CHARACTERIZING PEANO AND SYMMETRIC DERIVATIVES AND THE GGR CONJECTURE'S SOLUTION

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Abstract. We provide three characterizations of the nth symmetric (Peano) derivative $f_{(n)}^{s}(x)$ in terms of symmetric generalized Riemann derivatives of a function f at x, and a characterization of the nth Peano derivative $f_{(n)}(x)$ in terms of generalized Riemann derivatives of f at x. The latter has been a conjecture by Ginchev, Guerragio and Rocca since 1998.

About twenty years ago, three mathematicians studied the question of finding a necessary and sufficient condition for a function having $n-1$ Peano derivatives at a point to also have an nth Peano derivative at that point. They conjectured that the simultaneous existence of the nth forward Riemann derivative, together with the existence of *all* of its first $n-1$ backward integer shifts, with all these nth Riemann derivatives having the same common value, would be such a condition. The first seven cases of $n=2,3,\ldots,8$ were proved in [GR].

(Some terminology: An order n, or nth, *generalized Riemann derivative without excess* of a function f at x is given by the limit

$$D_{\mathcal{A}}f(x):=\lim_{h\to 0}h^{-n}\sum_{i=0}^{n}A_i f\left(x+a_i h\right),$$

where the data vector $\mathcal{A}$ consists of the distinct $n+1$ numbers $\{a_0,\ldots,a_n\}$, called the *base points*, which uniquely determine the coefficients by means of the defining linear system of equations $\sum_{i=0}^{n}A_i\left(a_i\right)^j=\delta_{jn}n!$, for $j=0,1,\ldots,n$. The definitions of a Peano and a generalized Riemann derivative are given a couple of paragraphs down below. If the base points are given by $\{a_i\}=\{0,1,\ldots,n\}$, then $D_{\mathcal{A}}$ is called the nth *forward Riemann derivative* and written as D_n, and the $\{A_i\}$ are given by $\binom{n}{i}(-1)^{n-i}$, $i=0,1,\ldots,n$. When the $\{a_i\}$ are shifted back by 1 to become $\{a_i\}=\{-1,0,1,\ldots,n-1\}$, the A_i are unchanged and the resultant generalized derivative will be called the first backward integer shift of the nth forward Riemann derivative and denoted $D_{n,-1}$; when the $\{a_i\}$ are shifted back by 2 to become $\{a_i\}=\{-2,-1,0,1,\ldots,n-2\}$, the A_i are unchanged and the resultant generalized derivative will be called the second backward integer shift of the nth forward Riemann derivative and denoted $D_{n,-2}$, and so on.)

Suppose that a function has $n-1$ Peano derivatives at a point x. Then a necessary and sufficient condition for that function to have an nth Peano derivative

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at the point x is that the function also has a certain nth generalized Riemann derivative at the point x, namely the one with base points $\{0, 2^0, 2^1, 2^2, \ldots, 2^{n-1}\}$, introduced by Marcinkiewicz and Zygmund in [MZ] (1936). This equivalence was observed in a recent preprint [ACF]. In that preprint appears another necessary and sufficient condition for extending Peano differentiation from order $n-1$ to order n, namely the simultaneous existence of the nth forward Riemann derivative, together with the existence of *all* of its first $n-2$ forward integer shifts.

Section 3 below contains the proof for all $n \geq 3$ of a slightly stronger statement than the conjecture. Suppose that a function has $n-1$ Peano derivatives at a point x. We weaken the additional assumptions made by the conjecture in two ways. First, we drop the assumption that the nth forward Riemann derivative exists, while retaining the assumed existence of all of its first $n-1$ backward shifts. Second, we do not assume that these backward shifts all have the same common value. We then prove that the function has n Peano derivatives at the point x.

To carry out the conjecture's proof, we needed to first establish a result for symmetric Peano derivatives that is very much like the conjecture itself. In doing so, we develop a whole theory of symmetric Peano and symmetric generalized Riemann derivatives in Sections 1 and 2. The major results in these two sections respectively amount to two characterizations of the symmetric Peano derivative in terms of symmetric generalized Riemann derivatives. As a consequence of the results in Section 3, a third such characterization is given in Section 4.

The next part of the introduction outlines the definitions, examples and basic properties needed to understand the details. At the end of the introduction we give more insight into the main results in each section.

Definitions and basic properties.

Peano derivatives. A real function f has n Peano derivatives at x if there exist real numbers $f_{(0)}(x), f_{(1)}(x), \ldots, f_{(n)}(x)$ such that

$$(1) \qquad f(x+h) = f_{(0)}(x) + \frac{f_{(1)}(x)}{1!}h + \frac{f_{(2)}(x)}{2!}h^2 + \cdots + \frac{f_{(n)}(x)}{n!}h^n + o(h^n).$$

The number $f_{(n)}(x)$ is the nth *Peano derivative* of f at x. The existence of the nth Peano derivative of f at x assumes the existence of all lower order Peano derivatives of f at x. By Taylor expansion, if the nth ordinary derivative $f^{(n)}(x)$ exists, then so does the nth Peano derivative $f_{(n)}(x)$ and $f_{(n)}(x) = f^{(n)}(x)$. The converse of this is, in general, false. For example, the function $f(x) = 0$ for x rational, and $f(x) = x^{n+1}$ for x irrational, is n times Peano differentiable at 0 and $f_{(0)}(0) = f_{(1)}(0) = \cdots = f_{(n)}(0) = 0$, while $f^{(n)}(0)$ does not exist for $n \geq 2$, since $f'(x)$ does not exist in a punctured neighborhood of 0.

The Peano derivatives were introduced by Peano in [P] (1891) and then studied by de la Vallée Poussin in [VP] (1908). For more on Peano derivatives, see [As1, Olv] and the survey article [EW] by Evans and Weil.

Symmetric Peano derivatives. A real function f has n symmetric Peano derivatives at x if there exist real numbers $f^s_{(0)}(x), f^s_{(1)}(x), \ldots, f^s_{(n)}(x)$ such that

$$(2) \quad \frac{1}{2}\{f(x+h) + (-1)^n f(x-h)\} = f^s_{(0)}(x) + \frac{f^s_{(1)}(x)}{1!}h + \cdots + \frac{f^s_{(n)}(x)}{n!}h^n + o(h^n).$$

In this case, $f^s_{(n)}(x)$ is called the nth *symmetric Peano derivative* of f at x. Definition (2) implies that $f^s_{(0)}(x) = f(x)$ for n even, and $f^s_{(n-1)}(x) = f^s_{(n-3)}(x) = f^s_{(n-5)}(x) = \cdots = 0$ for all n, hence f is also symmetric Peano differentiable at x of orders $n-2, n-4, \cdots$. In particular, $f^s_{(0)}(x) = 0$ for n odd.

It is clear that every n times Peano differentiable function at x is also n times symmetric Peano differentiable at x. The converse of this is in general false. For example, the function $f(x) = x^n$ for $x \geq 0$, and $f(x) = -x^n$ for $x < 0$, is n times symmetric Peano differentiable at 0 and $f^s_{(n)}(0) = f_{(n-2)}(0) = \cdots = 0$, while $f_{(n)}(0)$ does not exist, since $h \to 0^+$ (resp. $h \to 0^-$) would force $f_{(n)}(0)$ to be $n!$ (resp. $-n!$). Every even (odd) function being symmetric Peano differentiable at 0 of any odd (even) order makes symmetric Peano differentiabilities of f at x of different parity orders incomparable. (Except when $f(0+h) = o(h^n)$.)

The symmetric Peano derivatives were invented by de la Vallée Poussin in [VP], in 1908, and by this they should have been called de la Vallée Poussin derivatives. In the literature they were called generalized symmetric derivatives in [Z], whose first edition appeared in 1935, and simply symmetric derivatives in [Ws] (1964) and all later references. Our choice for further name change here is to distinguish them from other symmetric derivatives that we will frequently use throughout the paper. The symmetric Peano derivatives have many applications in the theory of trigonometric series.[SZ, Z]

Generalized Riemann derivatives. For a given function f and point x, the difference

$$\Delta_{\mathcal{A}}(x,h;f) = \sum_{i=0}^{d} A_i f(x + a_i h), \qquad \text{for } d \geq n,$$

is an nth *generalized Riemann difference*, if its data $\mathcal{A} = \{A_0, \ldots, A_d; a_0, \ldots, a_d\}$ satisfies the nth Vandermonde conditions $\sum_i A_i (a_i)^j = \delta_{ij} \cdot n!$, for $j = 0, 1, \ldots, n$. In this case, the nth *generalized Riemann derivative*, or the $\mathcal{A}$-*derivative* of f at x, is defined by the limit

$$D_{\mathcal{A}} f(x) = \lim_{h \to 0} \Delta_{\mathcal{A}}(x,h;f)/h^n. \tag{3}$$

For simplicity, throughout the paper, we will write $\Delta_{\mathcal{A}}(h)$ to denote $\Delta_{\mathcal{A}}(x,h;f)$.

The most known examples of nth generalized Riemann derivatives are the earlier mentioned nth *forward Riemann derivative* $D_n f(x)$, corresponding to the nth Riemann difference

$$\Delta_n(h) = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f(x + ih),$$

where $\mathcal{A} = \{(-1)^{n-i}\binom{n}{i}; i \mid i = 0, \ldots, n\}$, and the nth *symmetric Riemann derivative* $D^s_n f(x)$, corresponding to the nth symmetric Riemann difference

$$\Delta^s_n(h) = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f(x + (i - n/2)h),$$

where $\mathcal{A} = \{(-1)^{n-i}\binom{n}{i}; i - \frac{n}{2} \mid i = 0, \ldots, n\}$.

More examples of generalized Riemann derivatives are obtained by taking shifts of known generalized Riemann derivatives. By linear algebra, for each real number r, the nth Vandermonde relations holding for $\mathcal{A} = \{A_i; a_i\}$ is equivalent to their holding for the forward and backward r-*shifts* $\mathcal{A}, \pm r := \{A_i; a_i \pm r\}$ of $\mathcal{A}$. In this way, $D_n^s f(x) = D_{n,-n/2} f(x)$, or the nth symmetric Riemann derivative is the $n/2$-backward shift of the nth forward Riemann derivative, and $D_n f(x) = D_{n,n/2}^s f(x)$, or the nth forward Riemann derivative is the $n/2$-forward shift of the nth symmetric Riemann derivative.

Another way of making new generalized Riemann derivatives from old is by scaling known generalized Riemann derivatives. An *r-scale* of an nth generalized Riemann difference $\Delta_{\mathcal{A}}(h)$ with $\mathcal{A} = \{A_i; a_i\}$ is the nth generalized Riemann difference $\Delta_{\mathcal{A}_r}(h)$ with $\mathcal{A}_r = \{r^{-n}A_i; ra_i\}$. The process of scaling is different from other processes used to create new generalized derivatives. To see this, look at the simplest case of $\lim_{h\to 0} \frac{f(x+rh)-f(x)}{rh} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ to see that scaling is never anything more than applying the change of variable $h \to rh$ before letting h tend to 0. So whenever a property is enjoyed by exactly the set $\{D_{\mathcal{A}_r} : r > 0\}$ we may say that $D_{\mathcal{A}}$ is the unique generalized derivative with that property. Nevertheless, scaling is often a useful tool because if $\Delta_{\mathcal{A}}$ is the difference associated with $D_{\mathcal{A}}$ and $r \neq s$ then $\Delta_{\mathcal{A}_r}(h)$ and $\Delta_{\mathcal{A}_s}(h)$ are distinct.

When $d = n$, the nth generalized Riemann difference $\Delta_{\mathcal{A}}(h)$ has no excess: given $a_0, \dots, a_n$, the above nth Vandermonde relations for $\mathcal{A}$ form a system of $n+1$ linear equations in $n+1$ unknowns $A_0, \dots, A_n$ with non-singular Vandermonde coefficient matrix, hence it has a unique solution. In particular, the nth forward Riemann difference $\Delta_n(h)$ is the unique nth generalized Riemann difference based at $a_0 = 0, a_1 = 1, \dots, a_n = n$, and the nth symmetric Riemann difference $\Delta_n^s(h)$ is the unique nth generalized Riemann difference based at $a_0 = -n/2, a_1 = -n/2 + 1, \dots, a_n = n/2$.

Riemann derivatives were introduced in 1892 by Riemann in [R]. Generalized Riemann derivatives were formalized in 1935 by Denjoy in [D]. These were shown to satisfy properties similar to those for the ordinary derivatives, such as monotonicity, convexity, or the mean value theorem.[AJ, FFR, HL, HL1, T, W] For more on Riemann derivatives, see [AC1, BK].

Symmetric generalized Riemann differences. A (not necessarily generalized Riemann) difference $\Delta_{\mathcal{A}}(h)$ is *even* if $\Delta_{\mathcal{A}}(-h) = \Delta_{\mathcal{A}}(h)$, and *odd* if $\Delta_{\mathcal{A}}(-h) = -\Delta_{\mathcal{A}}(h)$. An nth generalized Riemann difference $\Delta_{\mathcal{A}}(h)$ is *symmetric* if $\Delta_{\mathcal{A}}(-h) = (-1)^n \Delta_{\mathcal{A}}(h)$, meaning that $\Delta_{\mathcal{A}}(h)$ is even when n is even, and odd when n is odd. For example, the nth symmetric Riemann difference $\Delta_n^s(h)$ is symmetric for all n.

Each nth generalized Riemann difference $\Delta_{\mathcal{A}}(h)$ gives rise to an nth symmetric generalized Riemann difference, its *symmetrization*, defined as

$$\Delta_{\mathcal{A}}^s(h) = \{\Delta_{\mathcal{A}}(h) + (-1)^n \Delta_{\mathcal{A}}(-h)\}/2.$$

Denoting $\Delta_{\mathcal{A}^s}(h) = \Delta_{\mathcal{A}}^s(h)$, if f is $\mathcal{A}$-differentiable at x, then f is $\mathcal{A}^s$-differentiable at x and $D_{\mathcal{A}} f(x) = D_{\mathcal{A}^s} f(x)$.

More examples of symmetric generalized Riemann differences are obtained from shifts of either forward or symmetric Riemann differences. Since $\Delta_{n,-j}(-h) = (-1)^n \Delta_{n,-n+j}(h)$, for $j = 0, 1, \dots, n$, the differences

$$\Delta_{n,j}^s(h) = \{\Delta_{n,-j}(h) + \Delta_{n,-n+j}(h)\}/2,$$

for $j = 0, 1, \dots, n$, are symmetric nth generalized Riemann differences. Note that $\Delta_{n,j}^s(h)$ is not the same as the j-shift of $\Delta_n^s(h)$, as the notation might suggest.

The nth generalized Riemann difference without excess based at a symmetric relative to the origin point set $\{a_0, a_1, \dots, a_n\}$ is an nth symmetric generalized

Riemann difference. Denote $m = \lfloor (n+1)/2 \rfloor$, and relabel the base points as $\{(a_0 = 0), \pm a_1, \pm a_2, \ldots, \pm a_m\}$, where (a_0) means a_0 appears only when n is even, and $0 < a_1 < \cdots < a_m$. By eliminating redundancies, the data vector $\mathcal{A}$ is simplified to an increasingly ordered set $S = \{(a_0 = 0), a_1, a_2, \ldots, a_m\}$ of non-negative real numbers. In this way, $\mathcal{A}$-differentiation will be the same as *symmetric S-differentiation*, and $\Delta_{\mathcal{A}}(h)$ will also be denoted as $\Delta_S(h)$.

Implication and equivalence of generalized derivatives. We say that a generalized derivative of a function f at x implies or is equivalent to another generalized derivative of f at x, if the existence of the first generalized derivative of f at x implies or is equivalent to the existence of the other generalized derivative of f at x.

Let $\mathcal{A}$ be the data vector of an nth generalized Riemann derivative. Taylor expansion about x shows that every n times Peano differentiable function f at x is $\mathcal{A}$-differentiable at x and $D_{\mathcal{A}}f(x) = f_{(n)}(x)$. The converse of this is in general false, and the result of [ACCs, Theorem 1], which we will invoke again at the end of Section 1, classifies all $\mathcal{A}$ for which the generalized Riemann derivative $D_{\mathcal{A}}f(x)$ implies, hence is equivalent to, the Peano derivative $f_{(n)}(x)$, for all functions f and points x.

By Taylor expansion, the nth symmetric Peano derivative $f^s_{(n)}(x)$ implies every symmetric nth generalized Riemann derivative $D_{\mathcal{A}}f(x)$. Theorem 1.5 determines all cases where the reverse of this implication is true. Theorem 1.3 in particular shows that, likewise symmetric Peano differentiations, symmetric generalized Riemann differentiations of different parity orders are incomparable.

The earliest equivalence of generalized derivatives was proved in 1927 by Kintchine in [Ki], who showed that the first symmetric Peano derivative, hence the first symmetric Riemann derivative, is a.e. equivalent to the first ordinary derivative. This was extended by Marcinkiewicz and Zygmund in [MZ], proving that the nth symmetric Riemann derivative and the nth Peano derivative are a.e. equivalent, and then further extended by Ash in [As] (1967), who showed that each nth generalized Riemann derivative is a.e. equivalent to the nth Peano derivative. More equivalences between symmetric, Peano and Riemann derivatives and their quantum analogues are given in [AC, ACR, GGR1].

Until very recently, the equivalence of generalized derivatives was largely viewed as an almost everywhere equivalence. The above mentioned result of [ACCs] paved the way to a more explicit pointwise theory of equivalences between generalized derivatives. Article [ACCh] classified all pairs of generalized Riemann derivatives that either pointwisely imply or are pointwisely equivalent to each other, and we will describe that result in Section 1. This classification was extended to complex functions in [ACCh1], and an application of it to continuity is given in [AAC]. The present article is a part of the same pointwise theory of equivalences between generalized derivatives. With the exception of Lemma 1.1 and all of Section 3, where the generalized derivatives are Peano and generalized Riemann, the generalized derivatives involved here are the symmetric Peano and the symmetric generalized Riemann.

Results. As we said earlier, Section 3 proves the conjecture by Ginchev, Guerragio and Rocca, a characterization of (ordinary) Peano differentiation by generalized Riemann differentiations, and Sections 1, 2 and 4 provide three characterizations of symmetric Peano differentiation by symmetric generalized Riemann differentiations.

Section 1. The existence of every nth symmetric generalized Riemann derivative $D_{\mathcal{A}}f(x)$ easily follows from the existence of the corresponding nth symmetric Peano derivative $f_{(n)}^{s}(x)$. In short, $f_{(n)}^{s}(x) \Rightarrow D_{\mathcal{A}}f(x)$. We first find the set of pairs $(n, \mathcal{A})$ for which the implication $f_{(n)}^{s}(x) \Rightarrow D_{\mathcal{A}}f(x)$ is reversible. When this occurs, we have a characterization of $f_{(n)}^{s}(x)$ by a single symmetric generalized Riemann derivative $D_{\mathcal{A}}f(x)$. This happens in a trivial way when $n = 1$ or 2. In fact, $f_{(1)}^{s}(x)$ is the number b satisfying $\frac{1}{2}\{f(x+h) - f(x-h)\} = bh + o(h)$ while $D_1^s f(x)$ is the number $b = \lim_{h\to 0} \frac{f(x+h)-f(x-h)}{2h}$ so that $f_{(1)}^{s}(x)$ and $D_1^s f(x)$ have exactly the same definition. Similarly, $f_{(2)}^{s}(x)$ is the number c such that $\frac{1}{2}\{f(x+h) + f(x-h)\} = f(x) + \frac{1}{2}ch^2 + o(h^2)$ and $D_2^s f(x)$ is the number $c = \lim_{h\to 0} \frac{f(x+h)-2f(x)+f(x-h)}{h^2}$ so that $f_{(2)}^{s}(x)$ and $D_2^s f(x)$ also have exactly identical definitions. We have then found the pairs $(1, \{\pm 1/2; \pm 1\})$ and $(2, \{1, -2, 1; 1, 0, -1\})$.

Theorem 1.5 says that, with the exception of these two trivial examples, there are no other pairs $(n, \mathcal{A})$ producing a characterization of $f_{(n)}^{s}(x)$ by a single symmetric generalized Riemann derivative, neither for any $n \geq 3$, nor for any symmetric generalized Riemann derivative other than D_1^s when $n = 1$ and D_2^s when $n = 2$. In other words, for $n \geq 3$, every nth symmetric generalized Riemann derivative is strictly more general than the nth symmetric Peano derivative.

The result of Theorem 1.5 is that for only a very slender set of orders of differentiation (namely $n = 1, 2$) a characterization of the nth symmetric Peano differentiation by a single symmetric generalized Riemann differentiation is possible. As such, Theorem 1.5 provides the motivation for the next two characterizations of each higher order symmetric Peano differentiation by means of a small system of symmetric generalized Riemann differentiations.

Section 2. Since no non-trivial symmetric generalized Riemann differentiation is equivalent to the nth symmetric Peano differentiation, for all functions f and points x, and since the nth symmetric Peano differentiation is incomparable to the $n-1$st symmetric Peano differentiation, a natural question to ask that might have a positive answer is the following:

> Are there non-trivial symmetric generalized Riemann differentiations that are equivalent to nth symmetric Peano differentiation, for all $n-2$ times symmetric Peano differentiable functions f and points x?

This is what we call a characterization by symmetric generalized Riemann differentiations of the nth symmetric Peano differentiation modulo $n-2$nd symmetric Peano differentiation.

The second characterization of the nth symmetric Peano differentiation is a positive characterization by a single symmetric generalized Riemann differentiation modulo $n-2$nd symmetric Peano differentiation. We show in Theorem 2.2 that the nth symmetric generalized Riemann derivative without excess, $\widetilde{D}_n f(x) = D_S f(x)$, corresponding to the simplified data vector $S = \{(a_0 = 0), a_1 = 1, a_2 = 2, a_3 = 4, \ldots, a_m = 2^{m-1}\}$, for $m = \lfloor (n+1)/2 \rfloor$, is equivalent to the nth symmetric Peano derivative $f_{(n)}^{s}(x)$, for all $n-2$ times symmetric Peano differentiable functions f at x. This means that $f_{(n)}^{s}(x)$ is equivalent to the system consisting of both $f_{(n-2)}^{s}(x), \widetilde{D}_n f(x)$. Corollary 2.3 contains an equivalent result, namely, that the

nth symmetric Peano derivative $f^s_{(n)}(x)$ is equivalent to the system consisting of all of $\widetilde{D}_n f(x), \widetilde{D}_{n-2} f(x), \widetilde{D}_{n-4} f(x)$, and so on.

Section 3. This section proves a characterization of the nth Peano differentiation modulo $n-1$st Peano differentiation, by backward shifts of the nth forward Riemann differentiation. In Theorem 3.1 we show that the system $D^{\text{sh}-}_n f(x)$ consisting of all backward shifts $D_{n,-j} f(x)$, for $j = 1, \ldots, n-1$, of the nth forward Riemann derivative $D_n f(x)$ is equivalent to the nth Peano derivative $f_{(n)}(x)$, for all $n-1$ times Peano differentiable functions f at x. Equivalently, $f_{(n-1)}(x)$ and $D^{\text{sh}-}_n f(x)$ together are equivalent to $f_{(n)}(x)$. Corollary 3.2 provides an equivalent statement of Theorem 3.1: the nth Peano derivative $f_{(n)}(x)$ is equivalent to $f_{(1)}(x)$ and all of $D^{\text{sh}-}_k f(x)$, for $k = 2, \ldots, n$, that is, to a system consisting of $1 + 2 + \cdots + (n-1) = n(n-1)/2$ shifts of forward Riemann derivatives of orders up to n.

Corollary 3.2, and implicitly Theorem 3.1, has been a conjecture by Ginchev, Guerragio and Rocca since 1998. They proved it for $n \leq 4$ in [GGR] and, with the use of a computer, they proved the result for $n \leq 8$ in [GR], and left the general case as an open problem. Their method is different than ours.

Section 4. Motivated by Conjecture 4.1, asserting that the nth symmetric Riemann derivative $D^s_n f(x)$, or the most common example of a symmetric generalized Riemann derivative, does not characterize the nth symmetric Peano derivative $f^s_{(n)}(x)$ modulo $f^s_{(n-2)}(x)$ in the same way as $\widetilde{D}^s_n f(x)$ did in Section 2, and in the light of the results in Section 3 for the Peano derivative, the natural question to ask next is the following:

> Are there sets of symmetric generalized Rieman derivatives, which are closely related to the symmetric Riemann derivative $D^s_n f(x)$, that are equivalent to the nth symmetric Peano derivative $f^s_{(n)}(x)$ modulo $f^s_{(n-2)}(x)$?

The third characterization of the symmetric Peano derivative $f^s_{(n)}(x)$ is the result of Theorem 4.3, showing that $f^s_{(n)}(x)$ is equivalent modulo $f^s_{(n-2)}(x)$ to the set of all consecutive symmetrizations $D^s_{n,j} f(x)$, for $j = 1, 2, \ldots, \lfloor n/2 \rfloor$, of backward shifts $D_{n,-j} f(x)$ of the forward Riemann derivative $D_n f(x)$, which are also shifts of the symmetric Riemann derivative $D^s_n f(x)$.

1. First characterization of the symmetric Peano derivative

The first characterization of the symmetric Peano differentiation is the question of finding all single symmetric generalized Riemann differentiations $\mathcal{A}$ of order m such that, for all f and x, the symmetric generalized Riemann derivative $D_{\mathcal{A}} f(x)$ implies, hence is equivalent to, the nth symmetric Peano derivative $f^s_{(n)}(x)$. This question is answered in Theorem 1.5 of Section 1.2. The proof of Theorem 1.5 relies on the classification of symmetric generalized Riemann derivatives, given in Section 1.1, which is the question of characterizing all pairs $(\mathcal{A}, \mathcal{B})$ of symmetric generalized Riemann differentiations such that, for each function f and point x, the derivative $D_{\mathcal{A}} f(x)$ either implies or is equivalent to the derivative $D_{\mathcal{B}} f(x)$. This question is answered in Theorem 1.3, whose proof relies in part on a highly non-trivial theorem, the analogue result for generalized Riemann derivatives, proved in [ACCh] and conveniently restated here as Lemma 1.1.

1.1. **The equivalence of symmetric generalized Riemann derivatives.** Recall that the symmetrization $\Delta^s_{\mathcal{A}}(h) = \{\Delta_{\mathcal{A}}(h) + (-1)^n \Delta_{\mathcal{A}}(-h)\}/2$ of any nth generalized Riemann difference $\Delta_{\mathcal{A}}(h)$ is an nth symmetric generalized Riemann difference. The *anti-symmetrization* of $\Delta_{\mathcal{A}}(h)$ is the difference

$$\Delta^{s'}_{\mathcal{A}}(h) = \{\Delta_{\mathcal{A}}(h) + (-1)^{n+1} \Delta_{\mathcal{A}}(-h)\}/2.$$

When this is non-zero, it is a scalar multiple of a symmetric generalized Riemann difference whose order is both larger than n and with parity opposite to n; see [ACCh, Theorem 4]. Furthermore, since $\Delta_{\mathcal{A}}(h) = \Delta^s_{\mathcal{A}}(h) + \Delta^{s'}_{\mathcal{A}}(h)$ is the unique expression of $\Delta_{\mathcal{A}}(h)$ as a sum of an even difference and an odd diference, a generalized Riemann difference $\Delta_{\mathcal{A}}(h)$ is symmetric if and only if $\Delta_{\mathcal{A}}(h) = \Delta^s_{\mathcal{A}}(h)$, that is, if and only if $\Delta^{s'}_{\mathcal{A}}(h) = 0$.

Also recall that the r-scale ($r \neq 0$) of an nth generalized Riemann difference $\Delta_{\mathcal{A}}(h)$ is the difference $\Delta_{\mathcal{A}_r}(h) = r^{-n}\Delta_{\mathcal{A}}(rh)$, and that $\mathcal{A}_r$-differentiation is equivalent to $\mathcal{A}$-differentiation. In addition, a linear combination $\Delta_{\bar{\mathcal{A}}}(h) := \sum_k R_k \Delta_{\mathcal{A}_{r_k}}(h)$ of scales of $\Delta_{\mathcal{A}}(h)$ is an nth generalized Riemann difference if and only if it is normalized, or $\sum_k R_k = 1$. In this case $\mathcal{A}$-differentiation implies $\bar{\mathcal{A}}$-differentiation. These are the two obvious ways to provide generalized Riemann differentiations $\mathcal{B}$ that are respectively equivalent to or implied by a given generalized Riemann differentiation $\mathcal{A}$.

The following lemma characterizes, in terms of symmetrizations and anti-symmetrizations of differences, all pairs $(\Delta_{\mathcal{A}}, \Delta_{\mathcal{B}})$ of generalized Riemann differences for which $\mathcal{A}$-differentiation either implies or is equivalent to $\mathcal{B}$-differentiation, for all functions f and points x. This combined result of two theorems in [ACCh] is the classification of generalized Riemann derivatives, which can be rephrased as follows:

Lemma 1.1. [ACCh, Theorems 2 and 3] *Let $\mathcal{A}$ and $\mathcal{B}$ be two generalized Riemann derivatives of orders m and n. Then, for each function f and point x,*

(i) *$D_{\mathcal{A}}f(x)$ is equivalent to $D_{\mathcal{B}}f(x)$ if and only if $m = n$ and there exist non-zero constants A, p and q such that*

$$\Delta^s_{\mathcal{B}}(h) = \Delta^s_{\mathcal{A}_p}(h) \quad \textit{and} \quad \Delta^{s'}_{\mathcal{B}}(h) = A\Delta^{s'}_{\mathcal{A}_q}(h).$$

(ii) *$D_{\mathcal{A}}f(x)$ implies $D_{\mathcal{B}}f(x)$ if and only if $m = n$ and there exist constants $\{P_j; p_j \neq 0 \,|\, j = 1, \dots, k\}$ and $\{Q_j; q_j \neq 0 \,|\, j = 1, \dots, \ell\}$, with $\sum_{j=1}^k P_j = 1$, so that*

$$\Delta^s_{\mathcal{B}}(h) = \sum_{j=1}^k P_j \Delta^s_{\mathcal{A}_{p_j}}(h) \quad \textit{and} \quad \Delta^{s'}_{\mathcal{B}}(h) = \sum_{j=1}^{\ell} Q_j \Delta^{s'}_{\mathcal{A}_{q_j}}(h).$$

An important feature of Lemma 1.1 is its use in producing non-obvious examples of generalized Riemann differentiations $\mathcal{B}$ that are either implied by or equivalent to a given generalized Riemann differentiation $\mathcal{A}$, for all functions f at x.

Example 1.2. Consider the following first generalized Riemann differentiations:

1. $D_{\mathcal{A}}f(x) = \lim\limits_{h\to 0} \dfrac{\Delta_{\mathcal{A}}(h)}{h} = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$;
2. $D_{\mathcal{B}}f(x) = \lim\limits_{h\to 0} \dfrac{\Delta_{\mathcal{B}}(h)}{h} = \lim\limits_{h\to 0} \dfrac{4f(x+h) - 7f(x) + 3f(x-h)}{h}$;
3. $D_{\mathcal{C}}f(x) = \lim\limits_{h\to 0} \dfrac{\Delta_{\mathcal{C}}(h)}{h} = \lim\limits_{h\to 0} \dfrac{f(x+2h) + f(x+h) - 3f(x) + f(x-2h)}{h}$.

The symmetrizers and anti-symmetrizers of their defining differences are:

$$\Delta^s_{\mathcal{A}}(h) = \Delta^s_{\mathcal{B}}(h) = \Delta^s_{\mathcal{C}}(h) = \{f(x+h) - f(x-h)\}/2,$$
$$\Delta^{s'}_{\mathcal{A}}(h) = \{f(x+h) - 2f(x) + f(x-h)\}/2,$$
$$\Delta^{s'}_{\mathcal{B}}(h) = \frac{7}{2}\{f(x+h) - 2f(x) + f(x-h)\},$$
$$\Delta^{s'}_{\mathcal{C}}(h) = f(x+2h) + \frac{1}{2}f(x+h) - 3f(x) + \frac{1}{2}f(x-h) + f(x-2h).$$

Since $\Delta^s_{\mathcal{A}}(h) = \Delta^s_{\mathcal{B}}(h)$ and $\Delta^{s'}_{\mathcal{B}}(h) = 7\Delta^{s'}_{\mathcal{A}}(h)$, by Lemma 1.1(i), $\mathcal{A}$-differentiation is equivalent to $\mathcal{B}$-differentiation, for all functions f at x. Since $\Delta^s_{\mathcal{C}}(h) = \Delta^s_{\mathcal{A}}(h)$ and $\Delta^{s'}_{\mathcal{C}}(h) = 2\Delta^{s'}_{\mathcal{A}}(2h) + \Delta^{s'}_{\mathcal{A}}(h) = 4\Delta^{s'}_{\mathcal{A}_2}(h) + \Delta^{s'}_{\mathcal{A}}(h)$, by Lemma 1.1(ii), $\mathcal{A}$-differentiation implies $\mathcal{C}$-differentiation. And since $\Delta^{s'}_{\mathcal{C}}(h)$ is not a non-zero scalar multiple of a scale of $\Delta^{s'}_{\mathcal{A}}(h)$, $\mathcal{A}$-differentiation is not equivalent to $\mathcal{C}$-differentiation.

We shall see next that the analogue result of Lemma 1.1 for symmetric generalized Riemann derivatives does no longer have a surprise factor: all symmetric generalized Riemann derivatives that are either implied by or equivalent to a given symmetric generalized Riemann derivative are precisely the expected ones.

The following theorem characterizes all pairs $(\Delta_{\mathcal{A}}, \Delta_{\mathcal{B}})$ of symmetric generalized Riemann differences with the property that $\mathcal{A}$-differentiation either implies or is equivalent to $\mathcal{B}$-differentiation, for all functions f and points x. This is the classification of symmetric generalized Riemann derivatives.

Theorem 1.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be the data vectors for two symmetric generalized Riemann derivatives of orders m and n. Then, for all functions f and real numbers x,*

(i) *$D_{\mathcal{A}}f(x)$ is equivalent to $D_{\mathcal{B}}f(x)$ if and only if $m = n$ and there is a non-zero real number p, such that*

$$\Delta_{\mathcal{B}}(h) = \Delta_{\mathcal{A}_p}(h).$$

(ii) *$D_{\mathcal{A}}f(x)$ implies $D_{\mathcal{B}}f(x)$ if and only if $m = n$ and there exist constants $P_1, \dots, P_k$, with $\sum_{j=1}^k P_j = 1$, and non-zero constants $p_1, \dots, p_k$, such that*

$$\Delta_{\mathcal{B}}(h) = \sum_{j=1}^{k} P_j \Delta_{\mathcal{A}_{p_j}}(h).$$

Proof. This is an easy consequence of Lemma 1.1 and our earlier observation that a difference $\Delta_{\mathcal{A}}(h)$ is symmetric if and only if $\Delta_{\mathcal{A}}(h) = \Delta^s_{\mathcal{A}}(h)$. □

Part (i) of Theorem 1.3 says that the differences corresponding to equivalent symmetric generalized Riemann derivatives are scales of each other. Part (ii) says that a symmetric generalized Riemann differentiation implies another symmetric generalized Riemann differentiation precisely when the difference corresponding to the second differentiation is a normalized linear combination of scales of the difference corresponding to the first differentiation.

Example 1.4. Consider the second symmetric generalized Riemann differences:

$$\Delta_{\mathcal{A}}(h) = f(x+h) - 2f(x) + f(x-h),$$
$$\Delta_{\mathcal{B}}(h) = \{f(x+2h) - f(x+h) - f(x-h) + f(x-2h)\}/3.$$

Since $\Delta_{\mathcal{B}}(h) = \frac{1}{3}\Delta_{\mathcal{A}}(2h) - \frac{1}{3}\Delta_{\mathcal{A}}(h) = \frac{4}{3}\Delta_{\mathcal{A}_2}(h) - \frac{1}{3}\Delta_{\mathcal{A}}(h)$ is a linear combination of scales of $\Delta_{\mathcal{A}}(h)$, by Theorem 1.3(ii), $\mathcal{A}$-differentiation implies $\mathcal{B}$-differentiation.

And since $\Delta_{\mathcal{B}}(h)$ is not a non-zero scalar multiple of a scale of $\Delta_{\mathcal{A}}(h)$, by Part (i) of the same result, $\mathcal{A}$-differentiation is not equivalent to $\mathcal{B}$-differentiation.

1.2. **First characterization of the symmetric Peano differentiation.** In this section, for all positive integers m and n, we determine all symmetric generalized Riemann differences $\Delta_{\mathcal{A}}(h)$ of order m for which the derivative $D_{\mathcal{A}}f(x)$ implies, hence is equivalent to, the symmetric Peano derivative $f^s_{(n)}(x)$, for all f and x. This is achieved in the following theorem:

Theorem 1.5. *When $n = 1$ or 2, the nth symmetric Riemann differentiation and the nth symmetric Peano differentiation have the same definition. With the exception of these two trivial cases, for $n \geq 1$, any nth symmetric generalized Riemann differentiation is more general than the nth symmetric Peano differentiation.*

Proof. We will prove the following more specific result: For all f and x, an mth symmetric generalized Riemann derivative $D_{\mathcal{A}}f(x)$ is equivalent to the nth symmetric Peano derivative $f^s_{(n)}(x)$ if and only if $m = n \leq 2$ and $\Delta_{\mathcal{A}}f(x)$ is a scale of the nth (non-generalized) symmetric Riemann difference $\Delta^s_n f(x)$.

The result for $n = 1, 2$ comes from Theorem 1.3, since the first symmetric Peano derivative $f^s_{(1)}(x)$ is identical to the first symmetric Riemann derivative $D^s_1 f(x)$, and the second symmetric Peano derivative $f^s_{(2)}(x)$ is identical to the second symmetric Riemann derivative $D^s_2 f(x)$, for all f and x. For this, see the Results/Section 1/Paragraph 1 part of the introduction.

Suppose now that $\mathcal{A} = \{A_i, a_i \mid i = 0, 1, \ldots, d\}$ is the data vector of a symmetric generalized Riemann difference of order $n \geq 3$, and let K be the field generated over $\mathbb{Q}$ by all the a_i's. Denote $j = n \mod 2$ and define the real function f_j by setting $f_j(x) = 0$, if $x \in K$, and $f_j(x) = x^j$, if $x \in \mathbb{R} \setminus K$. Then f_j has no symmetric Peano derivative of order n at 0. On the other hand,

$$\Delta_{\mathcal{A}}(0, h; f_j) = \sum_{i=0}^{d} A_i f_j(a_i h) = \begin{cases} \sum_{i=0}^{d} A_i a_i^j h^j & \text{if } h \in \mathbb{R} \setminus K, \\ 0 & \text{if } h \in K. \end{cases}$$

By the jth Vandermonde condition, $\Delta_{\mathcal{A}}(0, h; f) = 0$, for all h, so f is $\mathcal{A}$-differentiable at 0 and $D_{\mathcal{A}}f(0) = 0$. Thus $D_{\mathcal{A}}f(0)$ does not imply $f^s_{(n)}(0)$, for $n \geq 3$. $\square$

Our motivation for the classification result in Theorem 1.5 comes from Theorem 1 in [ACCs], asserting that the only generalized Riemann derivatives of every orders that are equivalent to the Peano derivative $f_{(n)}(x)$ are the first order $\mathcal{A}$-derivatives which are dilates ($h \to sh$, for some $s \neq 0$) of limits of the form

$$\lim_{h \to 0} \frac{Af(x + rh) + Af(x - rh) + f(x + h) - f(x - h) - 2Af(x)}{2h}, \text{where } Ar \neq 0.$$

2. Second characterization of the symmetric Peano derivative

Theorem 1.5 showed that, for each n, there are no *single* non-trivial symmetric generalized Riemann derivatives that are equivalent to the symmetric Peano derivative $f^s_{(n)}(x)$. The second characterization of the symmetric Peano derivative, for each n, will provide a *set* of non-trivial symmetric generalized Riemann derivatives that is equivalent to the symmetric Peano derivative $f^s_{(n)}(x)$.

Consider the sequence of differences $\widetilde{\Delta}^s_n(h) = \widetilde{\Delta}^s_n(x, h; f)$ of a function f at x, defined recursively as follows:

$$\begin{aligned}
&\widetilde{\Delta}^s_1(h) = f(x+h) - f(x-h),\\
&\widetilde{\Delta}^s_2(h) = f(x+h) - 2f(x) + f(x-h),\\
&\widetilde{\Delta}^s_n(h) = \widetilde{\Delta}^s_{n-2}(2h) - 2^{n-2}\widetilde{\Delta}^s_{n-2}(h), \text{ for } n > 2.
\end{aligned} \tag{4}$$

The recursive relation implies that

$$\widetilde{\Delta}^s_n(h) = 2^{n-2}\{\widetilde{\Delta}^s_{n-2}(2h)/(2h)^{n-2} - \widetilde{\Delta}^s_{n-2}(h)/h^{n-2}\}h^{n-2}, \tag{5}$$

and that the difference $\widetilde{\Delta}^s_n(h)$ is even when n is even, and odd when n is odd. We can write this difference explicitly as

$$\widetilde{\Delta}^s_n(h) = A_0 f(x) + \sum_{i=1}^{m} A_i\{f(x+2^{i-1}h) + (-1)^n f(x-2^{i-1}h)\}, \tag{6}$$

where the coefficients $A_0, A_1, \ldots, A_m$ satisfy $A_0 = 0$ for odd n and $A_m = 1$ for all n. This is a symmetric Peano difference based at $(x), x\pm h, x\pm 2h, x\pm 4h, \ldots, x\pm 2^{m-1}h$, that is, at $S = \{(0), 1, 2, 4, \ldots, 2^{m-1}\}$.

Part (ii) of the following lemma shows that the difference $\widetilde{\Delta}^s_n(h)$ satisfies all but the last of the nth Vandermonde conditions,

$$A_0 + \{1 + (-1)^n\} \cdot \sum_{i=1}^{m} A_i = 0 \text{ and } \{1 + (-1)^{n-j}\} \cdot \sum_{i=1}^{m} A_i 2^{(i-1)j} = 0, \tag{7}$$

for $j = 1, \ldots, n-1$, hence is a scalar multiple of an nth generalized Riemann derivative. Note that the equations (7) for j with $n - j$ odd are trivially satisfied.

Lemma 2.1. *For a function f at x, if the symmetric Peano derivative $f^s_{(n)}(x)$ exists, then:*

(i) $\widetilde{\Delta}^s_n(h)$ *is a scalar multiple of an nth generalized Riemann difference;*

(ii) *the limit* $\lim_{h\to 0} \widetilde{\Delta}^s_n(h)/h^n$ *exists.*

Proof. Induct on n. Cases $n = 1, 2$ are clear. For $n > 2$, suppose the result is true for $n - 2$ and that $f^s_{(n)}(x)$ exists. Then, by (5) and the inductive hypothesis,

$$\widetilde{\Delta}^s_n(h) = \widetilde{\Delta}^s_{n-2}(2h) - 2^{n-2}\widetilde{\Delta}^s_{n-2}(h) = o(h^{n-2}). \tag{8}$$

And by (4) and the inductive hypothesis, $\widetilde{\Delta}^s_n(h)$ is a scalar multiple of a generalized Riemann difference of order $n-2$. Recall that the existence of the nth symmetric Peano derivative (2) always implies that $f^s_{(j)}(x) = 0$, for $j = n-1, n-3, \cdots$. Moreover, substitution of (2) in (6) yields

$$\begin{aligned}
\widetilde{\Delta}^s_n(h) = &\left(A_0 + 2\cdot\{1 + (-1)^n\} \cdot \sum_{i=0}^{m} A_i\right) f(x)\\
&+ \sum_{j=1}^{n} \left(2 \cdot \{1 + (-1)^{n-j}\} \cdot \sum_{i=1}^{m} 2^{(i-1)j} A_i\right) \frac{f^s_{(j)}(x)}{j!} h^j + o(h^n).
\end{aligned} \tag{9}$$

By (8) and (9), all but the last equation of the Vandermonde system (7) are clearly satisfied. The remaining equation, the one for $j = n-1$, is trivially satisfied. This proves (7), hence (i). Moreover, equation (9) is reduced to $\widetilde{\Delta}^s_n(h) = Ah^n + o(h^n)$, for some real number A, completing the remaining Part (ii) of the inductive step. □

Lemma 2.1 implies that a unique scalar multiple $\lambda_n\widetilde{\Delta}_n^s(h)$ of the difference $\widetilde{\Delta}_n^s(h)$ is an nth generalized Riemann difference. In particular, the derivative defined as

$$\widetilde{D}_n^s f(x) := \lim_{h\to 0} \lambda_n \widetilde{\Delta}_n^s(h)/h^n$$

is an nth generalized Riemann derivative and $\widetilde{D}_n f(x) = f_{(n)}^s(x)$. Moreover, since the number of its base points is $n+1$, by the Vandermonde relations, the difference $\lambda_n\widetilde{\Delta}_n^s(h)$ is the unique nth generalized Riemann derivative based at these points.

The following theorem asserts that the special symmetric Riemann derivative $\widetilde{D}_n^s f(x)$ is equivalent to the symmetric Peano derivative $f_{(n)}^s(x)$, for all $n-2$ times symmetric Peano differentiable functions f at x. This will also be referred to simply as $\widetilde{D}_n^s f(x)$ is equivalent to $f_{(n)}^s(x)$ modulo $f_{(n-2)}^s(x)$

From now on, unless otherwise specified, all results assume $n \geq 3$.

Theorem 2.2. *For each function f and real number x,*

$$\text{both derivatives } f_{(n-2)}^s(x) \text{ and } \widetilde{D}_n^s f(x) \text{ exist} \iff f_{(n)}^s(x) \text{ exists.}$$

Proof. Since the definition of the nth symmetric Peano derivative $f_{(n)}^s(x)$ both assumes the existence of any symmetric Peano derivative of f at x of the same parity lower order, and implies any nth symmetric generalized Riemann derivative $D_{\mathcal{A}}f(x)$, one implication is clear. For the converse, an eventual translation of $f(x)$ by x reduces the problem to the case $x=0$, and an eventual subtraction from $f(x)$ of a degree n polynomial in x reduces it further to the case where $f(0) = f_{(1)}^s(0) = \cdots = f_{(n-1)}^s(0) = 0$ and $\widetilde{D}_n^s f(0) = 0$. The last condition means that $\widetilde{\Delta}_n^s(h) = o(h^n)$, or, for each $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that $|h| < \delta \Rightarrow |\widetilde{\Delta}_n^s(h)| < \varepsilon|h|^n$. Then

$$\left|\widetilde{\Delta}_{n-2}^s(2h) - 2^{n-2}\widetilde{\Delta}_{n-2}^s(h)\right| \leq \varepsilon|h|^n, \quad \left|\widetilde{\Delta}_{n-2}^s(h) - 2^{n-2}\widetilde{\Delta}_{n-2}^s\left(\frac{h}{2}\right)\right| \leq \varepsilon\left|\frac{h}{2}\right|^n, \ldots$$

$$\ldots, \left|\widetilde{\Delta}_{n-2}^s\left(\frac{h}{2^{k-1}}\right) - 2^{n-2}\widetilde{\Delta}_{n-2}^s\left(\frac{h}{2^k}\right)\right| \leq \varepsilon\left|\frac{h}{2^k}\right|^n.$$

Multiply these equations by $1, 2^{n-2}, 2^{2(n-2)}, \ldots, 2^{k(n-2)}$, respectively, and add. Further use of the triangle inequality on the left side yields

$$\left|\widetilde{\Delta}_{n-2}^s(2h) - 2^{(k+1)(n-2)}\widetilde{\Delta}_{n-2}^s\left(\frac{h}{2^k}\right)\right| \leq 2\varepsilon|h^n|.$$

Since $\widetilde{D}_{n-2}^s f(0) = f_{(n-2)}^s(0) = 0$, without loss of generality, by choosing k sufficiently large, the second term on the left side above can be made $\leq \varepsilon|h^n|$ and, by the triangle inequality, this leads to

$$\left|\widetilde{\Delta}_{n-2}^s(2h)\right| \leq 3\varepsilon|h^n|, \text{ or simply } \widetilde{\Delta}_{n-2}^s(h) = o(h^n).$$

Similarly, each term of the sequence of $\widetilde{\Delta}_{n-4}^s(h), \widetilde{\Delta}_{n-6}^s(h)$, and so on, is $o(h^n)$. Depending on the parity of n, the last of these is either $\widetilde{\Delta}_1^s(h) = f(0+h) - f(0-h) = o(h^n)$ or $\widetilde{\Delta}_2^s(h) = f(0+h) - 2f(0) + f(0-h) = o(h^n)$, leading to $\frac{1}{2}\{f(0+h) + (-1)^n f(0-h)\} = o(h^n)$, regardless of the parity of n. Then $f_{(n)}^s(0)$ exists, as needed, and is equal to zero. □

Theorem 1.5 showed that no single generalized symmetric Riemann derivative is equivalent to the nth symmetric Peano derivative $f^s_{(n)}(x)$ when $n > 2$. The following corollary *does* characterize the symmetric Peano derivative $f^s_{(n)}(x)$ when $n > 2$. For each such n, it provides a set of $\lfloor \frac{n+1}{2} \rfloor$ symmetric generalized Riemann derivatives which is equivalent to $f^s_{(n)}(x)$.

Corollary 2.3. *For each real function f and point x,*

$$f^s_{(n)}(x) \text{ exists} \iff \text{all derivatives } \widetilde{D}^s_k f(x) \text{ exist, for } k = n, n-2, n-4, \cdots.$$

Proof. Induct on n with step 2. The equivalence is clear when $n = 1, 2$. The inductive step is the result of Theorem 2.2. □

The above proof shows that Corollary 2.3 is actually equivalent to Theorem 2.2. An easy consequence of Corollary 2.3 is the following result which highlights the case of functions f for which both symmetric derivatives $f^s_{(n)}(x)$ and $f^s_{(n-1)}(x)$ exist. This case will play an important role in the next section in the proof of the GGR conjecture (Theorem 3.1).

Corollary 2.4. *For all functions f and real numbers x,*

$$\text{both } f^s_{(n)}(x) \text{ and } f^s_{(n-1)}(x) \text{ exist} \iff \text{all } \widetilde{D}^s_k f(x) \text{ exist, for } k = 1, 2, \ldots, n.$$

Proof. This follows from the result of Corollary 2.3 for both n and $n-1$. □

3. Proof of Ginchev-Guerragio-Rocca conjecture

This section proves the conjecture by Ginchev, Guerragio and Rocca on characterizing the Peano derivative by backward shifts of the forward Riemann derivative.

Theorem 2.2 is the symmetric analogue of a result given in [MZ] (1936) that we state later on as Lemma 3.8 and which asserts that a special nth forward generalized Riemann derivative $\widetilde{D}_n f(x)$ is equivalent to $f_{(n)}(x)$ modulo $f_{(n-1)}(x)$. The analogous result for the symmetric or forward Riemann derivatives $D^s_n f(x)$ or $D_n f(x)$ in place of $\widetilde{D}^s_n f(x)$ are not true; see [ACF] and Conjecture 4.2. This points at the fact that the cases when a single generalized Riemann derivative characterizes $f_{(n)}(x)$ modulo $f_{(n-1)}(x)$ are very scarce, and so the derivative $\widetilde{D}_n f(x)$ is really special. (The same can be said in the symmetric case about $\widetilde{D}^s_n f(x)$ via reference to Theorem 2.2 and Conjecture 4.1.)

Our next focus will then be on characterizing $f_{(n)}(x)$ modulo $f_{(n-1)}(x)$ by sets of nth generalized Riemann differentiations. As $D^s_n f(x)$ and $D_n f(x)$ are shifts of each other, and inspired by how the second characterization of the symmetric Peano derivative in Section 2 was built out of the failure of the first characterization in Section 1, the next theorem characterizes the nth Peano derivative $f_{(n)}(x)$, modulo the $n-1$-st Peano derivative $f_{(n-1)}(x)$, in terms of sets of shifts of either $D_n f(x)$ or $D^s_n f(x)$.

The theorem has been a conjecture by Ginchev, Guerragio and Rocca since 1998, saying that the nth Peano derivative $f_{(n)}(x)$ is equivalent to the system of all $n-1$ consecutive backward shifts of the nth forward Riemann derivative $D_n f(x)$, for all $n-1$ times Peano differentiable functions f at x. The theorem is easiest stated in terms of the set $D^{\text{sh}-}_n f(x) = \{D_{n,-j} f(x) \mid j = 1, \ldots, n-1\}$ of the first $n-1$ backward shifts of the Riemann derivative $D_n f(x)$ of f at x. We say that $D^{\text{sh}-}_n f(x)$ exists if all $D_{n,-j} f(x)$, for $j = 1, \ldots, n-1$, exist. The result goes as follows:

Theorem 3.1. *For each function f, real number x, and integer n at least 2,*

$$\text{both } f_{(n-1)}(x) \text{ and } D_n^{\text{sh}-}f(x) \text{ exist} \Longleftrightarrow f_{(n)}(x) \text{ exists.}$$

The result is slightly stronger than the original Ginchev–Guerragio–Rocca conjecture, where the $j = 0$ shift was also included in the left side of the equivalence. In addition, we do not need to assume that all values of the elements of $D_n^{\text{sh}-}f(x)$ are equal. The reason for this is because their mere existence guaranteed that they will be equal; see pages 3-5 of the thesis of Patrick J. O'Connor.[O]

The example of the function $f(x) = x^2 \cdot \text{sgn}(x)$ for which both symmetric derivatives $f_{(1)}^s(0)$ and $f_{(2)}^s(0)$ exist and are zero due to respectively $f(h) = o(h)$ as $h \to 0$ and f being an odd function, while the Peano derivative $f_{(2)}(0)$ does not exist due to its defining limit becoming 2 when $h \to 0^+$ and -2 when $h \to 0^-$, shows that the Peano derivative $f_{(n)}(x)$ is in general not equivalent to the compound of symmetric derivatives $f_{(n)}^s(x)$ and $f_{(n-1)}^s(x)$. However, for the purpose of the proof of Theorem 3.1, we chose to first prove the reverse implication in general and the direct implication for functions f for which the nth Peano is equivalent to both nth and $n-1$-st symmetric derivatives; we refer to this as the restricted proof of the theorem. In this way, we are able to develop in an easier setting most of the techniques needed in the general proof and also add more results to the theory of symmetric derivatives, which is consistent with the main goal of the article. The general proof is given at the end of the section.

Proof. By definition, the nth Peano derivative $f_{(n)}(x)$ implies the $n-1$-st Peano derivative $f_{(n-1)}(x)$ and, by Taylor expansion, the same $f_{(n)}(x)$ implies every nth generalized Riemann derivative $D_{\mathcal{A}}f(x)$, so the reverse implication is clear. For the direct implication, suppose that the Peano derivative $f_{(n)}(x)$ is equivalent to the conjunction of symmetric Peano derivatives $f_{(n)}^s(x)$ and $f_{(n-1)}^s(x)$. Then, by Theorem 2.2 and since $f_{(n-1)}(x)$ implies $f_{(n-2)}^s(x)$ via $f_{(n-2)}(x)$, it suffices to show that the system $D_n^{\text{sh}-}f(x)$ implies $\widetilde{D}_n^s f(x)$, for each f at x. This is the result of Lemma 3.3. □

The next consequence of Theorem 3.1 is actually equivalent to the theorem. It shows that the nth Peano derivative $f_{(n)}(x)$ of a function f at x can be viewed as a system of backward shifts of the first n Riemann derivatives of f at x.

Corollary 3.2. *For each function f, real number x, and integer n at least 2,*

$$f_{(1)}(x) \text{ exists and all } D_k^{\text{sh}-}f(x), \text{ for } k = 2, \ldots, n, \text{ exist} \Longleftrightarrow f_{(n)}(x) \text{ exists.}$$

Proof. Induct on n. Both the initial case $n = 2$ and the inductive step follow easily from Theorem 3.1. □

Recall that the proof of Theorem 3.1 was reduced to the following lemma:

Lemma 3.3. *For each function f, real number x, and integer n at least 2,*

$$D_n^{\text{sh}-}f(x) \text{ exists} \Longrightarrow \widetilde{D}_n^s f(x) \text{ exists.}$$

The rest of the section is dedicated to proving Lemma 3.3. As an example, we first prove the result for $n = 5$.

Case $n = 5$. Eventually by subtracting a degree 5 polynomial from $f(x)$ and then shifting f by x, without loss of generality, we may assume that $x = 0$ and $D_5^{\text{sh}-} f(0)$ exists and all its components $D_{5,-j} f(0)$, for $j = 1, \dots, 4$, are zero. The existence of the degree 5 polynomial depends on all components $D_{5,-j} f(0)$ having the same value. This always happens due to the result of [O] that we discussed above. The hypothesis translates into all differences

$$\begin{aligned}\Delta_{5,-1}(h) &= f(4h) - 5f(3h) + 10f(2h) - 10f(h) + 5f(0) - f(-h),\\ \Delta_{5,-2}(h) &= f(3h) - 5f(2h) + 10f(h) - 10f(0) + 5f(-h) - f(-2h),\\ \Delta_{5,-3}(h) &= f(2h) - 5f(h) + 10f(0) - 10f(-h) + 5f(-2h) - f(-3h),\\ \Delta_{5,-4}(h) &= f(h) - 5f(0) + 10f(-h) - 10f(-2h) + 5f(-3h) - f(-4h)\end{aligned}$$

being $o(h^5)$. Then the same is true about the 5th symmetric generalized Riemann differences $\Delta^s_{5,j}(h) = \{\Delta_{5,-j}(h) + \Delta_{5,j-5}(h)\}/2$, for $j = 1, 2$, written explicitly as

$$\begin{aligned}2\Delta^s_{5,1}(h) =& f(4h) - 5f(3h) + 10f(2h) - 9f(h) + 9f(-h) - 10f(-2h) + 5f(-3h) - f(-4h)\\ 2\Delta^s_{5,2}(h) =& f(3h) - 4f(2h) + 5f(h) - 5f(-h) + 4f(-2h) - f(-3h),\end{aligned}$$

as well as the linear combination $\frac{1}{6}\Delta^s_{5,1}(h) + \frac{5}{6}\Delta^s_{5,2}(h)$. This has the coefficients add up to 1 and eliminates both the term in $f(3h)$ and the term in $f(-3h)$, so is the symmetric 5th generalized Riemann difference $\lambda_5 \widetilde{\Delta}^s_5(0, h; f)$, based at $\pm h, \pm 2h, \pm 4h$, where $\lambda_5 = 1/12$ and

$$\widetilde{\Delta}^s_5(h) = f(4h) - 10f(2h) + 16f(h) - 16f(-h) + 10f(-2h) - f(-4h) = o(h^5).$$

We conclude that $\widetilde{D}^s_5 f(0) = \lim_{h\to 0} \lambda_5 \widetilde{\Delta}^s_5(h)/h^5$ exists and is equal to 0.

Polynomials and generalized Riemann differences. In order to proceed with the general proof of Lemma 3.3, we need to simplify notation at this point. The mapping

$$\Delta_{\mathcal{A}}(h) = \sum_i A_i f(x + a_i h) \mapsto P_{\mathcal{A}}(y) = \sum_i A_i y^{a_i}$$

induces a linear isomorphism between the space of all differences $\Delta_{\mathcal{A}}(h)$, based at half integers a_i, and the space $\mathbb{R}[y^{1/2}, y^{-1/2}]$ of Laurent polynomials with real coefficients in variable $y^{1/2}$. This correspondence has numerous interesting properties, as follows:

- The nth Riemann difference $\Delta_n(h)$ corresponds to the polynomial
$$P_n(y) = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} y^i = (y-1)^n.$$
- The nth symmetric Riemann difference $\Delta^s_n(h)$ corresponds to the polynomial
$$P^s_n(y) = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} y^{i-n/2} = y^{-n/2}(y-1)^n = (y^{1/2} - y^{-1/2})^n.$$
- A difference $\Delta_{\mathcal{A}}(h)$ is an nth generalized Riemann difference if and only if n is the highest power of $y - 1$ dividing $P_{\mathcal{A}}(y)$. See Lemma 2 on page 134 of the Collected Works of Marcinkiewicz and Zygmund.[MZ]
- If r is a half integer, then the Laurent polynomial corresponding to the r-shift $\Delta_{\mathcal{A},r}(h)$ of a difference $\Delta_{\mathcal{A}}(h)$ is $P_{\mathcal{A},r}(y) = y^r P_{\mathcal{A}}(y)$.

- $\Delta_{\mathcal{A}}(h)$ is an even difference if $P_{\mathcal{A}}(y^{-1}) = P_{\mathcal{A}}(y)$, and an odd difference if $P_{\mathcal{A}}(y^{-1}) = -P_{\mathcal{A}}(y)$.
- An nth generalized Riemann difference $\Delta_{\mathcal{A}}(h)$ is symmetric if and only if $P_{\mathcal{A}}(y^{-1}) = (-1)^n P_{\mathcal{A}}(y)$.
- The symmetrization $\Delta_{\mathcal{A}^s}(h) = \Delta^s_{\mathcal{A}}(h)$ of $\Delta_{\mathcal{A}}(h)$ corresponds to the polynomial $P_{\mathcal{A}^s}(y) = P^s_{\mathcal{A}}(y) = \{P_{\mathcal{A}}(y) + (-1)^n P_{\mathcal{A}}(y^{-1})\}/2$.
- The r-dilate $\Delta_{\mathcal{A}}(rh)$ of a difference $\Delta_{\mathcal{A}}(h)$, for r integer, corresponds to the polynomial $P_{\mathcal{A}}(y^r)$.
- The r-scale $\Delta_{\mathcal{A}_r}(h) = r^{-n}\Delta_{\mathcal{A}}(rh)$ of an nth generalized Riemann difference $\Delta_{\mathcal{A}}(h)$, for r integer, corresponds to the polynomial $P_{\mathcal{A}_r}(y) = r^{-n}P_{\mathcal{A}}(y^r)$.

Based on these properties, we can prove the following lemma:

Lemma 3.4. *Suppose $m = \lfloor (n+1)/2 \rfloor$ and $S = \{(0), a_1, a_2, \dots, a_m\}$ is any set of integers with $0 < a_1 < a_2 < \cdots < a_m < n$. Then, for each function f and point x, the symmetric generalized Riemann derivative D_S satisfies the following property:*

$$D_n^{\text{sh}-} f(x) \text{ exists} \Longrightarrow D_S f(x) \text{ exists.}$$

Proof. As in the Case $n = 5$, without loss of generality, we may assume that f is $D_n^{\text{sh}-}$-differentiable at 0 and $D_n^{\text{sh}-} f(0) = 0$, or $\Delta_{n,-j}(h) = o(h^n)$, for $j = 1, \dots, n-1$. Then $\Delta^s_{n,j}(h) = \{\Delta_{n,-j}(h) + \Delta_{n,-n+j}(h)\}/2 = o(h^n)$, for $j = 1, \dots, \lfloor n/2 \rfloor$. By the above properties, the polynomial corresponding to $\Delta_{n,-j}(h)$ is $P_{n,-j}(y) = y^{-j}(y-1)^n$, and the polynomial corresponding to $\Delta^s_{n,j}(h)$ is

$$P^s_{n,j}(y) = \frac{1}{2}(y^{-j} + y^{-n+j})(y-1)^n = \frac{1}{2}(y^{\frac{n}{2}-j} + y^{-\frac{n}{2}+j})(y^{\frac{1}{2}} - y^{-\frac{1}{2}})^n.$$

On the other hand, the polynomial $P_S(y)$ corresponding to the unique nth symmetric generalized Riemann difference $\Delta_S(h)$ is a Laurent polynomial of positive degree $t = a_m$, with $t \le n-1$, the degree of $P^s_{n,1}(y)$. Two extra properties, $P_S(y^{-1}) = (-1)^n P_S(y)$ and $(y-1)^n = y^{n/2}(y^{1/2} - y^{-1/2})^n$ divides $P_S(y)$, make

$$P_S(y) = \left(\alpha_t(y^{t-\frac{n}{2}} + y^{-t+\frac{n}{2}}) + \alpha_{t-1}(y^{t-\frac{n}{2}-1} + y^{-t+\frac{n}{2}+1}) + \cdots\right)(y^{\frac{1}{2}} - y^{-\frac{1}{2}})^n.$$

Since this is a linear combination of the $P^s_{n,j}(y)$, the difference $\Delta_S(h)$ is a linear combination of the $\Delta^s_{n,j}(h) = o(h^n)$, hence $\Delta_S(h) = o(h^n)$, i.e., $D_S f(0)$ exists. □

The following corollary provides a reduction of the proof of Lemma 3.3.

Corollary 3.5. *Suppose $m = \lfloor (n+1)/2 \rfloor$, and for $k = 1, \dots, \lfloor m/2 \rfloor$, denote*

$$S_k = \{(0), 1, 2, \dots, 2k, 2(k+1), \dots, 2(m-k)\}.$$

Then, for each function f and point x,

$$D_n^{\text{sh}-} f(x) \text{ exists} \Longrightarrow D_{S_k} f(x) \text{ exists.}$$

Proof. This is an easy consequence of Lemma 3.4, since for each k, $S_k \setminus \{0\}$ consists of m positive integers, the largest of whom is $a_m = 2(m-k) < n$. □

By Corollary 3.5, the proof of Lemma 3.3 is reduced to the following lemma:

Lemma 3.6. *Suppose $m = \lfloor (n+1)/2 \rfloor$, and for $k = 1, \dots, \lfloor m/2 \rfloor$, denote*

$$S_k = \{(0), 1, 2, \dots, 2k, 2(k+1), \dots, 2(m-k)\}.$$

Then, for each function f and point x,

$$D_{S_k} f(x) \text{ exists, for each } k = 1, \dots, \lfloor m/2 \rfloor \Longrightarrow \widetilde{D}_n f(x) \text{ exists.}$$

The proof of Lemma 3.6 is based on the following result of recursive set theory, whose proof is omitted, due to the proof of an equivalent version of it, obtained by adding 0 to each set in $\mathcal{S}$, being a part of the proof of [ACF, Lemma 5].

Lemma 3.7. *Suppose a collection $\mathcal{S}$ of sets, each consisting of m positive integers, is defined by the following properties:*

(i) $\{1, 2, \ldots, 2k, 2(k+1), 2(k+2), \ldots, 2(m-k)\} \in \mathcal{S}$, *for* $k = 1, \ldots, \lfloor m/2 \rfloor$;

(ii) *if* $S \in \mathcal{S}$, *then* $2S := \{2s \mid s \in S\} \in \mathcal{S}$;

(iii) *if* $S, T \in \mathcal{S}$ *have* $|S \cap T| = m - 1$, *then for each* $a \in S \cap T$, $S \cup T \setminus \{a\} \in \mathcal{S}$.

Then $\{1, 2, 4, \ldots, 2^{m-1}\} \in \mathcal{S}$.

Proof of Lemma 3.6. Suppose n is odd. Then $0 \notin S_k$, for all k. Let f be a function satisfying the left side of the implication, and let $\mathcal{S}$ be the set of all strictly increasing ordered sets S, each consisting of m positive integers, for which $D_S f(x)$ exists. The assumption that $D_{S_k} f(x)$ exists, for each $k = 1, \ldots, \lfloor m/2 \rfloor$, makes the hypothesis (i) in Lemma 3.7 satisfied for this $\mathcal{S}$. Hypothesis (ii) is trivially satisfied, due to $\Delta_{2S}(h)$ being the scale by 2 of $\Delta_S(h)$, making S-differentiability of f at x equivalent to $2S$-differentiability of f at x. For (iii), suppose $D_S f(x)$ and $D_T f(x)$ exist, for S, T with $|S \cap T| = m$, and let $a \in S \cap T$. Then the linear combination $\alpha\Delta_S(h) + \beta\Delta_T(h)$ with $\alpha + \beta = 1$ that eliminates $f(x+ah)$, by symmetry, also eliminates $f(x-ah)$ and is an nth symmetric generalized Riemann difference based at $n+1$ points, so it must be $\Delta_{S\cup T\setminus\{a\}}(h)$. In addition, $D_{S\cup T\setminus\{a\}} f(x) = \alpha D_S f(x) + \beta D_T f(x)$ exists, so $S \cup T \setminus \{a\} \in \mathcal{S}$, proving hypothesis (iii). Lemma 3.7 implies that $\{1, 2, 4, \ldots, 2^{m-1}\} \in \mathcal{S}$, translating into $\widetilde{D}_n f(x)$ exists. The case when n is even is similar. □

Proof of Theorem 3.1: the unrestricted case. In the remaining part of the section we complete the proof of the direct implication in Theorem 3.1 for general test functions f at x.

Recall that the restricted proof relies on Theorem 2.2 and Lemma 3.3 which are true for general functions f. The proof of Lemma 3.3 is based on proving the linear algebra result that "$\Delta_n^{\text{sh}-}(x, h; f) \implies \widetilde{\Delta}_n^s(x, h; f)$", which means that the difference $\widetilde{\Delta}_n^s(x, h; f)$ is a linear combination of dilates by various powers of 2 of the differences in the set $\Delta_n^{\text{sh}-}(x, h; f)$ of all $n-1$ consecutive backward shifts of the nth forward Riemann difference $\Delta_n(x, h; f)$.

The general proof of Theorem 3.1 is pretty much the same as the restricted proof: it uses Lemma 3.8 instead of Theorem 2.2 and Lemma 3.9 instead of just Lemma 3.3. The difference between the general proof and the restricted proof is that the restricted proof relies only on nth symmetric differences, while the general proof relies on both nth and $n+1$-st symmetric differences.

Let $\widetilde{D}_n f(x)$ be the unique nth generalized Riemann derivative of f at x based at $x, x+h, x+2h, \ldots, x+2^{n-1}h$. The the proof of the first of the following two lemmas is given by Marcinkiewicz and Zygmund in [MZ].

Lemma 3.8. [MZ] *For each function f and real number x,*

$$\textit{both derivatives } f_{(n-1)}(x) \textit{ and } \widetilde{D}_n f(x) \textit{ exist} \iff f_{(n)}(x) \textit{ exists.}$$

Lemma 3.9. *For each function f, real number x, and integer n at least 2,*

$$D_n^{\text{sh}-} f(x) \textit{ exists} \Longrightarrow \widetilde{D}_n f(x) \textit{ exists.}$$

Proof. By Lemmas 3.3 and 3.11, it suffices to show that $\Delta_n^{\text{sh}-}(x, h; f) \Rightarrow \widetilde{\Delta}_{n+1}^s(x, h; f)$, which is the result of Lemma 3.10. □

We are now ready to provide the general proof of the direct implication in Theorem 3.1. This is as easy as the restricted proof.

General Proof of Theorem 3.1. By Lemma 3.8, it suffices to show that the system $D_n^{\mathrm{sh}-}f(x)$ implies $\widetilde{D}_n f(x)$, for each f at x, which is the result of Lemma 3.9. $\square$

Recall that the proof of Lemma 3.3 was based on proving the linear algebra result that $\Delta_n^{\mathrm{sh}-}(x,h;f) \Longrightarrow \widetilde{\Delta}_n^s(x,h;f)$. The next lemma shows that the same result holds true for $\widetilde{\Delta}_{n+1}^s(x,h;f)$ in place of $\widetilde{\Delta}_n^s(x,h;f)$, with essentially the same proof.

Lemma 3.10. *For each function f, real numbers x and h, and integer n at least 2,*

$$\Delta_n^{\mathrm{sh}-}(x,h;f) \Longrightarrow \widetilde{\Delta}_{n+1}^s(x,h;f).$$

Proof. For simplicity we write $\Delta(h)$ to mean $\Delta(x,h;f)$. Let $\Delta_{n,j}^{s'}(h) := \{\Delta_{n,-j}(h) - \Delta_{n,-n+j}(h)\}/2$, for $j = 1,\dots,\lfloor n/2 \rfloor$, be the skew-symmetrizations of the first half of differences in the set $\Delta_n^{\mathrm{sh}-}(x,h;f)$. These are symmetric differences of order $n+1$. By the same proof of the implication $\Delta_n^{\mathrm{sh}-}(x,h;f) \Longrightarrow \widetilde{\Delta}_n^s(x,h;f)$ in Lemma 3.3 that uses $\Delta_{n,j}^{s'}(h)$ instead of $\Delta_{n,j}^s(h)$ starting with line 3 of the proof of Lemma 3.4 one deduces that $\widetilde{\Delta}_{n+1}^s(x,h;f)$ is implied by $\Delta_n^{\mathrm{sh}-}(x,h;f)$. $\square$

Lemma 3.11. *For each function f, real numbers x and h, and integer n at least 2,*

$$\widetilde{\Delta}_n^s(x,h;f) \text{ and } \widetilde{\Delta}_{n+1}^s(x,h;f) \Longrightarrow \widetilde{\Delta}_n(x,h;f).$$

Proof. Suppose n is odd and let $m = (n+1)/2$ as before. Denote $y_0 = x$ and $y_{\pm k} = x \pm 2^{k-1}h$, for $k = 1,2,\cdots$. Then $\widetilde{\Delta}_n(h)$ is based at $y_0, y_1, \dots, y_n$, $\widetilde{\Delta}_n^s(h)$ is based at $y_{-m},\dots,y_{-1},y_1,\dots,y_m$ and $\widetilde{\Delta}_{n+1}^s(h)$ is based at $y_{-m},\dots,y_{-1},y_0,y_1,\dots,y_m$. Define the sequence of differences $\delta_k = \delta_k(h)$ of f at x and h, for $k = 0,1,\dots,m$, as follows: Take $\delta_0 = \widetilde{\Delta}_{n+1}^s(h)$, $\delta_1 = \widetilde{\Delta}_n^s(h)$, and for $k \ge 2$, δ_k is the unique nth generalized Riemann difference based at $y_{-m+k-1}, y_{-m+k}, \dots, y_{-m+k-1+n}$. Then $\delta_m = \widetilde{\Delta}_n(h)$.

What we need to prove is that δ_0 and δ_1 imply δ_m. For this it suffices to show that δ_k and δ_{k+1} implies δ_{k+2}, for $k = 0,1,\dots,m-2$. There are two different cases to consider.

When $k = 0$, by looking at the base points sets for δ_0 and δ_1 that were outlined above, we see that any linear combination of δ_0 and δ_1 that eliminates the base point y_{-m} will be a scalar multiple of an nth generalized Riemann difference based and the $n+1$ points $y_{-m+1},\dots,y_m$, so it must be a scalar multiple of δ_2. This proves that δ_0 and δ_1 imply δ_2.

Suppose $k > 0$. Then the base points of the nth differences δ_k, δ_{k+1} and $\delta_{k+1}(2h)$ are respectively described by the rows in the following diagram:

$$\begin{array}{ccccccccccc} y_{-m+k-1} & y_{-m+k} & \cdots & y_{-1} & y_0 & y_1 & \cdots & y_{-m+k-1+n} & & \\ & y_{-m+k} & \cdots & y_{-1} & y_0 & y_1 & \cdots & y_{-m+k-1+n} & y_{-m+k+n} & \\ y_{-m+k-1} & y_{-m+k} & \cdots & y_{-2} & y_0 & y_2 & \cdots & y_{-m+k-1+n} & y_{-m+k+n} & y_{-m+k+1+n} \end{array}$$

Any non-zero linear combination of these three differences that eliminates the base points y_{-m+k-1} and y_{-m+k} is a non-zero scalar multiple of an nth generalized Riemann difference based at the $n+1$ points $y_{-m+k+1},\dots,y_{-m+k+1+n}$, hence is a non-zero scalar multiple of δ_{k+2}. Thus δ_k and δ_{k+1} implies δ_{k+2}, as needed. The case n even has a similar proof. $\square$

4. Third characterization of the symmetric Peano derivative

This section peels off from the characterization of the Peano derivative $f_{(n)}(x)$, given in Section 3, the part that pertains to the symmetric Peano derivative $f^s_{(n)}(x)$, producing a rightful third characterization of the symmetric Peano derivative in terms of symmetric generalized Riemann derivatives.

The motivation behind such a characterization comes from the following conjecture which highlights the failure of an analogue of Theorem 2.2 to hold true for the symmetric Riemann derivative $D^s_n f(x)$ in place of $\widetilde{D}^s_n f(x)$.

Conjecture 4.1. *For all functions f and points x,*

$$f^s_{(n-2)}(x) \text{ and } D^s_n f(x) \text{ exist } \not\Longrightarrow f^s_{(n)}(x) \text{ exists.}$$

Theorem 1(i) in [ACF] says that $f_{(n-1)}(x)$ and $D^s_n f(x) \quad \not\Longrightarrow f_{(n)}(x)$, for all functions f and points x. The counterexample used in there does not have the restricted condition in the first part of the proof of Theorem 3.1 that the nth Peano derivative is equivalent to both the nth and $n-1$-st symmetric derivatives. In the hypothetical assumption that a new counterexample satisfying the above restricted condition is found, then a stronger version of Theorem 1(i) in [ACF] will hold true.

If the stronger version of Theorem 1(i) in [ACF] is true, then the following is a proof that Conjecture 4.1 is true for all functions f whose nth Peano derivative is equivalent to both the nth and $n-1$-st symmetric derivatives.

Restricted proof of Conjecture 4.1. Suppose that Conjecture 4.1 is false. Then the weaker statement that $f_{(n-1)}(x)$ and $D^s_n f(x)$ exist implies that $f_{(n)}(x)$ exist would have to be true for all f that have the restricted condition and x, contradicting the stronger version of Theorem 1(i) in [ACF]. □

Conjecture 4.1 is the symmetric analogue of the following conjecture from [ACF] on Peano derivatives:

Conjecture 4.2 ([ACF]). *For all functions f and points x,*

$$f_{(n-1)}(x) \text{ and } D_n f(x) \text{ exist } \not\Longrightarrow f_{(n)}(x) \text{ exists.}$$

The evidence for this conjecture comes from its smallest non-trivial case of $n = 3$, proved in [ACF, Theorem 1] via a clever example that does not extend to the general case n. The relationship between Conjecture 4.2 and Conjecture 4.1 is similar but not quite the same as the one between Lemma 1.1 and Theorem 1.3. Based on this, if the conjecture will turn out to be true, then this would shed more light into how the theory of symmetric Peano derivatives relates to the theory of Peano derivatives. And the main principle in this paper was to respectively view these two theories as theories of either sets of symmetric generalized Riemann derivatives or sets of generalized Riemann derivatives.

The following theorem provides a third characterization of the nth symmetric Peano derivative $f^s_{(n)}(x)$, modulo $f^s_{(n-2)}(x)$, in terms of symmetrizations of backward shifts of the nth forward Riemann derivative $D_n f(x)$, which are the same as forward shifts of the nth symmetric Riemann derivative $D^s_n f(x)$

Theorem 4.3. *For all functions f and points x,*

$$f^s_{(n-2)}(x) \text{ and all of } D^s_{n,-j} f(x), \text{ for } j = 1, \ldots, \lfloor n/2 \rfloor, \text{ exist} \Longleftrightarrow f^s_{(n)}(x) \text{ exists.}$$

Proof. As we have done several times so far, the reverse implication is clear. For the direct implication, by Theorem 2.2, it suffices to show that the set consisting of all $D^s_{n,-j}f(x)$, for $j = 1,\dots,\lfloor n/2\rfloor$, implies $\widetilde{D}^s_n f(x)$. This is the compound of the last part of the proof of Lemma 3.4, Corollary 3.5, Lemma 3.7, and Lemma 3.6. □

We end this article with an equivalent statement of Theorem 4.3, characterizing the nth symmetric Peano derivative $f^s_{(n)}(x)$ as being equivalent to a set of symmetric generalized Riemann derivatives of f at x.

Corollary 4.4. *For all functions f and points x,*

$$\text{all } D^s_{k,-j}f(x) \text{ exist, for } k = n, n-2, \dots \text{ and } j = 1,\dots,\lfloor k/2\rfloor \iff f^s_{(n)}(x) \text{ exists.}$$

Proof. This follows easily from Theorem 4.3, by induction on n with step 2. □

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