplus 2\mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$ is not. Divisible groups have the property that they are *injective* $\mathbb{Z}$-modules. Equivalently, they are direct summands in all abelian groups that contain them. The fundamental theorem of divisible groups says that an indecomposable divisible group is either isomorphic to $\mathbb{Q}$ or to C_{p^∞} for some p, and each divisible group is a direct sum of indecomposable divisible subgroups. In particular, each divisible group G is the direct sum

$$G = G_1 \oplus G_2$$

of two characteristic divisible subgroups: G_1 is the *torsion subgroup* of G and consists of all elements of G of finite order; it is the sum of all indecomposable divisible subgroups of G that are isomorphic to C_{p^∞} for some p. G_2 is torsion free; it is the sum of all indecomposable divisible subgroups of G that are isomorphic to $\mathbb{Q}$.

Example 6.1. (i) Let $G = (\mathbb{R}, +)$. Then $G_1 = (0)$ and $G_2 = \mathbb{Q}^{(c)}$ where $c = 2^{\aleph_0}$. This means that $\mathbb{R} = G_2$ is the direct sum of continuum many copies of $\mathbb{Q}$.

(ii) Let $G = U$. Then $G_1 = U_{\text{fin}} (\cong \mathbb{Q}/\mathbb{Z} \cong \bigoplus_{p \text{ prime}} C_{p^\infty})$ and $G_2 \cong \mathbb{Q}^{(c)} (\cong \mathbb{R})$.

(iii) Let $G = \mathbb{C}^\times$. Then $G \cong U \times \mathbb{R}^+$, hence $G_1 = U_{\text{fin}}$ and $G_2 \cong \mathbb{Q}^{(c)}$.

Note that each type C_{p^∞} indecomposable summand of G_1 in Parts (ii) and (iii) of the above example has multiplicity one.

6.2. Group algebras

Let $\Bbbk$ be a field and G is a group. The *group algebra* $\Bbbk G$ is the vector space

$$\Bbbk G = \text{span}_{\Bbbk} G = \left\{ \sum_{g \in G} a_g g : a_g \in \Bbbk, \text{ all but finitely many non-zero} \right\}$$

of basis G, where the multiplication is given by the multiplication of basis elements, which is the multiplication in G. When G is a semigroup or a monoid, the same definition makes $\Bbbk G$ a *semigroup algebra* or a *monoid algebra*. For example, the monoid algebra $\Bbbk M$ of the multiplicative monoid $M = \{1, x, x^2, \dots\} = x^{\mathbb{N}}$ of nonnegative powers of a variable x is nothing but the polynomial algebra $\Bbbk[x]$. Since $(M, \cdot) \cong (\mathbb{N}, +)$, we have $\Bbbk\mathbb{N} \cong \Bbbk[x]$. In the same way, the group algebra $\Bbbk\mathbb{Z} \cong \Bbbk[x^{\mathbb{Z}}]$ is isomorphic to the Laurent polynomial algebra $\Bbbk[x, x^{-1}]$, and the group algebra $\Bbbk\mathbb{Q}$ is isomorphic to the generalized polynomial algebra $\Bbbk[x^{\mathbb{Q}}]$ in one variable, where rational exponents are allowed. Based on these observations, for an abelian group G whose operation is written additively, we shall prefer its isomorphic multiplicative copy x^G. The group algebra of a direct sum of groups is the tensor product of the group algebras of the individual terms. The same is true for semigroups or monoids. For example, $\Bbbk[\mathbb{N} \times \mathbb{N}] \cong \Bbbk[x^{\mathbb{N}} y^{\mathbb{N}}] = \Bbbk[x, y] \cong \Bbbk[x] \otimes_{\Bbbk} \Bbbk[y] \cong \Bbbk[\mathbb{N}] \otimes \Bbbk[\mathbb{N}]$.

An element e of a ring R is an *idempotent* if $e^2 = e$. The elements 0 and 1 are always idempotents. An idempotent is *primitive* if it cannot be written as a sum of non-zero idempotents. A *complete system of primitive idempotents* of R is a set of primitive idempotents that add up to 1. It is the set consisting of the identity elements of the summands in a decomposition of R as a direct sum of indecomposable ideals. For example, a complete set of primitive idempotents in $\mathbb{Z}_6$ is $\{e_1 = 3, e_2 = 4\}$. They correspond to the decomposition $\mathbb{Z}_6 = 3\mathbb{Z}_6 \oplus 4\mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$ of $\mathbb{Z}_6$ as a direct sum of indecomposable ideals. A ring R is indecomposable if and only if 0 and 1 are its only idempotents; equivalently, if and only if 1 is a primitive idempotent. The ring $\mathbb{Z}_n$ is indecomposable if and only if n is a power of a prime. If $n = p_1^{a_1} \cdots p_r^{a_r}$, then $\mathbb{Z}_n$ has 2^r idempotents and r primitive idempotents; in this case $\mathbb{Z}_n$ is a direct sum of r indecomposable ideals.

For the remaining of the section, $\Bbbk = \mathbb{C}$ is the field of complex numbers and $G = \mathbb{C}^\times$ is the multiplicative group of non-zero complex numbers. To avoid confusion with complex numbers being both scalars and group elements, we denote the field elements $a, b, \dots \in \Bbbk$ and the group elements $x_a, x_b, \dots \in G$. The group algebra $\Bbbk G$ is the $\Bbbk$-algebra generated by elements x_a with $a \in G$ and subject to the relations $x_a x_b = x_{ab}$, for all $a, b \in \mathbb{C}^\times$. Recall that the group G decomposes as the direct sum $G = G_1 \oplus G_2$ of its torsion subgroup

$$G_1 = U_{\text{fin}} = \bigcup_{n=1}^{\infty} U_n \cong \bigoplus_{p \text{ prime}} U_{p^\infty} \cong \mathbb{Q}/\mathbb{Z} \cong \bigoplus_{p \text{ prime}} C_{p^\infty}$$

and the torsion-free subgroup $G_2 \cong \mathbb{Q}^{(c)} \cong \left(x^{\mathbb{Q}}\right)^{(c)}$. This in turn gives the tensor product decomposition $\Bbbk G \cong \Bbbk G_1 \otimes_{\Bbbk} \Bbbk G_2$.

The algebraic structure of $\Bbbk G_2$ is quite clear: this is a tensor product of continuum many factors all isomorphic to $\Bbbk[x^{\mathbb{Q}}]$, or a generalized polynomial algebra over $\Bbbk$ in continuum many indeterminates where rational exponents are allowed.

Turning to the factor $\Bbbk G_1 \cong \bigotimes_{p \text{ prime}} \Bbbk U_{p^\infty}$, we first notice that $m|n$ is equivalent to $U_m \subseteq U_n$, which in turn yields $\Bbbk U_m \subseteq \Bbbk U_n$. If $\omega_n = e^{2\pi i/n}$ is a primitive nth root of unity, then $\omega_n^{n/m} = \omega_m$. The cyclic group $U_n = \langle \omega_n \rangle$ has order n and the group algebra $\Bbbk U_n$ is n-dimensional over $\Bbbk$.

The following proposition gives information about the idempotents of $\Bbbk U_n$.

Proposition 6.2. (i) *The elements*

$$e_{k,n} := \frac{1}{n} \sum_{i=0}^{n-1} \omega_n^{ki} x_{\omega_n^i}, \qquad \textit{for } k = 0, 1, \dots, n-1,$$

form a complete set of primitive idempotents and a $\Bbbk$*-basis of the group algebra* $\Bbbk U_n$.

(ii) *If* $n = \ell m$ *and* $t = 0, 1, \ldots, m-1$, *then*

$$e_{t,m} = \sum_{j=0}^{\ell-1} e_{mj+t,n}.$$

In particular, the group algebra $\Bbbk U_{p^\infty} = \Bbbk\left[\bigcup_{n=1}^{\infty} U_{p^n}\right]$ *has no primitive idempotents.*

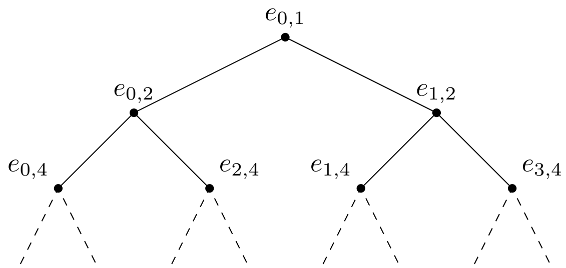
Proof. (i) It is routine computation checking that the $e_{k,n}$ are orthogonal idempotents. In particular, they form a linearly independent subset, hence a basis, of the n-dimensional group algebra $\Bbbk U_n$. (ii) We have

$$\sum_{j=0}^{\ell-1} e_{mj+t,n} = \sum_{j=0}^{\ell-1}\left(\frac{1}{n}\sum_{i=0}^{n-1}\omega_n^{(mj+t)i}x_{\omega_n^i}\right) = \frac{1}{n}\sum_{i=0}^{n-1}\left(\sum_{j=0}^{\ell-1}\omega_n^{mji}\right)\omega_n^{ti}x_{\omega_n^i}.$$

The sum in the last parenthesis is $\sum_{j=0}^{\ell-1}\omega_\ell^{ji}$. This equals ℓ when $\ell | i$, that is, when $i = \ell s$ for some $s = 0, 1, \ldots, m-1$, and it equals zero in all other cases. Then

$$\sum_{j=0}^{\ell-1} e_{mj+t,n} = \frac{1}{n}\sum_{s=0}^{m-1}(\ell)\,\omega_n^{t\ell s}x_{\omega_n^{\ell s}} = \frac{1}{m}\sum_{s=0}^{m-1}\omega_m^{ts}x_{\omega_m^s} = e_{t,m}. \quad \square$$

We let the idempotents in Proposition 6.2(ii) label the vertices of a simple two-level tree graph, with a parent $e_{t,m}$ and ℓ children $e_{mj+t,n}$, for $j = 0, \ldots, \ell-1$. For a prime p and various nonnegative integers r, by taking $n = p^r$, $m = p^{r-1}$ and $\ell = p$, these simple trees match together to form an infinite *p-splitting tree* of idempotents in $\Bbbk C_{p^\infty}$ where the sum of the children of each node equals the node. The tree is rooted at 1 and has p^r nodes at the rth level, corresponding to the p^r primitive idempotents in $\Bbbk C_{p^r}$. The figure below shows this graph when $p = 2$.



The next corollary gives a surprising expression of the group algebra $\Bbbk\mathbb{C}^\times$ as a tensor product of a group algebra of a torsion group and a group algebra of a torsion-free group.

Corollary 6.3. *For each prime* p, *we have* $\Bbbk\mathbb{C}^\times \cong \Bbbk U_{p^\infty} \otimes \Bbbk\mathbb{R}^+$.

Proof. Recall from earlier in the section that $\Bbbk U_{\text{fin}} \cong \bigotimes_{p \text{ prime}} \Bbbk U_{p^\infty}$. By Proposition 6.2, both algebras $\Bbbk U_{\text{fin}}$ and $\Bbbk U_{p^\infty}$ have countable dimensions over $\Bbbk$, are generated by idempotents, and do not have any primitive idempotents. A theorem of D. Berman (see [41, Chapter 14, Theorem 3.8]) makes them isomorphic. The rest comes from the isomorphism $\Bbbk\mathbb{C}^\times \cong \Bbbk U_{\text{fin}} \otimes \Bbbk\mathbb{R}^+$ deduced earlier. $\square$

6.3. The group algebra $\mathbf{A} = \Bbbk\mathbb{C}^\times$ *and the monoid algebra* $\mathbf{B} = \Bbbk\mathbb{C}$

Let $\mathbf{A}$ be the group algebra over $\Bbbk$ of the multiplicative group of non-zero complex numbers introduced in the previous section and let $\mathbf{B}$ denote the $\Bbbk$-monoid algebra of the multiplicative monoid of all complex numbers. Then $\mathbf{B}$ is obtained from $\mathbf{A}$ by adjoining the absorbing basis element x_0. As in real case, see [13], we show that the questions of implication or equivalence of complex generalized Riemann differentiation translate into principal ideals in $\mathbf{A}$ or $\mathbf{B}$.

Adding to the properties of $\mathbf{A}$ deduced in Section 6.2, here are a few more basic facts about the algebras $\mathbf{A} \subset \mathbf{B}$:

- The identity element in both $\mathbf{A}$ and $\mathbf{B}$ is x_1.
- The basis elements x_r, for $r \in \mathbb{C}^\times$, are units (invertible elements) of $\mathbf{A}$, but these are not the only ones. We call them the *trivial units* of $\mathbf{A}$.
- If r is a complex nth root of unity, then $x_1 - x_r$ is a zero-divisor of $\mathbf{A}$; we have $(x_1 - x_r)(x_1 + x_r + x_r^2 + \cdots + x_r^{n-1}) = x_1^2 - x_r^n = x_{1^2} - x_{r^n} = x_1 - x_1 = 0$.
- Let $V = \sum_{r\in\mathbb{k}} \mathbb{k}x_r$. Then the dual $\mathbb{k}$-space V^* is isomorphic to the $\mathbb{k}$-space of functions $\mathbb{k} \to \mathbb{k}$ via $\varphi \in V^* \leftrightarrow f : \mathbb{k} \to \mathbb{k}$, $f(r) = \varphi(x_r)$.
- The element $\sigma = x_{-1}$ is the unique element of $\mathbf{B}$ of order 2.
- The sets $\omega(\mathbf{A}) = \{\sum A_i x_{a_i} \in \mathbf{A} : \sum A_i = 0\}$ and $\omega(\mathbf{B})$, defined in a similar manner, are ideals of codimension one in $\mathbf{A}$ and $\mathbf{B}$, called the *augmentation ideals* of these algebras.

The following lemma gives an expression of all units in $\mathbf{A}$ as linear combinations of trivial units of the indecomposable summands of $\mathbf{A}$ determined by the idempotents $e_{k,n}$. These trivial units are expressed as

$$e_{k,n}x_r = \frac{1}{n}\sum_{i=0}^{n-1} \omega_n^{ki} x_{r\omega_n^i},$$

where n and k are integers with $0 \le k \le n-1$, and r is a positive real number.

Lemma 6.4. *For each unit u of the group algebra $\mathbf{A} = \mathbb{k}\mathbb{C}^\times$, there exists a positive integer n such that u is expressed as the linear combination*

$$u = \sum_{k=0}^{n-1} A_k e_{k,n} x_{r_k},$$

where the coefficients A_k are non-zero complex numbers and r_k are positive real numbers, for all k.

Proof. Recall that $\mathbf{A} \cong \mathbb{k}U_{\text{fin}} \otimes_{\mathbb{k}} \mathbb{k}\mathbb{R}^+ = \bigcup_{n=1}^\infty (\mathbb{k}U_n \otimes_{\mathbb{k}} \mathbb{k}\mathbb{R}^+)$ and let $\mathbf{A}_n$ be the subalgebra of $\mathbf{A}$ that corresponds to $\mathbb{k}U_n \otimes_{\mathbb{k}} \mathbb{k}\mathbb{R}^+$ under this isomorphism. Then $u \in \mathbf{A}_n$, for some n. The primitive idempotents $e_{k,n}$ give rise to a direct sum decomposition $\mathbf{A}_n = \oplus_{k=0}^{n-1} e_{k,n}\mathbf{A}_n$ of indecomposable ideals $e_{k,n}\mathbf{A}_n$. In particular, $u = \sum_{k=0}^{n-1} u_k$, where u_k is a unit in $e_{k,n}\mathbf{A}_n \cong e_{k,n}\mathbb{k}U_n \otimes_{\mathbb{k}} \mathbb{k}\mathbb{R}^+$. Since $e_{k,n}\mathbb{k}U_n$ is one-dimensional and $\mathbb{k}\mathbb{R}^+$ is a torsion free group, a theorem of Bovdi in [18] (see also Passman [41, Chapter 13, Theorem 1.11]) makes all units in $e_{k,n}\mathbf{A}_n$ trivial, that is, non-zero scalar multiples of group elements. Then $u_k = A_k e_{k,n} x_{r_k}$, for a non-zero complex number A_k and a positive real number r_k. □

6.4. Group algebras and generalized Riemann derivatives

For a complex function f, each element $\alpha = \sum A_i x_{a_i}$ of $\mathbf{B}$ corresponds uniquely to the difference $\Delta_\alpha f(z,h) = \sum_i A_i f(z + a_i h)$. If this is an nth generalized Riemann difference, then $\Delta_\alpha f(z,h) = \Delta_{\mathcal{A}} f(z,h)$, where $\mathcal{A} = \{A_i; a_i| \text{ for all } i\}$, and the *$\alpha$-derivative* of f at z, given by

$$D_\alpha f(z) := \lim_{h\to 0} \frac{\Delta_\alpha f(z,h)}{h^n},$$

is nothing but the generalized Riemann derivative $D_{\mathcal{A}}f(z)$.

For each non-zero r in $\Bbbk^\times$, define $d_r = \frac{1}{2}(x_r - x_{-r})$ and $e_r = \frac{1}{2}(x_r + x_{-r})$. Then $\alpha = d_1$ corresponds to the first symmetric difference $\Delta_\alpha f(z,h) = \frac{1}{2}\{f(z+h) - f(z-h)\}$; $\alpha = x_1 - x_0$ corresponds to the ordinary first difference $\Delta_\alpha f(z,h) = f(z+h) - f(z)$; and when $\Bbbk = \mathbb{R}$, $\alpha = d_1 + A(e_r - x_0)$, where $A, r \in \Bbbk^\times$, corresponds to the first order generalized Riemann difference

$$\Delta_\alpha f(z,h) = \frac{f(x+h) - f(x-h) + A[f(x+rh) + f(x-rh) - 2f(x)]}{2}$$

considered in [12]. Note that d_r does not correspond to a generalized Riemann derivative, for $r \neq 1$, and the same is true about e_r, for each complex number r.

The following lemma relates generalized Riemann differences to elements of the augmentation ideal $\omega(\mathbf{B})$. In this way, it provides the tool for separating the α-differences that are scalar multiples of generalized Riemann differences from those that are not.

Lemma 6.5. *Let $\alpha = \sum A_i x_{a_i}$ be a non-zero element of $\mathbf{B}$ and let $\Delta_\alpha f(z,h)$ be the difference corresponding to it. Then*

(i) *If $\Delta_\alpha f(z,h)$ is a generalized Riemann difference, then $\alpha \in \omega(\mathbf{B})$.*

(ii) *If $\alpha \in \omega(\mathbf{B})$, then there exists a unique nonzero complex number a such that $a^{-1}\Delta_\alpha f(z,h)$ is a generalized Riemann difference.*

Proof. (i) If $\Delta_\alpha f(z,h)$ is a generalized Riemann difference, then $\alpha \in \omega(\mathbf{B})$ by the first Vandermonde condition.

(ii) Suppose α is a non-zero element of $\omega(\mathbf{B})$. Then by the linear algebra of Vandermonde systems, there exists a positive integer n such that $\sum A_j a_j^n$ is a non-zero number a. If n is taken minimal with this property, then $a^{-1}\Delta_\alpha f(z,h)$ is a generalized Riemann difference. □

Suppose $\alpha, \beta \in \omega(\mathbf{B})$ and z is a complex number. We will write $\alpha \vdash \beta$ if for each measurable function f, $D_\alpha f(z)$ exists $\Rightarrow$ $D_\beta f(z)$ exists, and $\beta \sim \alpha$ if the converse also holds. The ideal of $\mathbf{B}$ generated by α is denoted by (α). Write $\alpha = \sum A_i x_{a_i}$ and let $\alpha_r = \alpha x_r = \sum A_i x_{a_i r}$ be the *dilate* of α by $r \in \Bbbk^\times$. Note especially that $(\alpha) = (\alpha_r)$ is the span of the dilates of α, i.e., dilates are associate. For each $r \in \Bbbk$, a function f is α-differentiable at z if and only if it is α_r-differentiable at z. This means that the equality of some principal ideals of $\mathbf{B}$ corresponds to the equivalence of the generalized differentiations corresponding to their generators. We shall see that the same is true in general. The *support* of α is the set $\text{supp}(\alpha) = \{a_i \mid A_i \neq 0\}$.

The following theorem writes the implication and equivalence of complex generalized Riemann differentiations in terms of inclusion and equality of principal ideals of the algebra $\mathbf{B} = \Bbbk\mathbb{C}$ of the multiplicative monoid $\mathbb{C}$.

Theorem 6.6. *Let $\mathcal{A}, \mathcal{B}$ be data vectors of complex generalized Riemann derivatives of orders m and n, and let α, β be the elements of $\omega(\mathbf{B})$ corresponding to them. Then*

(i) *$\alpha \vdash \beta$ if and only if $m = n$ and $(\alpha) \supseteq (\beta)$;*

(ii) *$\alpha \sim \beta$ if and only if $m = n$ and $(\alpha) = (\beta)$.*

Proof. Part (ii) is a consequence of Part (i). We first prove Part (i) under the assumption $m = n$, and then show that the remaining cases $m > n$ and $m < n$ of the direct implication are impossible.

Suppose $m = n$. When $(\alpha) \supseteq (\beta)$, write $\beta = \alpha \sum_r A_r x_r = \sum_r A_r \alpha_r$ as a linear combination of translates of α. If a function f is α-differentiable at z then f is β-differentiable at z and $D_\beta f(z) = \sum_r A_r D_{\alpha_r} f(z)$.

Conversely, we assume $(\alpha) \not\supseteq (\beta)$ and construct a function f such that $D_\alpha f(0)$ exists but $D_\beta f(0)$ does not exist. Let G be the subgroup of $\Bbbk^\times$ generated by all nonzero a_i's and b_i's. Then G is countable while the factor group $\Bbbk^\times/G$ is not. Then there exists a sequence $\{s_n\}_{n\geq 1}$ of representatives of cosets of G in $\Bbbk^\times$ such that $\lim_{n\to\infty} s_n = 0$. Since x_{s_n} is a (trivial) unit in $\mathbf{B}$, we have $\beta_{s_n} := \beta x_{s_n} \notin (\alpha)$, for all n. We claim that the β_{s_n}'s are linearly independent modulo (α). Indeed, if $\sum \lambda_n \beta_{s_n} \in (\alpha)$, with $\lambda_n \in \Bbbk$, then this can be expressed as $\sum \lambda_n \beta_{s_n} = \alpha \sum \mu_i x_{r_i}$, for $\mu_i \in \Bbbk$ and $r_i \in \Bbbk^\times$. For each n, let σ_{s_n} be the sum of all terms $\mu_i x_{r_i}$ with $r_i \in G s_n$. Then $\sum \lambda_n \beta_{s_n} = \alpha \sum \sigma_{s_n} = \sum \alpha\sigma_{s_n}$. Since the supports of α and β are contained in G, the supports of β_{s_n} and $\alpha\sigma_{s_n}$ are included in Gs_n. By the uniqueness of expression of a group algebra element as a sum of elements with supports in distinct cosets of a subgroup, we deduce $\lambda_n \beta_{s_n} = \alpha\sigma_{s_n}$ for each n. If $\lambda_n \neq 0$ for some n, then $\beta_{s_n} = \lambda_n^{-1}\alpha\sigma_{s_n} \in (\alpha)$, a contradiction, and the claim is proved.

Let $V = \sum_{r\in\Bbbk} \Bbbk x_r$, and recall that the dual space $V^* = \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$ is linearly isomorphic to the space of functions $\mathrm{Func}(\Bbbk) = \{f | f : \Bbbk \to \Bbbk\}$ via the map $\theta \in V^*$ maps to $f \in \mathrm{Func}(\Bbbk)$ such that $f(r) = \theta(x_r)$, for all $r \in \Bbbk$. Let W be a complement of the subspace $(\alpha) \oplus \sum \Bbbk\beta_{s_n}$ in V, and let $\theta : V \to \Bbbk$ be the linear map defined as zero on both (α) and W, and $\theta(\beta_{s_n}) = 1$, for all n. Then the complex function f that corresponds to θ under the above isomorphism has $\Delta_\alpha f(0, h) = \sum A_i f(a_i h) = \sum A_i \theta(x_{a_i h}) = \theta\left(\sum A_i x_{a_i h}\right) = \theta(\alpha_h)$. As $\alpha_h = \alpha x_h \in (\alpha)$, the last term of the above chain is zero, so $D_\alpha f(0) = \lim_{h\to 0} \Delta_\alpha f(0, h)/h^n = 0$. Also, $\Delta_\beta f(0, s_n) = \sum B_i f(b_i s_n) = \sum B_i \theta(x_{b_i s_n}) = \theta\left(\sum B_i x_{b_i s_n}\right) = \theta(\beta_{s_n}) = 1$, for all n, implies that the limit $\lim_{n\to\infty} \Delta_\beta f(0, s_n)/s_n^m = \lim_{n\to\infty} 1/s_n^m$ is not a finite number, and so $D_\beta f(0)$ does not exist.

Suppose $m > n$ and $\alpha \vdash \beta$. Let $\mathbb{F}$ be the subfield of $\mathbb{C}$ generated over the rationals by all a_i's and all b_i's. Define a function f by $f(z) = z^m \chi(z)$, where χ is the characteristic function of the subset $\mathbb{F}$ of $\mathbb{C}$. Then $f(h) = o(h^n)$ as $h \to 0$, making f is n times Peano differentiable at 0, hence β-differentiable at 0 and $D_\beta f(0) = f_n(0) = 0$. On the other hand, taking $h \in \mathbb{F}$ makes $D_\alpha f(0) = m!$ and taking $h \in \mathbb{C} \setminus \mathbb{F}$ makes $D_\alpha f(0) = 0$. Then f is not α-differentiable at 0, a contradiction with the hypothesis (i).

Suppose $m < n$ and $\alpha \vdash \beta$. Then the mth Vandermonde conditions, $\sum A_i a_i^m = m!$ for α and $\sum B_i b_i^m = 0$ for β, show that α is not a linear combination of dilates of β. This means that $(\beta) \not\supseteq (\alpha)$ and, as we did in the first part of the proof, this leads to contradiction by constructing a function f such that $D_\beta f(0)$ exists but $D_\alpha f(0)$ does not exist. □

6.5. Proof of the main classification theorems

Before we proceed with the proofs of the Theorems 2.4 and 2.8, the following two examples both illustrate the result of Theorem 6.6 and give insight on how this is going to be used in the above mentioned proofs. Given an element α in $\omega(\mathbf{B})$, both examples classify all elements β in $\omega(\mathbf{B})$ for which β-differentiation is equivalent to α-differentiation. Recall from Lemma 6.5 that for each β in $\omega(\mathbf{B})$, there exists a unique nonzero scalar multiple of β that corresponds to a generalized Riemann derivative. Also, if α corresponds to a generalized Riemann derivative, then so too does $\frac{1}{r}\alpha x_r = \frac{1}{r}\alpha_r$, for $r \in \Bbbk$.

Example 6.7. Consider the cyclic subgroup $U_2 = \{\pm 1\}$ of $\Bbbk^\times$. Pick $\ell = 2$ and observe that the primitive idempotents

$$\varepsilon_0 = e_{0,2} = \frac{1}{2}(x_1 + x_{-1}) = e_1 \quad \text{and} \quad \varepsilon_1 = e_{1,2} = \frac{1}{2}(x_1 - x_{-1}) = d_1$$

of the group algebra $\Bbbk U_\ell$ extend to orthogonal idempotents of $\mathbf{B}$ that add up to 1.

Let $\alpha = x_1 - x_0$ be the element of $\mathbf{B}$ corresponding to the first difference $\Delta_\alpha f(z, h) = f(z + h) - f(z)$. Then α-differentiation is the same as ordinary differentiation. We compute

$$\varepsilon_0\alpha = \frac{1}{2}(x_1 + x_{-1})(x_1 - x_0) = \frac{1}{2}(x_1 + x_{-1}) - x_0 = e_1 - x_0,$$
$$\varepsilon_1\alpha = \frac{1}{2}(x_1 - x_{-1})(x_1 - x_0) = \frac{1}{2}(x_1 - x_{-1}) = d_1$$

to deduce that $(\alpha) = (\varepsilon_0\alpha) \oplus (\varepsilon_1\alpha) = (e_1 - x_0) \oplus (d_1) = (e_1 - x_0, d_1)$.

An element β of $\mathbf{B} = \mathbf{B}\varepsilon_0 \oplus \mathbf{B}\varepsilon_1$ has the expression $\beta = \varepsilon_0\beta + \varepsilon_1\beta = (\sum A_r x_r)e_1 + (\sum B_s x_s)d_1 + Cx_0$, where $r, s \neq 0$. The condition $\beta \in \omega(\mathbf{B})$ makes $\sum A_r + C = 0$, so

$$\beta = (\sum A_r x_r)(e_1 - x_0) + (\sum B_s x_s)d_1.$$

This leads to $\varepsilon_0\beta = (\sum A_r x_r)(e_1 - x_0)$ and $\varepsilon_1\beta = (\sum B_s x_s)d_1$, which in turn yields $(\beta) = ((\sum A_r x_r)(e_1 - x_0)) \oplus ((\sum B_s x_s)d_1)$. By Theorem 6.6(ii), α-differentiation equivalent to β-differentiation translates into the equality of ideals $(\alpha) = (\beta)$. This is the same as the equality of their components

$$(e_1 - x_0) = ((\sum A_r x_r)(e_1 - x_0)) \text{ and } (d_1) = ((\sum B_s x_s)d_1).$$

It follows that $\sum A_r x_r$ and $\sum B_s x_s$ are units in $\mathbf{B}$, so by Lemma 6.4 they are sums of trivial units of the indecomposable components of $\mathbf{B}$. Writing $\sum A_r x_r = Ax_u(e_1 - x_0) + Bx_v d_1$, for complex numbers u and v, allows a recalculation of the first component of β: $(\sum A_r x_r)(e_1 - x_0) = (Ax_u(e_1 - x_0) + Bx_v d_1)(e_1 - x_0) = Ax_u(e_1 - x_0) = A(e_u - x_0)$. A similar computation simplifies the second component of β as $(\sum B_s x_s)d_1 = Bd_v$. Thus $\beta = A(e_u - x_0) + Bd_v$. It corresponds to a first generalized Riemann difference precisely when the first Vandermonde condition $A(\frac{u}{2} + \frac{-u}{2}) + B(\frac{v}{2} - \frac{-v}{2}) = 1$, or $B = \frac{1}{v}$, is satisfied. It follows that the first order derivatives D_β, for

$$\beta = A(e_u - x_0) + \frac{1}{v}d_v,$$

are all possible generalized Riemann differentiations of a function f at z that are equivalent to ordinary differentiation. When $\Bbbk = \mathbb{R}$, we recover the result of [12, Theorem 1].

The next example shows that the symmetric derivative is implied by many generalized Riemann derivatives, but it implies only rescales of it.

Example 6.8. Let $\alpha = d_1$ and let $\beta = (\sum A_r x_r)(e_1 - x_0) + (\sum B_s x_s)d_1$ be a generic element of $\omega(\mathbf{B})$ equivalent to α. As in Example 6.7, we first deduce $(\beta) = ((\sum A_r x_r)(e_1 - x_0)) \oplus ((\sum B_s x_s)d_1)$, and then write the equality of ideals $(\beta) = (\alpha)$ implied by Theorem 6.6, componentwise, as

$$((\sum A_r x_r)(e_1 - x_0)) = (0) \text{ and } ((\sum B_s x_s)d_1) = (d_1).$$

The first equality makes $\sum A_r x_r = 0$; the second makes $\sum B_s x_s$ a unit in $\mathbf{B}$, hence by Lemma 6.4, it is a sum $\sum B_s x_s = A_u x_u(e_1 - x_0) + B_v x_v d_1$ of trivial units of the indecomposable components of $\mathbf{B}$. Multiplication by d_1 yields $(\sum B_s x_s)d_1 = B_v x_v d_1 = B_v d_v$. All these simplify the expression of β as $\beta = B_v d_v$. Finally, the first Vandermonde condition, $B_v(\frac{v}{2}) - B_v(\frac{-v}{2}) = 1$ or $B_v = \frac{1}{v}$, makes $\beta = \frac{1}{v}d_v$. In conclusion, the generalized Riemann derivatives equivalent to the symmetric derivative are all its rescales.

We are now ready to give the proofs of the Theorems 2.4 and 2.8. Both proofs rely on the result of Theorem 6.6 and the decomposition $\mathbf{B} = \oplus_{k=0}^{\ell-1} e_{k,\ell}\mathbf{B}$ of the algebra $\mathbf{B}$ as a direct sum of ideals. This decomposition is inherited from a similar decomposition in $\mathbf{A}$, which in turn comes from the decomposition of $\Bbbk U_\ell$. If α is an element of $\mathbf{B}$, then $\alpha = \sum_{k=0}^{\ell-1} e_{k,\ell}\alpha$ is the unique decomposition of α as a sum $\alpha = \sum_{k=0}^{\ell-1} \alpha_k$, with $\alpha_k \in e_{k,\ell}\mathbf{B}$. The term $\alpha_k = e_{k,\ell}\alpha$ is the kth *component* of α.

Proof of Theorem 2.4. Let α, β be the elements of $\omega(\mathbf{B})$ corresponding to $\mathcal{A}$ and $\mathcal{B}$. Theorem 6.6(ii) translates the equivalence of $\mathcal{A}$-differentiation and $\mathcal{B}$-differentiation into $m = n$ and the equality $(\alpha) = (\beta)$ of principal ideals of $\mathbf{B}$. This in turn says that $\beta = u\alpha$, for some unit u of $\mathbf{B}$. By Lemma 6.4, u is expressed as $u = \sum_{k=0}^{\ell-1} A_k e_{k,\ell} x_{r_k}$, with $A_k \in \Bbbk^\times$ and $r_k \in \mathbb{R}^+$. It follows that $\beta = \sum_{k=0}^{\ell-1} A_k e_{k,\ell} x_{r_k} \alpha$. By taking the kth components of α and β, this reads as $\beta_k = A_k e_{k,\ell} x_{r_k} \alpha = A_k x_{r_k} \alpha_k$, or β_k is a nonzero scalar multiple of the r_k-dilate of α_k, for each k. Since $m = n$, Proposition 2.2 makes the components α_k and β_k, when $k = n$ mod ℓ, correspond to nth generalized Riemann derivatives, so $\beta_k = r_k^{-n} \alpha_k x_{r_k}$. □

Proof of Theorem 2.8. By ring theory, each ideal I of $\mathbf{B} = \oplus_{k=0}^{\ell-1} e_{k,\ell}\mathbf{B}$ has a unique decomposition $I = \oplus_{k=0}^{\ell-1} e_{k,\ell} I$ as a direct sum of ideals of the components of $\mathbf{B}$. In particular, $(\alpha) = \oplus_{k=0}^{\ell-1}(\alpha_k)$ and $(\beta) = \oplus_{k=0}^{\ell-1}(\beta_k)$. By Theorem 6.6(i), the assertion in Part (i) translates into $m = n$ and $(\alpha) \supseteq (\beta)$. Basic ideal theory makes this is equivalent to $(\alpha_k) \supseteq (\beta_k)$, for all k. In particular, β_k is a linear combination $\beta_k = \sum_i R_i \alpha_k x_{r_i}$ of dilates of α_k. This is equivalent to the first identity in Part (ii). The second identity comes from Lemma 3.6(ii). □

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